

LOSS OF MEMORY IN THE PERIODIC EHRENFEST MODEL WITH SMALL POLYHEDRAL SCATTERERS

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ABSTRACT. We determine the Boltzmann–Grad limit of the Ehrenfest “wind-tree” model with a periodic configuration of polyhedral scatterers. The central result is a loss-of-memory effect in dimension $d \geq 3$ for a generic choice of polyhedron. This yields a Markovian limiting transport process on an extended state space and a generalised linear Boltzmann equation for the time evolution of the particle density.

1. INTRODUCTION

The original wind-tree model, introduced by P. and T. Ehrenfest in their classic 1911 encyclopedia article on the conceptual foundations of statistical mechanics [EE11], describes a cloud of non-interacting point particles in an array of identically oriented, congruent square scatterers placed at random locations in the two-dimensional plane $\mathbb{R}^2$. It was designed as a more accessible variant of the Lorentz gas [Lor05] (which has spherical scatterers), given the simpler reflection law for the velocities. Gallavotti [Gal69] proved that in the Boltzmann–Grad limit, the evolution of the particle density is described—in both models—by the linear Boltzmann equation whose collision kernel is the appropriate differential scattering cross section of the single scatterer. Subsequently, Gallavotti’s kinetic result was extended to an intermediate kinetic-diffusive time-scale in [LT20, LT21] to prove invariance principles for both models.

The objective of the present paper is to study the analogous question for the Ehrenfest model in the periodic setting. We work in general dimension $d \geq 2$, with scatterers located at the points of a covolume-one Euclidean lattice $\mathcal{L}$, and consider general polyhedral scatterers $\mathcal{P}$. We assume throughout that $\mathcal{P} \subset \mathbb{R}^d$ is a bounded open convex polyhedron with $0 \in \mathcal{P}$ and $\text{vol}(\mathcal{P}) = 1$. We will prove that, for $d \geq 3$ and a “generic” polyhedron, the Boltzmann–Grad limit is a Markovian random flight on an extended state space. Besides the usual phase coordinates $(Q, V) \in T^1(\mathbb{R}^d)$, the extended space retains the time ξ until the next collision and the labelled impact parameter $W = (\ell, b)$. We refer to the billiard dynamics lifted to this extended state space as the *lifted Ehrenfest flow*; note that a trajectory of the lifted flow is fully determined by initial data in (Q, V) . The precise hypothesis on the polyhedron is stated in Section 4 in terms of an $\mathcal{L}$ -*nonresonance condition*. This nonresonance mechanism has no two-dimensional analogue (Remark 3).

The main result of this paper is the following.

Theorem 1. *Let $d \geq 3$, let $\mathcal{L}$ be a Euclidean lattice of covolume one, and let $\mathcal{P} \subset \mathbb{R}^d$ be as above and $\mathcal{L}$ -nonresonant. For random initial data (Q, V) distributed according to*

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an absolutely continuous probability measure, the lifted Ehrenfest flow converges under the Boltzmann–Grad scaling in finite-dimensional distributions to a time-homogeneous Markov process. The evolution of the lifted particle density f is given by the generalised linear Boltzmann equation

$$(1.1) \quad (\partial_t + V \cdot \nabla_Q - \partial_\xi) f(t, Q, V, \xi, W) = [\mathcal{C}_P f](t, Q, V, \xi, W)$$

where $\mathcal{C}_P$ is the collision operator (8.7).

Perhaps surprisingly, the generalised Boltzmann equation for the periodic Ehrenfest model with generic polyhedra is of the same type as the one found for the periodic Lorentz gas [CG10, MS10, MS11, MS24]. As we will show, its collision term combines the polyhedron’s scattering law with the free-path kernel.

The paper is organised as follows. Section 2 introduces the periodic Ehrenfest billiard, including the impact and departure coordinates used throughout the paper. Section 3 recasts a single free flight as a first-point problem for an affine lattice. The geometric and measure-theoretic properties of this construction yield the joint Boltzmann–Grad limit of the first free path and its labelled impact parameter, valid in every dimension $d \geq 2$.

