SIGN CHANGES ALONG GEODESICS OF MODULAR FORMS

DUBI KELMER, ALEX KONTOROVICH, AND CHRISTOPHER LUTSKO

ABSTRACT. Given a compact segment, β, of a cuspidal geodesic on the modular surface, we study the number of sign changes of cusp forms and Eisenstein series along β. We prove unconditionally a sharp lower bound for Eisenstein series along a full density set of spectral parameters. Conditioned on certain moment bounds, we extend this to all spectral parameters, and prove similar theorems for cusp forms. The arguments rely in part on the authors' mean square bounds [KKL24], and on removing the assumption of the Lindelöf hypothesis from recent work of Ki [Ki23].

1. INTRODUCTION

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a discrete, cofinite group acting on the upper half-plane $\mathbb{H}$ by fractional linear transformations. Given a real-valued automorphic function $f : \Gamma\backslash\mathbb{H} \to \mathbb{R}$, we denote by Z_f its zero set, which separates the space into connected nodal domains. A key question in the analysis of f is to consider the number of nodal domains. Of particular interest, with applications to quantum chaos, is to study the number of nodal domains of eigenfunctions of the hyperbolic Laplace-Beltrami operator, as the eigenvalue goes to infinity. Henceforth we work specifically with the modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$; the proofs below can be generalized to congruence subgroups, as long as they include reflection symmetries.

Recall the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$ into cusp forms and Eisenstein series. The Eisenstein series for the modular group Γ is given by

$$E(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathfrak{Im}(\gamma z)^s,$$

where Γ_∞ is the stabilizer of ∞ in Γ. This series converges absolutely for $\mathfrak{Re}(s) > 1$, and has meromorphic continuation for all $s \in \mathbb{C}$. For $s = \frac{1}{2} + it$, the function

$$E_t(z) := E(z, \frac{1}{2} + it)$$

is an eigenfunction of the Laplacian (as well as all Hecke operators), and has Laplace eigenvalue $\lambda = \frac{1}{4} + t^2$.

Moreover, a *Maass cusp form* is a function $\phi : \mathbb{H} \to \mathbb{R}$ satisfying

(i) $\Delta\phi + \lambda\phi = 0, \qquad \lambda = \lambda_\phi > 0,$
(ii) $\phi(\gamma z) = \phi(z), \qquad \gamma \in \Gamma,$
(iii) and $\phi \in L^2(\Gamma\backslash\mathbb{H})$ with L^2 norm 1.

Date: September 25, 2024.

Kelmer is partially supported by NSF CAREER grant DMS-1651563.

Kontorovich is partially supported by NSF grant DMS-2302641, BSF grant 2020119, and a Simons Fellowship.

Given such a cusp form ϕ, we write its eigenvalue as $\lambda_\phi = \frac{1}{4} + t_\phi^2$.

A heuristic argument of Bogomolny and Schmit [BS02] gives a very precise prediction for the asymptotic number $N^\Omega(\phi)$ of nodal domains of a Maass cusp form ϕ in a compact domain $\Omega \subseteq \Gamma\backslash\mathbb{H}$, namely that $N^\Omega(\phi)$ grows like a constant times λ_ϕ, as $\lambda_\phi \to \infty$. While their prediction is supported by numerics, it seems currently out of reach, and even the weaker claim that $N^\Omega(\phi) \to \infty$ as $\lambda_\phi \to \infty$ is not currently known unconditionally (and may not be true for general surfaces, see [GRS13, p.3]).

The space $\Gamma\backslash\mathbb{H}$ has an orientation reversing isometry, $\sigma(x+iy) = -x+iy$. We say that a nodal domain is inert if it is preserved by σ, and split if it is paired with another domain. We denote by $N_{\text{in}}(f)$ and $N_{\text{sp}}(f)$ the number of inert and split domains. Let $\delta \subset \mathcal{F}_\Gamma$ denote the set of fixed points of σ, which is naturally partitioned as

$$\delta = \delta_1 \cup \delta_2 \cup \delta_3,$$

with $\delta_1 = \{iy : y \geq 1\}, \delta_2 = \{\frac{1}{2} + iy : y \geq \frac{\sqrt{3}}{2}\}$ and $\delta_3 = \{x + iy : 0 < x < \frac{1}{2}, x^2 + y^2 = 1\}$. It was then observed in [GRS13] that for an even cusp form (i.e., a cusp form satisfying $\phi(\sigma z) = \phi(z)$), one can bound $N_{\text{in}}(\phi)$ by counting the number of sign changes of ϕ along δ, or more generally, along a non-empty compact segment $\beta \subseteq \delta$. Explicitly, given a segment $\beta \subseteq \delta$, let $K^\beta(\phi)$ denote the number of sign changes of ϕ along β, and $N_{\text{in}}^\beta(\phi)$ the number of nodal domains intersecting β; then

$$1 + \frac{1}{2}K^\beta(\phi) \leq N_{\text{in}}^\beta(\phi) \leq |Z_\phi \cap \beta|. \tag{1.1}$$

It is thus possible to reduce the problem of studying the number of (inert) nodal domains to studying the number of sign changes/zeros. For this problem, [GRS13] proved, assuming the Lindelöf hypothesis for the L-functions attached to ϕ, that, given a compact geodesic segment β in δ_1 or δ_2,

$$t_\phi^\nu \ll |Z_\phi \cap \beta| \ll t_\phi,$$

for any $\nu < 1/12$. (Note that the upper bound here is unconditional and follows from general complexification techniques [TZ09].) In addition, these techniques can be applied to give a similar, although still conditional, lower bound for the same problem on Eisenstein series. Following this Jang and Jung [JJ18] used arithmetic quantum unique ergodicity, to prove qualitatively that the number of nodal domains goes to ∞ with the eigenvalue. Moreover, Jung and Young [JY19] proved an unconditional but weaker lower bound for Eisenstein series with $\nu < 1/51$.

Recently, Ki [Ki23, Theorem 1] proved an essentially sharp (in the exponent) lower bound for both Maass forms and the Eisenstein series, conditional on both the Lindelöf hypothesis for the associated L-function and a fourth moment bound along β. Explicitly, Ki shows that for any $\varepsilon > 0$,

$$|Z_f \cap \beta| \gg_\varepsilon t_f^{1-\varepsilon}, \tag{1.2}$$

where f is either a Maass form or the Eisenstein series (Ki's technique can also be applied to sign changes, $K^\beta(f)$). Our Theorem 1.3 recovers this sharp lower bound for Eisenstein series *without* the assumption of the Lindelöf hypothesis, and in Theorem 1.7 we also remove assumption on the fourth moment bound, by restricting to a full-density subset of forms. Moreover, Theorems 1.9 and 1.12 show similar results for cusp forms, conditioned on an L^2 estimate for L-functions (namely, Conjecture 2.12).

While we specialize to the modular surface, we can extend this work to congruence subgroups with reflection symmetries. In addition, we specialize our analysis to the central line $z = iy$, but this can also be extended to any cuspidal geodesic, see Remark 1.14.

1.1. **Main results.** The main goal of this paper is to prove the same bound as Ki's (1.2) for the Eisenstein series, *without* assuming the Lindelöf hypothesis.

Theorem 1.3. *Let $\beta = i[a,b]$ be a compact segment of the imaginary line, and suppose that there is some $p > 2$ such that for all $\varepsilon > 0$,*

$$\left(\int_a^b |E_t(iy)|^p \frac{\mathrm{d}y}{y}\right)^{1/p} \ll_\varepsilon t^\varepsilon, \tag{1.4}$$

as $t \to \infty$. Then for any $\varepsilon > 0$,

$$K^\beta(E_t) \gg_\varepsilon t^{1-\varepsilon}, \tag{1.5}$$

as $t \to \infty$.

Remark 1.6. Explicitly what we show is that the bound of order t^ϵ for the L^p norm, implies a lower bound of order $t^{1-\epsilon'}$ for $K^\beta(E_t)$ with any $\epsilon' > \frac{8p}{p-2}\epsilon$ (see §3.3). In particular, a sufficiently strong subconvex bound for the sup norm of E_t of order t^ν with $\nu < \frac{1}{8}$ is already sufficient to obtain a non trivial lower bound for $K^\beta(E_t)$. We note however that with the current best known bound for the sup norm of E_t we can only take $\nu > \frac{1}{3}$, which is not sufficient to get an unconditional improvement here.

While (1.4) is beyond the reach of current techniques, it follows from the sup-norm conjecture. We can show that the sup norm bounds do hold for a full-density set of spectral parameters; here we say that $\mathcal{A} \subset \mathbb{R}$ is of full density to mean that $\frac{|\mathcal{A}\cap[T,2T]|}{T} \to 1$ as $T \to \infty$. This yields the following *unconditional* estimate.

Theorem 1.7. *Let $\beta = i[a,b]$ be a compact segment of the imaginary line. For any $m > 17$ there is a set $\mathcal{A} = \mathcal{A}(m) \subseteq \mathbb{R}$ of full density, such that for all $t \in \mathcal{A}$,*

$$K^\beta(E_t) \geq \frac{t}{(\log t)^m}. \tag{1.8}$$

The key insight in the proof of Theorem 1.3 is to show that, rather than the Lindelöf hypothesis, one can make do with an estimate on the L^2 norm of the L-function associated to the Eisenstein series, which translates to a fourth moment estimate on the Riemann zeta function. For Maass forms, we can make the same simplification. However, while the L^2 estimate for the associated L-function is certainly weaker than the Lindelöf hypothesis and is known in many instances, it is still not known in the precise setup needed in our context. In fact, such estimates also appear in the study of restricted quantum unique ergodicity for Maass forms and would be of interest there (see [You18]). We state the requisite L^2 estimate below as Conjecture 2.12. Assuming this conjecture holds, we can prove the analogues of Theorem 1.3 and Theorem 1.7 in the context of cusp forms:

Theorem 1.9. *Fix $\beta = i[a,b]$ a compact segment of the imaginary line, and assume that there is some $p > 2$ such that for any even Hecke cusp form ϕ and any $\varepsilon > 0$,*

$$\left(\int_a^b |\phi(iy)|^p \frac{\mathrm{d}y}{y}\right)^{1/p} \ll_\varepsilon t_\phi^\varepsilon. \tag{1.10}$$

Further, assume Conjecture 2.12. Then for any $\varepsilon > 0$,

$$K^{\beta}(\phi) \gg_{\varepsilon} t_{\phi}^{1-\varepsilon}. \tag{1.11}$$

Once again, we can prove the sup-norm conjecture for ϕ for a set of forms of full density, as follows.

Theorem 1.12. *Fix β, a compact segment of the imaginary line $i[a,b]$, and assume Conjecture 2.12. For any $\varepsilon > 0$, there is a full density set $\mathcal{A} = \mathcal{A}(\varepsilon) \subseteq \mathbb{N}$ such that for all $j \in \mathcal{A}$,*

$$K^{\beta}(\phi_j) \geq t_{\phi_j}^{1-\varepsilon}. \tag{1.13}$$

Note that for certain real Riemann surfaces, Zelditch showed logarithmic growth of the number of nodal domains, along a full-density sequence of eigenvalues, see [Zel16]. Using the bound (1.1), our Theorem 1.12 (conditionally) produces nearly linear growth.

Remark 1.14. As stated, the above theorems concern the geodesic $z = iy$. In fact, the proof below works for any cuspidal geodesic $x + iy$ with $x = \frac{p}{q}$ a rational number. For this, we require estimates on the second moment of the series

$$\sum_{n} \frac{a_f(n)e(nx)}{n^s}$$

and a lower bound on the L^2-norm of the Eisenstein series/cusp form along $\beta = \{x + iy : a < y < b\}$.

The lower bound is proved in [You18] for Eisenstein series, and in [GRS13] (although this is only proved for the lines $x = 0$, and $x = \frac{1}{2}$) for cusp forms.

For the estimates on the twisted L series, we split into congruence classes modulo q using Dirichlet characters. This allows us to write the L function as

$$\frac{1}{\phi(q)} \sum_{a \bmod q} e_q(aq) \sum_{\chi} \overline{\chi}(a) \sum_{n} \frac{a_f(n)\chi(n)}{n^s}.$$

Now for cusp forms, bounding the inner twisted L-function requires us to extend Conjecture 2.12 to these. For the Eisenstein series, this requires known estimates for the 4th moment of Dirichlet L-functions [Top21].

1.2. **Proof strategy.**

For both Eisenstein series and Maass forms, the proofs of Theorems 1.9, 1.12, 1.3 and 1.7 follow the same strategy. The starting point is [Ki23, Proof of Theorem 1], wherein Ki conditionally proves the inequality (1.13) for all cusp forms (the method also applies to Eisenstein series).

The first key idea in our proof is a modification of Ki's argument, allowing us to replace the full strength of the Lindelöf hypothesis with corresponding bounds on the second moment of the associated L-function. For cusp forms, this is Conjecture 2.12, while for the Eisenstein series, this boils down to fourth moment estimates on the Riemann zeta function which are well-known (see §2.2).

The second point (necessary only for the proofs of Theorems 1.7 and 1.12) is that, while the sup norm bound remains open for both cusp forms and the Eisenstein series, it is known on average over the spectral parameter. For the Eisenstein series, the authors [KKL24] proved a mean square bound which implies the sup norm bound for almost all spectral parameters (see §2.2). For cusp forms, a simple argument using the pre-trace formula gives similar bounds on average (see §2.3).

The proof of Theorem 1.9 is identical, but (1.10) allows one to avoid appealing to sup norm bounds on average. We focus on Theorems 1.12 and 1.7 below since the proofs include one extra step.

Notation. We use standard Vinogradov notation that $f \ll g$ if there is a constant $C > 0$ so that $f(x) \leq Cg(x)$ for all x.

Acknowledgements. We thank Valentin Blomer, Henryk Iwaniec, and Matt Young for many insightful discussions. This paper was written while the second-named author was visiting Princeton University; he would like to express his gratitude for their hospitality.