The passage from one flight to a collision history requires simultaneous equidistribution of the lattices obtained by unfolding the successive reflections. Section 4 therefore formulates a lattice-relative nonresonance condition on the face reflections, proves that it is generic in both the measure-theoretic and Baire-category senses when $d \geq 3$, and explains the obstruction in dimension $d = 2$. In Section 5, product equidistribution is combined with a recursive first-point construction to obtain the impact-resolved law of every fixed finite collision record. The factorisation of this law is the loss-of-memory mechanism behind the limiting dynamics.

Section 6 pushes the impact-resolved law forward to velocity and segment variables. In particular, it records the differential scattering cross section of a polyhedron and explains why discarding the impact coordinate does not in general give a closed Markov description. Section 7 then passes from a fixed number of collisions to macroscopic continuous time, proving finite-dimensional convergence and excluding explosion of the limiting flight process. Finally, Section 8 constructs the extended Markov process and proves the full convergence statement. The Kolmogorov forward equation for the process yields the generalised Boltzmann equation (1.1). This completes the proof of Theorem 1.

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2. THE DYNAMICAL SETTING

Let $d \geq 2$, let $S_1^{d-1} = \{v \in \mathbb{R}^d : \|v\| = 1\}$, and fix a covolume-one Euclidean lattice $\mathcal{L} = \mathbb{Z}^d M_0$, with $M_0 \in \text{SL}(d, \mathbb{R})$. All vectors are represented as row vectors. Define the

regular boundary of the polyhedron $\mathcal{P}$ as a labelled disjoint union

$$\partial_* \mathcal{P} = \bigsqcup_{\ell=1}^s F_\ell,$$

where $F_1, \dots, F_s$ are the relative interiors of the codimension-one faces (or facets) of $\overline{\mathcal{P}}$ and n_ℓ is the outward unit normal to F_ℓ . Write dA_ℓ for $(d-1)$ -dimensional Euclidean measure on F_ℓ .

For $\rho > 0$ define the open obstacle set and the completed billiard domain by

$$\mathcal{O}_\rho = \bigcup_{m \in \mathcal{L}} (m + \rho \mathcal{P}), \quad \mathcal{K}_\rho = \mathbb{R}^d \setminus \mathcal{O}_\rho.$$

We take ρ small enough that these obstacles are disjoint, and write $T^1(\mathcal{K}_\rho)$ for the usual completed billiard phase space. At a regular collision with F_ℓ , specular reflection sends the incoming velocity v to

$$vR_\ell = v - 2(v \cdot n_\ell)n_\ell,$$

where $R_\ell = I - 2n_\ell n_\ell$. Billiard trajectories and the corresponding initial data (q_{in}, v_0) are regular if they encounter only regular collisions. To define the billiard dynamics for all initial data, we agree that trajectories terminate when they hit a lower-dimensional face or graze a face. The initial data leading to such trajectories form a Liouville-null set and, as we shall see, do not affect the Boltzmann–Grad limit.

For regular initial data (q_{in}, v_0) , let t_j be the collision times, $\tau_j = t_j - t_{j-1}$ the free-flight times, q_j the collision points, and v_j the post-collision velocities. We also put $t_0 = 0$. For an initial free phase point (q, v) we write

$$\tau_1(q, v; \rho) = \inf\{t > 0 : q + tv \in \partial \mathcal{O}_\rho\},$$

with $\inf \emptyset = \infty$. If the j th collision is at the point $q_j \in m_j + \rho F_{\ell_j}$, set

$$x_j^{(\rho)} = \rho^{-1}(q_j - m_j) \in F_{\ell_j}, \quad v_j = v_{j-1}R_{\ell_j}.$$

If $\tau_j = \infty$, or if the dynamics for nonregular initial data terminates at time t_j , we formally declare all subsequent flight times, collision points, and velocities undefined.