2. Preliminaries

2.1. **Littlewood's sign changes lemma.** A key analytic ingredient in Ki's proof is [Ki23, Theorem 2.2], which is a variant on a theorem of Littlewood [Lit66] controlling the number of zeros of a real valued function. While Ki's formulation (as well as Littlewood's) discusses the number of zeros, we note that the argument actually controls the number of sign changes. For the sake of completeness, we include the proof of this result below.

Given a real valued function f on the interval $I = [a, b]$ let $M_p(f)$ denote the $L^p(I)$ norm:

$$M_p(f) := \left(\frac{1}{|I|}\int_I |f(y)|^p \,\mathrm{d}y\right)^{1/p}.$$

The following is a slight variant of [Ki23, Theorem 2.2].

Lemma 2.1. *Let f be a real valued function defined on an open interval containing $I = [a, b]$. Let $N \in \mathbb{N}$ be sufficiently large so that f is defined on $[a, b+\eta]$ with $\eta = \frac{|I|}{N}$, and define*

$$J(f,\eta) = \frac{1}{|I|}\int_I \left|\int_0^\eta f(y+v)\mathrm{d}v\right| \mathrm{d}y.$$

Suppose that there is some $c \in (0,1)$ such that $M_1(f) \geq cM_2(f)$ and that $J(f,\eta) < \frac{c^3\eta M_2(f)}{16}$. Then the number of sign changes, $K^I(f)$, of f on I satisfies

$$K^I(f) \geq \frac{c^2 N}{8}.$$

Proof. By scaling and shifting f we may assume that $I = [0, 1]$ and $\eta = \frac{1}{N}$. For any $1 \leq m \leq N$ let $I_m = [\frac{m-1}{N}, \frac{m}{N})$, and define

$$J_m(f,\eta) := \int_{I_m} \left|\int_0^\eta f(y+t)dt\right| dy.$$

Let $\mathcal{M}_1 = \{1 \leq m \leq N : \ f \text{ changes sign in } I_m\}$ and let $\mathcal{M}_2$ be its complement. Since in any interval I_m with $m \in \mathcal{M}_1$ there is at least one sign change of f, we have that $K^I(f) \geq |\mathcal{M}_1| = N - |\mathcal{M}_2|$. Let $E = \cup_{m\in\mathcal{M}_2} I_m$ so that $|E| = \frac{|\mathcal{M}_2|}{N}$ and the result will follow by showing that $|E| < 1 - \frac{c^2}{8}$. We assume now that $|E| > 1 - \frac{c^2}{8}$ and proceed by contradiction.

Let $H := \{y \in I : |f(y)| \geq \frac{cM_2(f)}{2}\}$, then the assumption $M_1(f) \geq cM_2(f)$ implies that $|H| \geq 1 - \frac{c^2}{4}$. Indeed, we can estimate

$$cM_2(f) \ \leq \ M_1(f) \ \leq \ \int_H f + \int_{H^c} f \ \leq \ |H|^{1/2} M_2(f) + (1-|H|)\frac{cM_2(f)}{2}.$$

Setting $X = \sqrt{|H|}$, then from the above display we see that $X^2 - \frac{2}{c}X + 1 \leq 0$ hence $X > \frac{c}{2}$ so $|H| > \frac{c^2}{4}$. For any $m \in \mathcal{M}_2$ we have that $J_m^*(f,\eta) := \int_{I_m}\int_0^\eta |f(y+t)|dtdy = J_m(f,\eta)$. We can estimate on one hand

$$\sum_{m\in\mathcal{M}_2} J_m^*(f,\eta) = \sum_{m\in\mathcal{M}_2} J_m(f,\eta) \leq J(f,\eta) < \frac{c^3\eta M_2(f)}{16}.$$

On the other hand, we have

$$\sum_{m\in\mathcal{M}_2} J_m^*(f,\eta) = \int_0^\eta \int_E |f(y+t)|dydt = \int_0^\eta \int_{E_t} |f(y)|dydt,$$

where E_t is the shift of E by t. By our assumption $|E_t| = |E| > 1 - \frac{c^2}{8}$ and since

$$\frac{c^2}{4} \leq |H| = |H \cap E_t| + |H \cap E_t^c| < |H \cap E_t| + \frac{c^2}{8},$$

using our bounds on $|H|$, we also have that $|H \cap E_t| > \frac{c^2}{8}$. We can thus bound

$$\int_0^\eta \int_{E_t} |f(y)|dydt \geq \int_0^\eta \int_{E_t\cap H} |f(y)|dydt > \frac{c^3\eta M_2(f)}{16},$$

in contradiction. □

2.2. **Preparation for Eisenstein series.** We now collect a number of results regarding the Eisenstein series and its L-function that will be needed in our proof.

In previous work, the authors proved the following mean square bounds on the Eisenstein series:

Theorem 2.2 ([KKL24, Theorem 1]). *Given a compact region $\Omega \subseteq \mathbb{H}$ there is $c > 0$ such that for all $z \in \Omega$*

$$\frac{1}{T}\int_T^{2T} \left|E(z, \frac{1}{2} + it)\right|^2 \mathrm{d}t \leq c\log^4(T). \tag{2.3}$$

Corollary 2.4. *For any compact set Ω and any $a > 2$ there is a set $\mathcal{A} = \mathcal{A}_{\Omega,a} \subseteq \mathbb{R}$ satisfying that*

(1) $|\mathcal{A} \cap [T, 2T]| = T(1 + O(\frac{1}{\log(T)^{2a-4}}))$

(2) For any $t \in \mathcal{A}$ for any $z \in \Omega$ we have $|E_t(z)| \leq \log(t)^a$.

The next result is a lower bound on $M_2(E_t)$ given in [You18] that is needed in order to apply Lemma 2.1.

Proposition 2.5 ([You18, Theorem 1.1]).

$$M_2(E_t) := \left(\frac{1}{b-a}\int_a^b |E_t(iy)|^2\,\mathrm{d}y\right)^{1/2} \gg (\log T)^{1/2}$$

for any $t \in \mathbb{R}$ and any fixed segment (a, b).

The final result we need is about the size of the L-function of the Eisenstein series on the critical line, which can be written explicitly in terms of the Riemmann zeta function. Recall, the Lindelöf hypothesis predicts that, for any $\varepsilon > 0$ and all $t \in \mathbb{R}$, one has $|\zeta(\frac{1}{2} + it)| = O((1 + |t|)^\varepsilon)$. While the Lindelöf hypothesis is far from reach of modern technology, there are some results concerning moment bounds on the zeta function which will suffice for our purposes. The following classical theorem was proven by Heath-Brown

Theorem 2.6 ([HB79])**.** *There is $\kappa > 0$ such that for any T large one has*

$$\frac{1}{T}\int_0^T \left|\zeta(\frac{1}{2}+it)\right|^4 \mathrm{d}t = P_4(\log(T)) + O(T^{-\kappa}), \tag{2.7}$$

with $P_4(x)$ a polynomial of degree 4.

2.3. **Preparation for Maass forms.** We now collect the corresponding results we need to apply the argument for Maass forms.

The first result regards the sup norm of Maass forms. While we cannot prove the conjectured sup norm bounds for Maass forms, we can prove the following mean square bounds on them, which imply the mean square bounds on average. While this result is not new (see e.g [Iwa02, Proposition 7.2] we include a proof for the sake of completeness.

Lemma 2.8. *There is $C > 0$ such that for all z in a compact set $\Omega \subset \mathbb{H}$ we have the following bounds:*

$$\sum_{t_{\phi_j} \leq T} |\phi_j(z)|^2 \leq CT^2$$

Proof. We recall some well known results on the pre-trace formula and refer to [Hej76] for more details. Given a point pair invariant $k(z,w) = k(\sinh^2(d(z,w)))$ with $d(z,w)$ the hyperbolic distance and $k \in C_c^\infty(\mathbb{R}^+)$, its spherical transform is defined as $H(s) = \int_{\mathbb{H}^2} k(z,i)\mathfrak{Im}(z)^s d\mu(z)$. By [Hej76, Proposition 4.1] the point pair invariant can be recovered from $H(s)$ as follows : Let $h(r) = H(\frac{1}{2}+ir)$ and let $g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty h(r)e^{-iru}dr$ denote its Fourier transform, then, defining the auxiliary function $Q \in C_c^\infty(\mathbb{R}^+)$ by $g(u) = Q(\sinh^2(\frac{u}{2}))$ we have that $k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(r)}{\sqrt{r-t}}$. We also recall that $k(0) = \frac{1}{2\pi}\int_0^\infty h(r)r\tanh(\pi r)dr$ (see [Hej76, Proposition 6.4]).

Given any such point pair invariant we have the pre-trace formula

$$\sum_{\gamma\in\Gamma} k(z,\gamma z) = \sum_j h(t_{\phi_j})|\phi_j(z)|^2 + \frac{1}{2\pi}\int_{\mathbb{R}} h(t)|E(z,\tfrac{1}{2}+it)|^2 dt.$$

Now, fix a smooth even compactly supported function $g(u) \in C_c^\infty((-1,1))$ with Fourier transform $h(r) \geq 0$ for $r \in \mathbb{R}$ and $h(r) \geq \frac{1}{2}$ for $|r| \leq 1$. For any $T \geq 1$ let $g_T(u) = Tg(Tu)$ so that $h_T(r) = h(\frac{r}{T})$ and $k_T(z,w)$ the corresponding point pair invariant. Since $g_T(u)$ is supported on $(-\frac{1}{T},\frac{1}{T})$ the point pair invariant $k_T(z,w)$ is supported on the set $\{(z,w)|d(z,w) \leq \frac{1}{T}\}$ with $d(z,w)$ the hyperbolic distance. Since Γ acts properly discontinuously on $\mathbb{H}^2$ for any fixed z there is $\delta = \delta(z)$ such that $d(z,\gamma z) \geq \delta$ for any $\gamma \in \Gamma$ with $\gamma z \neq z$. In particular taking $T_0 \geq \sup\{z \in \Omega : \frac{1}{\delta(z)}\}$, for any $T \geq T_0$ we have that $k_T(z,\gamma z) = 0$ if $\gamma z \neq z$. Hence for any $T \geq T_0$ we have

$$\sum_j h(\frac{t_{\phi_j}}{T})|\phi_j(z)|^2 + \frac{1}{2\pi}\int_{\mathbb{R}} h(\frac{t}{T})|E(z,\tfrac{1}{2}+it)|^2 dt \quad = \quad |\Gamma_z|k_T(0).$$

Since $h(t) \geq 0$ is positive we can bound

$$\begin{aligned}\sum_{t_{\phi_j}\leq T} |\phi_j(z)|^2 &\leq 2|\Gamma_z|k_T(0)\\ &= \frac{|\Gamma_z|}{\pi}\int_0^\infty h(\frac{r}{T})r\tanh(\pi r)dr \ll T^2.\end{aligned}$$

□

Corollary 2.9. *For any compact set $\Omega \subseteq \mathbb{H}^2$ and any $\varepsilon > 0$ there is a set $\mathcal{A} = \mathcal{A}_{\Omega,\varepsilon} \subseteq \mathbb{N}$ satisfying that*

(1) $|\mathcal{A} \cap [T, 2T]| = T(1 + O(T^{-\varepsilon}))$
(2) For any $j \in \mathcal{A}$ for any $z \in \Omega$ we have $|\phi_j(z)| \leq t_{\phi_j}^{\varepsilon}$.

Once again, the lower bound we need for $M_2(\phi)$ is known, this time having been proved by Ghosh, Reznikov and Sarnak [GRS13].

Proposition 2.10 ([GRS13, Theorem 6.1])**.**

$$M_2(\phi) := \left(\frac{1}{b-a}\int_a^b |\phi(iy)|^2 \,\mathrm{d}y\right)^{1/2} \gg 1$$

for any segment (a, b).

The final ingredient we need is an estimate for the L-function associated to the cusp for ϕ, we now describe. Given a cusp for ϕ, we consider the Fourier expansion

$$\sum_{n \neq 0} \rho_\phi(n) y^{1/2} K_{it_\phi}(2\pi |n| y) e(nx),$$

where K is the K-Bessel function. Furthermore, we let $\lambda_\phi(n) = \frac{\rho_\phi(n)}{\rho_\phi(1)}$ denote the eigenvalues of the Hecke operators.

With the Fourier coefficients in hand we define the associated L-function

$$L_\phi(s) := \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s}. \tag{2.11}$$

The following conjecture gives a mean square bound for this L-function.

Conjecture 2.12. *Let ϕ be a Maass form with spectral parameter t_ϕ. There exists a $\delta > 0$ such that, for $2T \leq t_\phi \leq T^{1+\delta}$ and every $\varepsilon > 0$, we have*

$$\frac{1}{T}\int_T^{2T} \left|L_\phi(\frac{1}{2} + it)\right|^2 \mathrm{d}t \ll t_\phi^{\varepsilon}, \tag{2.13}$$

as $T \to \infty$.

Such an estimate clearly follows from the Lindelöf hypothesis, and we note that for the range $2T > t_\phi$ the estimate (2.13) is known (see [GRS13, Section 6.1]). While it is possible that our range $2T \leq t_\phi \leq T^{1+\delta}$ is also within reach of current technology we were not able to establish it and thus leave it as an open conjecture.

3. Proof of Theorem 1.7

We start by proving Theorem 1.7. The proof for cusp forms is more or less identical; we explain the major differences in §4. The proof for both is an application of Theorem 2.1 for which we require a lower bound on $M_2(\cdot)$ (see Proposition 2.5) and an upper bound on $J(\cdot)$.