Let $e_1 = (1, 0, \dots, 0)$. Fix a measurable map $K : S_1^{d-1} \rightarrow \text{SO}(d)$, smooth away from one direction v_s , such that

$$vK(v) = e_1.$$

For an explicit construction, see [MS10]. Let ω denote the uniform probability measure on S_1^{d-1} , put $E = e_1^\perp$, and write $\pi_\perp$ and vol_E for orthogonal projection and Lebesgue measure on E . For $v \in S_1^{d-1}$ define the projected shadow and the impact shadow by

$$C_v = \pi_\perp(\mathcal{P}K(v)), \quad B_v = -C_v \subset E.$$

The sign in $B_v = -C_v$ is a coordinate convention. Put

$$A_{\mathcal{P}} = \sup_{v \in S_1^{d-1}} \text{vol}_E(B_v) < \infty.$$

The family $\{B_v\}$ has compact closure in the Hausdorff topology, with inradii uniformly bounded below and outradii uniformly bounded above. For $b \in B_v$ define

$$t_v^-(b) = \inf\{t \in \mathbb{R} : te_1 - b \in \mathcal{P}K(v)\}, \quad x_v^-(b) = (t_v^-(b)e_1 - b)K(v)^{-1}.$$

If $x_v^-(b)$ lies in the regular incoming boundary

$$\partial_- \mathcal{P}(v) = \bigsqcup_{v \cdot n_\ell < 0} F_\ell$$

it determines a unique face label $\ell_v^-(b) \in \{1, \dots, s\}$; on the remaining null set of b , extend $\ell_v^-(b)$ measurably by choosing the least index among the adjacent incoming faces. In this way, the face cells

$$B_{v,\ell} = \{b \in B_v : \ell_v^-(b) = \ell\}$$

partition B_v . Define the impact space at velocity v by

$$\mathcal{B}_v = \bigsqcup_{\ell=1}^s (\{\ell\} \times B_{v,\ell}).$$

Then $W = (\ell, b) \in \mathcal{B}_v$ is the *labelled impact parameter*. For regular b , it represents the point $x_v^-(b) \in F_\ell$; on the null cell boundaries, ℓ is the measurable tie-breaking label as specified above. The global impact bundle

$$\mathcal{B} = \{(v, W) : v \in S_1^{d-1}, W \in \mathcal{B}_v\}$$

is given the Borel structure and cellwise topology inherited from $S_1^{d-1} \times \{1, \dots, s\} \times E$.

If a particle with velocity v has next impact parameter b , define

$$\begin{aligned} \mathcal{R}(v, b) &= vR_{\ell_v^-(b)}, \\ \mathcal{Z}(v, b) &= \pi_\perp(x_v^-(b)K(\mathcal{R}(v, b))). \end{aligned}$$

If $b \in B_{v,\ell}$ is a regular impact and $u = vR_\ell$, then $\mathcal{Z}(v, b) \in \pi_\perp(F_\ell K(u))$. Modulo the null cell boundaries, the facewise map

$$B_{v,\ell} \longrightarrow \pi_\perp(F_\ell K(u)), \quad b \longmapsto \mathcal{Z}(v, b),$$

is measure preserving: the source and target are projections of the same face and, since $u \cdot n_\ell = -v \cdot n_\ell$, their flux measures are $(-v \cdot n_\ell)dA_\ell$. For a regular collision record we therefore put

$$(2.1) \quad \xi_j^{(\rho)} = \rho^{d-1}\tau_j, \quad b_j^{(\rho)} = -\pi_\perp(x_j^{(\rho)}K(v_{j-1})), \quad W_j^{(\rho)} = (\ell_j, b_j^{(\rho)}), \quad z_j^{(\rho)} = \mathcal{Z}(v_{j-1}, b_j^{(\rho)}).$$

Figure 1 summarises these coordinates.

Similarly, for $z \in C_v$ put

$$t_v^+(z) = \sup\{t \in \mathbb{R} : te_1 + z \in \mathcal{P}K(v)\}, \quad x_v^+(z) = (t_v^+(z)e_1 + z)K(v)^{-1}.$$

For almost every z , this point lies in the regular outgoing boundary

$$\partial_+ \mathcal{P}(v) = \bigsqcup_{v \cdot n_\ell > 0} F_\ell.$$

Furthermore, define

$$C_{v,\ell} = \{z \in C_v : x_v^+(z) \in F_\ell\}.$$

Thus, modulo the null cell boundaries, the target of the preceding facewise map is $C_{u,\ell}$, and b and z are the incoming impact and outgoing departure parameters. Our sign convention gives the terminal lattice point $y = \xi e_1 + b + z$, cf. Figure 2.