3.1. **Upper bound on J.** Rather than work with $E_t(z)$ it is more convenient to work with

$$f_t(z) = \frac{1}{\sqrt{y}} E_t(z),$$

since y is bounded away from 0 and ∞, any statement about zeroes or nodal lines for f_t holds equally well for E_t. Thanks to Theorem 2.1, our goal is now to bound

$$J(f_t, \eta) := \frac{1}{b-a} \int_a^b \left| \int_0^\eta f_t(i(y+v)) \mathrm{d}v \right| \mathrm{d}y. \tag{3.1}$$

Proposition 3.2. *Fix an interval $(a, b) \subset \mathbb{R}_{>0}$ for all $\frac{2}{t} < \eta < 1$ and $t \geq 10$ sufficiently large*

$$J(f_t, \eta) \ll \eta \left(\frac{\log(t)^9}{\sqrt{\eta t}} + \frac{\log(t)^7}{t^{\kappa/2}} \right) + \frac{(\log(t))^9}{t} \tag{3.3}$$

Proof. First, Fourier expand the Eisenstein series: [Iwa02, (3.20)]

$$E(z, s) = y^s + \varphi(s) y^{1-s} + \frac{4\sqrt{y}}{\theta(s)} \sum_{n=1}^\infty \eta_{s-1/2}(n) K_{s-1/2}(2\pi n y) \cos(2\pi n x)$$

with $\theta(s) = \pi^{-s}\Gamma(s)\zeta(2s)$ and $\varphi(s) = \theta(1-s)\theta(s)^{-1}$, and where

$$\eta_t(n) = \sum_{ab=n} \left(\frac{a}{b}\right)^t.$$

With that, we define the L-function

$$L(t, \nu) = \sum_{n \geq 1} \frac{\eta_{it}(n)}{n^\nu}$$

It's well-known that this L-function can be related to the Riemann zeta function:

$$\begin{aligned} L(t, \nu) &= \sum_{n \geq 1} \frac{1}{n^{\nu+it}} \sum_{d|n} d^{2it} \\ &= \sum_{d \geq 1} d^{2it} \sum_{n \equiv 0 \bmod d} \frac{1}{n^{\nu+it}} \\ &= \sum_{d \geq 1} d^{2it} \sum_{n \geq 1} \frac{1}{n^{\nu+it} d^{\nu+it}} = \zeta(\nu+it)\zeta(\nu-it). \end{aligned} \tag{3.4}$$

Following [Ki23, Proof of Lemma 4.1] we can relate $J(f_t, \eta)^2$ to this L-function. Specifically, we can write

$$\begin{aligned} J(f_t, \eta)^2 &= \left(\frac{1}{b-a} \int_a^b \left| \int_0^\eta f_t(i(y+v)) \mathrm{d}v \right| \mathrm{d}y \right)^2 \\ &\ll \left(\frac{1}{b-a} \int_a^b \left| \frac{(y+\eta)^{1+it} - y^{1+it}}{1+it} + \varphi(\frac{1}{2}+it) \frac{(y+\eta)^{1-it} - y^{1-it}}{1-it} \right| \mathrm{d}y \right)^2 \\ &\quad + \left(\frac{1}{b-a} \int_a^b \left| \int_0^\eta \left[\frac{4}{\theta(s)} \sum_{n=1}^\infty \eta_{it}(n) K_{it}(2\pi n(y+v)) e(nx) \right] \mathrm{d}v \right| \mathrm{d}y \right)^2 \\ &\ll \frac{1}{t^2} + \mathcal{J}(t), \end{aligned}$$

where

$$\mathcal{J}(t) = \left(\frac{1}{b-a}\int_a^b \left|\int_0^\eta \left[\frac{4}{\theta(s)}\sum_{n=1}^\infty \eta_{it}(n)K_{it}(2\pi n(y+v))e(nx)\right] \mathrm{d}v\right| \mathrm{d}y\right)^2.$$

From here we can expand the K-Bessel function [Olv76, (10.32.13)], that is,

$$K_{it}(z) = \frac{(z/2)^{it}}{4\pi i}\int_{(c)} \Gamma(\nu)\Gamma(\nu - it)\left(\frac{z}{2}\right)^{-2\nu} \mathrm{d}\nu,$$

and set $c = 1/4$, yielding

$$\begin{aligned}
\mathcal{J}(t) &\ll \int_a^b \left|\int_0^\eta \int_{(1/4)} \left[\frac{(y+v)^{-2\nu+it}}{\theta(\frac{1}{2}+it)}\Gamma(\nu)\Gamma(\nu-it)\sum_{n=1}^\infty \eta_{it}(n)\,(\pi n)^{-2\nu+it}\right] \mathrm{d}\nu \mathrm{d}v\right|^2 \mathrm{d}y \\
&= \int_a^b \left|\int_0^\eta \int_{(1/4)} \left[\frac{(y+v)^{-2\nu}}{\theta(\frac{1}{2}+it)}\Gamma(\nu+\frac{it}{2})\Gamma(\nu-\frac{it}{2})\sum_{n=1}^\infty \eta_{it}(n)\,(\pi n)^{-2\nu}\right] \mathrm{d}\nu \mathrm{d}v\right|^2 \mathrm{d}y \\
&\ll \int_a^b \int_{(1/2)} |I(\eta,\nu,y)\gamma(\nu,t)L(t,\nu)|^2 \,\mathrm{d}\nu \mathrm{d}y
\end{aligned}$$

where $\gamma(\nu,t) = \frac{\Gamma(\frac{\nu+it}{2})\Gamma(\frac{\nu-it}{2})}{\theta(\frac{1}{2}+it)}\pi^{-\nu}$ and $I(\eta,y;\nu) := \int_0^\eta (y+v)^{-\nu}\mathrm{d}v$. We now estimate the inner integral. Write $\nu = 1/2 + ir$, and using the invariance under $r \mapsto -r$ it is enough to estimate the integral

$$\int_0^\infty \left|I(\eta, y, \frac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)L(t,\frac{1}{2}+ir)\right|^2 \mathrm{d}r.$$

Noting that $\eta < 1$ and that the interval (a, b) is fixed, we can bound the integral, I by

$$I(\eta, y; \tfrac{1}{2}+ir) = \int_0^\eta (y+v)^{-1/2}e^{-i\log(y+v)r}\mathrm{d}v \ll \min(\eta, \frac{1}{|r|}). \tag{3.5}$$

Using Stirling's formula, the γ-factor can be bounded by

$$\begin{aligned}
\gamma(\frac{1}{2}+ir,t) &= \frac{\Gamma(\frac{1/2+ir+it}{2})\Gamma(\frac{1/2+ir-it}{2})}{\theta(\frac{1}{2}+it)}\pi^{-1-ir} \\
&\ll \frac{e^{-\pi|t+r|/4}e^{-\pi|r-t|/4}}{e^{-\pi t/2}\zeta(1+2it)}\frac{1}{((1+|r-t|)(r+t))^{1/4}} \\
&\ll (\log(t))^7\frac{e^{-\pi|t+r|/4}e^{-\pi|r-t|/4}e^{\pi t/2}}{((1+|r-t|)(1+|r+t|))^{1/4}},
\end{aligned}$$

where we used the bound $\zeta(1+2it) \gg \frac{1}{\log(t)^7}$ (see [Tit51, (3.6.5)]).

First when $r \geq t$ we can bound

$$|I(\eta,y;\tfrac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)|^2 \ll \log(t)^{14}\frac{e^{-\pi(r-t)}}{((1+(r-t))(1+(r+t))^{1/2}r^2},$$

and using the convexity bound $\zeta(\frac{1}{2}+it) \ll t^{1/4}$ for the zeta function we can bound

$$|L(t,\frac{1}{2}+ir)|^2 = |\zeta(\frac{1}{2}+i(t+r))\zeta(\frac{1}{2}+i(r-t))|^2 \ll ((1+(r-t))(1+(r+t))^{1/2},$$

hence in this range

$$|I(\eta, y; \tfrac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)|^2 \ |L(t,\frac{1}{2}+ir)|^2 \ll \ t^{-2}\log(t)^{14}e^{-\pi(r-t)},$$

and we can bound

$$\int_t^\infty \left| I(\eta, y, \frac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)L(t,\frac{1}{2}+ir)\right|^2 \mathrm{d}r \ll \frac{\log(t)^{14}}{t^2}. \tag{3.6}$$

Next for the range $r \leq \frac{1}{\eta} \leq \frac{t}{2}$ we can bound

$$|I(\eta, y; \tfrac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)|^2 \ll \frac{\eta^2\log(t)^{14}}{t},$$

to get

$$\int_0^{1/\eta} \left| I(\eta, y, \frac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)L(t,\frac{1}{2}+ir)\right|^2 \mathrm{d}r \ll \frac{\eta^2\log(t)^{14}}{t}\int_0^{1/\eta}|L(t,\frac{1}{2}+ir)|^2\mathrm{d}r.$$

Now use Cauchy-Schwarz for the inner integral together with (2.7) to bound

$$\begin{aligned}\int_0^{1/\eta}|L(t,\frac{1}{2}+ir)|^2\mathrm{d}r &\ll (\int_0^{1/\eta}|\zeta(\frac{1}{2}+i(t-r))|^4\mathrm{d}r)\int_0^{1/\eta}|\zeta(\frac{1}{2}+i(t+r)|^4\mathrm{d}r)^{1/2}\\ &\ll \int_{t-1/\eta}^{t+1/\eta}|\zeta(\frac{1}{2}+ir)|^4 dt\\ &\ll (t+1/\eta)P_4(\log(t+1/\eta))) - (t-1/\eta)P_4(\log(t-1/\eta)) + O(t^{1-\kappa})\\ &\ll \frac{\log(t)^4}{\eta} + t^{1-\kappa}\end{aligned}$$

to conclude that

$$\int_0^{1/\eta} \left| I(\eta, y, \frac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)L(t,\frac{1}{2}+ir)\right|^2 \mathrm{d}r \ll \eta^2\left(\frac{\log(t)^{18}}{\eta t}+\frac{\log(t)^{14}}{t^\kappa}\right).$$

Finally, in the range $\frac{1}{\eta} \leq r \leq t$ we first bound

$$|I(\eta, y; \tfrac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)|^2 \ll \frac{(\log(t))^{14}}{r^2((1+(t-r))(1+t+r))^{1/2}},$$

hence

$$\begin{aligned}\int_{1/\eta}^t \left| I(\eta, y, \frac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)L(t,\frac{1}{2}+ir)\right|^2 \mathrm{d}r &\ll (\log(t))^{14}\int_{1/\eta}^t \frac{|L(t,\frac{1}{2}+ir)|^2}{r^2((1+(t-r))(1+t+r)))^{1/2}}\mathrm{d}r\\ &\ll \frac{(\log(t))^{14}}{\sqrt{t}}\int_0^{t-1/\eta}\frac{|L(t,\frac{1}{2}+i(t-r))|^2}{(t-r)^2(1+r)^{1/2}}\mathrm{d}r.\end{aligned}$$

Split the integral into dyadic intervals to estimate

$$\int_0^{t-1/\eta} \frac{|L(t,\frac{1}{2}+i(t-r))|^2}{(t-r)^2(1+r)^{1/2}}\mathrm{d}r \ll t^{-2}\int_0^1 |L(t,\frac{1}{2}+i(t-r))|^2\mathrm{d}r + \sum_{k=1}^{\log(t-1/\eta)} \frac{1}{2^{k/2}(t-2^k)^2}\int_{2^{k-1}}^{2^k}|L(t,\frac{1}{2}+i(t-r))|^2\mathrm{d}r$$

We can bound the first integral by

$$\begin{aligned}\int_0^1 |L(t,\frac{1}{2}+i(t-r))|^2\mathrm{d}r &= \int_0^1 |\zeta(\frac{1}{2}+ir)|^2|\zeta(\frac{1}{2}+i(2t-r)|^2\mathrm{d}r\\ &\leq \left(\int_0^1 |\zeta(\frac{1}{2}+ir)|^4\mathrm{d}r\int_{2t-1}^{2t}|\zeta(\frac{1}{2}+ir|^4\mathrm{d}r\right)^{1/2}\\ &\ll t^{1/2}(\log t)^2\end{aligned}$$

and for each dyadic interval with $A=2^k\leq t$ we have

$$\begin{aligned}\int_A^{2A}|L(t,\frac{1}{2}+i(t-r))|^2\mathrm{d}r &\ll \int_A^{2A}|\zeta(\frac{1}{2}+i(2t-r)|^2|\zeta(\frac{1}{2}+ir)|^2\mathrm{d}r\\ &\ll \left(\int_{2t-2A}^{2t-A}|\zeta(\frac{1}{2}+ir|^4\mathrm{d}r\int_A^{2A}|\zeta(\frac{1}{2}+ir)|^4\mathrm{d}r\right)^{1/2}\\ &\ll (A\log^4(t)+t^{1-\kappa})^{1/2}(A\log^4(t))^{1/2} \ll \log^4(t)(A+t^{\frac{1-\kappa}{2}}A^{1/2}).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{t-1/\eta}\frac{|L(t,\frac{1}{2}+i(t-r))|^2}{(t-r)^2(1+r)^{1/2}}\mathrm{d}r &\ll t^{-3/2}(\log t)^2+(\log(t))^4\sum_{k=1}^{\log(t-1/\eta)}\frac{2^{k/2}+t^{\frac{1-\kappa}{2}}}{(t-2^k)^2}\\ &\ll t^{-3/2}+(\log t)^4\int_1^{\log(t-1/\eta)}\frac{2^{u/2}+t^{\frac{1-\kappa}{2}}}{(t-2^u)^2}du\\ &\ll \frac{(\log t)^4}{t^{3/2}},\end{aligned}$$

and

$$\int_{1/\eta}^t \left|I(\eta,y,\frac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)L(t,\frac{1}{2}+ir)\right|^2\mathrm{d}r \ll \frac{(\log(t))^{18}}{t^2}.$$

Combining the three terms and integrating over the outer interval (a,b) we get that

$$J(f_t,\eta)^2 \ll \eta^2\left(\frac{\log(t)^{18}}{\eta t}+\frac{\log(t)^{14}}{t^\kappa}\right)+\frac{(\log(t))^{18}}{t^2},$$

and taking a square root concludes the proof. □

3.2. **Proof of Theorem 1.7.** First, by Proposition 2.5, there is a constant C_1 such that

$$M_2(f_t) \geq C_1$$

uniform for all t. Let $\omega = \frac{1}{(b-a)}$, $N = \frac{t}{(\log t)^{2m}}$, and $\eta = \frac{1}{N}$. By Proposition 4.2, there is a constant C_2 so that

$$J(f_t, \eta) \leq C_2\, \eta \left(\log(t)^{9-m} + \frac{\log(t)^7}{t^{\kappa/2}} \right)$$

Let $a > 2$. Then by Theorem 2.2, there exists a set $\mathcal{A}_{\beta,a} \subseteq \mathbb{R}$ with $|\mathcal{A}_{\beta,a} \cap [T, 2T]| = T(1 + O(\frac{1}{\log(T)^{2a-4}}))$ so that that for any $t \in \mathcal{A}_{\beta,a}$ we have that

$$\sup_{y \in [a,b]} |f_t(iy)| \leq \log(t)^a.$$

Hence for any $t \in \mathcal{A}_{\beta,a}$ we can bound

$$M_1(f_t) \geq \frac{M_2(f_t)}{(\log t)^a}.$$

Let $c = (\log t)^{-a}$ then $M_1(f_t) \geq cM_2(f_t)$. Assuming $m > 9 + 3a$, for all sufficiently large t we can bound

$$J(f_t, \eta) \leq \frac{c^3}{16} \eta M_2(f_t),$$

and hence by Theorem 2.1 we can conclude that

$$N_\beta(f_t) \geq \frac{t}{(\log t)^{2(m+a)}}.$$

And so the same statement holds for $E_t(iy)$.

3.3. **Proof of Theorem 1.3.** Assume we have an L^p bound $M_p(f_t) \ll_\epsilon t^\epsilon$ and use L^p interpolation to bound

$$M_2(f_t)^2 \leq M_1(f_t)^{\frac{p-2}{p-1}} M_p(f_t)^{\frac{p}{p-1}}.$$

This combined with the lower bound $M_2(f_t) \gg 1$ implies that there is a constant $C_1 = C_1(\epsilon)$ so that

$$M_1(f_t) \geq C_1 M_2(f_t) t^{-\frac{\epsilon p}{p-2}}.$$

Let $c = C_1 t^{-\frac{p\epsilon}{p-2}}$ and $\eta = \frac{1}{N} = t^{\delta - 1}$, so that $c^3 \eta M_2(f_t) \geq C_2 \eta t^{-\frac{3p\epsilon}{p-2}}$. From the upper bound

$$J(f_t, \eta) \leq C_5 \eta \log(t)^9 t^{-\delta/2},$$

we see that $J(f_t, \eta) \leq \frac{c^3}{16} \eta M_2(f_t)$ as long as $\delta > \frac{6p\epsilon}{p-2}$ in which case Theorem 2.1 implies that

$$K^\beta(f_t) \geq C_3 t^{1-\delta-\frac{\kappa\epsilon\epsilon}{p-2}}$$

for an appropriate constant $C_3 > 0$. In particular, we see that for any $\kappa > 8$, for all sufficiently large t we have that

$$K^\beta(f_t) \geq t^{1-\frac{\kappa\epsilon\epsilon}{p-2}},$$

from which the claim follows.

4. Proof of Theorem 1.12

The proof for cusp forms follows along nearly identical lines. Once again, to apply Theorem 2.1 we require a lower bound on $M_2(\cdot)$ (see Proposition 2.10 and an upper bound on $J(\cdot)$).

4.1. **Upper bound on J.** For the bound on $J(\phi,\eta)$ we again renormalize

$$f(z) = \frac{1}{\sqrt{y}}\phi(z).$$

Hence our goal is to bound

$$J(f,\eta) := \frac{1}{b-a}\int_a^b \left|\int_0^\eta f(i(y+v))\mathrm{d}v\right| \mathrm{d}y, \tag{4.1}$$

as follows.

Proposition 4.2. *For any compact interval $(a,b)\subset\mathbb{R}_{>0}$ and $\eta\in(\frac{2}{t},1)$, for any $\varepsilon>0$ we have that*

$$J(f,\eta) \ll \eta\frac{t_\phi^\varepsilon}{\sqrt{\eta t_\phi}} + t^{\varepsilon-1}. \tag{4.3}$$

Proof. As for the Eisenstein series, we can again Fourier expand the Maass form,

$$f(iy) = \sum_{n\neq 0}\rho_\phi(n)K_{it_\phi}(2\pi\left|n\right|y),$$

with $\rho_\phi(n)=\rho_\phi(1)\lambda_\phi(n)$, and use the integral equation of the K-Bessel function to relate $J(f,\eta)$ to the L-function (2.11). That is, we have that

$$\begin{aligned} J(f,\eta)^2 &= \left(\frac{1}{b-a}\int_a^b\left|\int_0^\eta\left[\rho_\phi(1)\sum_{n=1}^\infty\lambda_\phi(n)K_{it}(2\pi n(y+v))e(nx)\right]\mathrm{d}v\right|\mathrm{d}y\right)^2 \\ &\ll \int_a^b\int_{(1/2)}\left|I(\eta,y;\nu)\gamma(\nu,t)L_\phi(\nu)\right|^2\mathrm{d}\nu\mathrm{d}y \end{aligned}$$

where $\gamma(\nu,t)=\rho_\phi(1)\Gamma(\frac{\nu+it}{2})\Gamma(\frac{\nu-it}{2})\pi^{-\nu}$ and $I(\eta,y;\nu):=\int_0^\eta(y+v)^{-\nu}\mathrm{d}v$. We have the bound (3.5) for $I(\eta,y;\frac{1}{2}+ir)$ as before and using the bound $\rho_\phi(1)\ll t_\phi^\varepsilon e^{\frac{\pi t_\phi}{2}}$ [GRS13, (14)], and Stirling's formula, we can similarly bound

$$\gamma(\frac{1}{2}+ir,t)\ll t_\phi^\varepsilon\frac{e^{-\pi|t_\phi+r|/4}e^{-\pi\left|r-t_\phi\right|/4}e^{\pi t/2}}{((1+|r-t_\phi|)(1+|r+t^\phi|))^{1/4}}.$$

We can again reduce the inner integral to the range $0<r<\infty$ and split it into three ranges

$$\int_0^\infty\left|I(\eta,y;\frac{1}{2}+ir)\gamma(\frac{1}{2}+ir,t)L_\phi(\frac{1}{2}+ir)\right|^2\mathrm{d}r = \mathcal{I}_0^{1/\eta}+\mathcal{I}_{1/\eta}^{t_\phi}+\mathcal{I}_{t_\phi}^\infty.$$

For the last the range $r\geq t_\phi$, we can use the convexity bound $|L_\phi(\frac{1}{2}+ir)|\ll(1+r+t_\phi)^{1/4+\varepsilon}$ (see, e.g., [IS00]) to estimate

$$\begin{aligned} \mathcal{I}_{t_\phi}^\infty &\ll t_\phi^{\varepsilon-2}\int_{t_\phi}^\infty\frac{e^{-\pi(r-t_\phi)}(1+r+t_\phi)^{1/2+\varepsilon}}{((1+|r-t_\phi|)(1+|r+t^\phi|))^{1/2}}\mathrm{d}r \\ &\ll t_\phi^{\varepsilon-2}\int_0^\infty\frac{e^{-\pi r}(1+r+2t_\phi)^{1/2+\varepsilon}}{(1+r)(r+2t^\phi|))^{1/2}}\mathrm{d}r \ll t_\phi^{\varepsilon-2}. \end{aligned}$$

In the first range when $r \leq 1/\eta$, we have that

$$\begin{aligned}\mathcal{I}_0^{1/\eta} &\ll t_\phi^{\varepsilon-1}\eta^2 \int_0^{1/\eta} |L_\phi(\frac{1}{2}+ir)|^2 \mathrm{d}r \\ &\ll \eta^2 \frac{t_\phi^\varepsilon}{\eta t_\phi}.\end{aligned}$$

Finally, for $1/\eta < t < t_\phi$, split to dyadic intervals and apply Conjecture 2.12:

$$\begin{aligned}\mathcal{I}_{1/\eta}^{t_\phi} &\ll t_\phi^\varepsilon \int_{1/\eta}^{t_\phi} \frac{|L_\phi(\frac{1}{2}+ir)|^2}{((r+t_\phi)(t_\phi-r))^{1/2}r^2}\mathrm{d}r \\ &\ll t_\phi^\varepsilon \sum_{k=\log(1/\eta)}^{\log(t_\phi)} \frac{1}{((2^k+t_\phi)(t_\phi-2^k))^{1/2}2^{2k}} \int_{2^{k-1}}^{2^k} |L_\phi(\frac{1}{2}+ir)|^2 \mathrm{d}r \\ &\ll t_\phi^{2\varepsilon} \sum_{k=\log(1/\eta)}^{\log(t_\phi)} \frac{1}{((2^k+t_\phi)(t_\phi-2^k))^{1/2}2^k} \\ &\ll t_\phi^{2\varepsilon} \int_{\log(1/\eta)}^{\log(t_\phi)} \frac{1}{((2^u+t_\phi)(t_\phi-2^u))^{1/2}2^u}du \\ &\ll t_\phi^{2\varepsilon} \int_{1/\eta}^{t_\phi} \frac{1}{((v+t_\phi)(t_\phi-v))^{1/2}v^2}dv \\ &\ll t_\phi^{2\varepsilon-2} \int_{1/\eta t}^{1} \frac{1}{((1+v)(1-v))^{1/2}v^2}dv \ll t_\phi^{2\varepsilon-2}.\end{aligned}$$

Integrating over (a,b) we see that

$$J(f,\eta)^2 \ll \eta^2 \frac{t_\phi^\varepsilon}{\eta t_\phi} + t_\phi^{\varepsilon-2},$$

and taking square roots concludes the proof. □

Proof of Theorem 1.12. As above, let $f_j(z) = y^{-1/2}\phi_j(z)$ and let $t_j = t_{\phi_j}$. Fix $\varepsilon > 0$ and let $\mathcal{A}_\varepsilon = \{j \in \mathbb{Z} : \sup_{z\in\beta}|f_j(z)|\} \leq t_j^{\varepsilon/16}$. By Corollary 2.9, we have that $\mathcal{A}_\varepsilon$ is of full density. Now for any $j \in \mathcal{A}_\varepsilon$, let $c_j = t_j^{-\varepsilon/16}$, let $N_j = t_j^{1-\varepsilon/2}$, and fix $\omega = \frac{1}{|\beta|}$ and $\eta_j = N_j^{-1}$, so that $\eta_j = \frac{N_j}{\omega(b-a)}$. Since $|f_j(iy)| \leq c_j$ for any $y \in [a,b]$, we have that $M_1(f_j) \geq c_j M_2(f_j)$. By Proposition 2.5 there is an absolute constant C_1 so that $M_2(f_j) \geq C_1$. Let $\varepsilon' < \varepsilon/16$; then by Proposition 4.2, there is a constant $C_2 > 0$ so that

$$J(f_j,\eta_j) \;\leq\; C_2\eta_j t_j^{\varepsilon'-\varepsilon/4} \;\leq\; \frac{c_j^3}{16}\eta_j M_2(f_j),$$

when t_j is sufficiently large. Hence by Theorem 2.1, we have that

$$N_\beta(f_j) \geq \frac{c_j^2 N_j}{10(\omega+2)} \geq t_j^{1-\varepsilon},$$

as claimed. □

References

[BS02] E. Bogomolny and C. Schmit. Percolation model for nodal domains of chaotic wave functions. *Phys. Rev. Lett.*, 88:114102, 2002. 2

[GRS13] A. Ghosh, A. Reznikov, and P. Sarnak. Nodal domains of Maass forms I. *Geom. Funct. Anal.*, 23(5):1515–1568, 2013. 2, 4, 8, 14

[HB79] D. R. Heath-Brown. The fourth power moment of the Riemann zeta function. *Proc. London Math. Soc. (3)*, 38(3):385–422, 1979. 7

[Hej76] D. Hejhal. *The Selberg trace formula for* $\mathrm{PSL}(2, R)$*. Vol. I.* Lecture Notes in Mathematics, Vol. 548. Springer-Verlag, Berlin-New York, 1976. 7

[IS00] H. Iwaniec and P. Sarnak. Perspectives on the analytic theory of L-functions. pages 705–741. 2000. GAFA 2000 (Tel Aviv, 1999). 14

[Iwa02] H. Iwaniec. *Spectral methods of automorphic forms*, volume 53 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, second edition, 2002. 7, 9

[JJ18] S. Jang and J. Jung. Quantum unique ergodicity and the number of nodal domains of eigenfunctions. *J. Amer. Math. Soc.*, 31(2):303–318, 2018. 2

[JY19] J. Jung and M. Young. Sign changes of the Eisenstein series on the critical line. *Int. Math. Res. Not. IMRN*, (3):641–672, 2019. 2

[Ki23] H. Ki. Lˆ4-norms and sign changes of Maass forms. *arXiv:2302.02625*, 2023. 1, 2, 4, 5, 9

[KKL24] D. Kelmer, K. Kontorovich, and C. Lutsko. Mean square bounds on Eisenstein series. *To Appear: Int. J. Number Theory*, 2024. 1, 4, 6

[Lit66] J. E. Littlewood. The real zeros and value distributions of real trigonometrical polynomials. *J. London Math. Soc.*, 41:336–342, 1966. 5

[Olv76] F. W. J. Olver. *Airy and related functions.* NIST handbook of mathematical functions. U.S. Dept. Commerce, 1976. 10

[Tit51] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function.* Oxford Univ. Press, London/New York, Oxford, 1951. 10

[Top21] Berke Topacogullari. The fourth moment of individual Dirichlet L-functions on the critical line. *Math. Z.*, 298(1-2):577–624, 2021. 4

[TZ09] J. A. Toth and S. Zelditch. Counting nodal lines which touch the boundary of an analytic domain. *J. Differential Geom.*, 81(3):649–686, 2009. 2

[You18] M. Young. Equidistribution of Eisenstein series on geodesic segments. *Adv. Math.*, 340:1166–1218, 2018. 3, 4, 6

[Zel16] Steve Zelditch. Logarithmic lower bound on the number of nodal domains. *J. Spectr. Theory*, 6(4):1047–1086, 2016. 4

Email address: kelmer@bc.edu

Department of Mathematics, Boston College, Boston, Massachusetts, United States

Email address: alex.kontorovich@rutgers.edu

Department of Mathematics, Rutgers University, New Brunswick, New Jersey, United States

Email address: clutsko@uh.edu

Department of Mathematics, University of Houston, Houston, Texas, United States