The projected measures are the billiard flux measures. Namely,

$$(2.2) \quad db = (-v \cdot n_\ell) dA_\ell(x), \quad x \in F_\ell, \quad v \cdot n_\ell < 0,$$

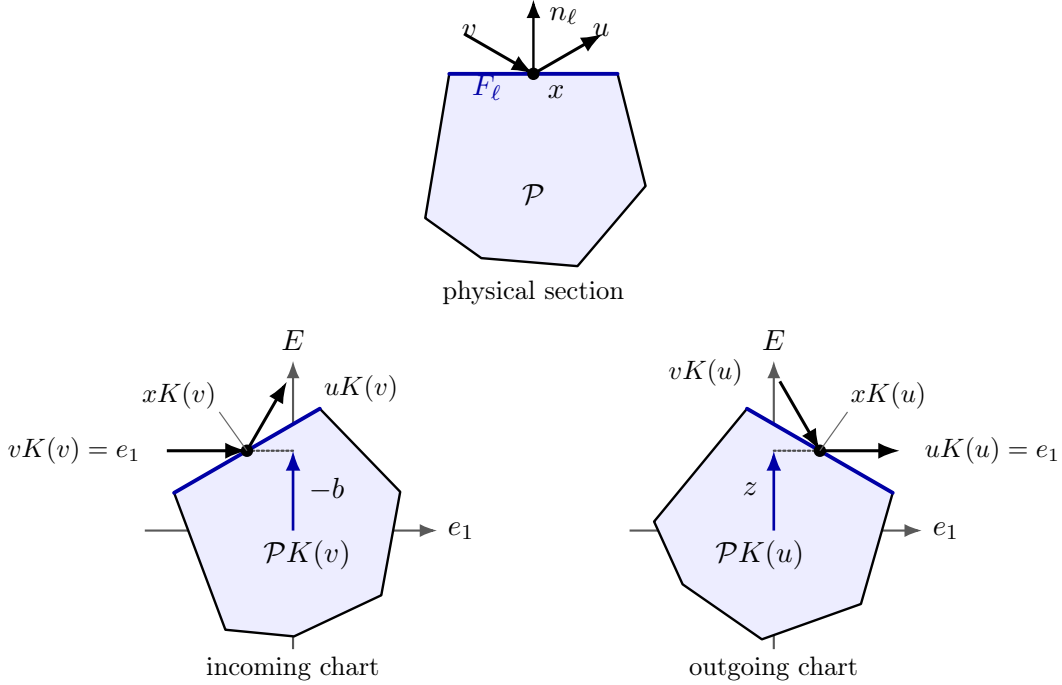


FIGURE 1. A planar section of one regular collision in physical coordinates and in the two straightened charts. Every panel shows the same polygonal section, collision point, and velocity pair, rotated together. In the incoming chart $vK(v) = e_1$, and the dotted segment projects $xK(v)$ orthogonally onto E , giving $-b$. In the outgoing chart $uK(u) = e_1$, and the corresponding projection gives z .

and on a regular outgoing face,

$$(2.3) \quad dz = (v \cdot n_\ell) dA_\ell(x), \quad x \in F_\ell, \quad v \cdot n_\ell > 0.$$

3. LIMIT LAW FOR THE FIRST COLLISION

We now adapt the first-collision construction of [MS10, Sections 4 and 7]. Put

$$\text{ASL}(d, \mathbb{R}) = \text{SL}(d, \mathbb{R}) \ltimes \mathbb{R}^d, \quad \text{ASL}(d, \mathbb{Z}) = \text{SL}(d, \mathbb{Z}) \ltimes \mathbb{Z}^d,$$

with multiplication law $(M_1, \eta_1)(M_2, \eta_2) = (M_1 M_2, \eta_1 M_2 + \eta_2)$, and define

$$X = \text{ASL}(d, \mathbb{Z}) \backslash \text{ASL}(d, \mathbb{R}), \quad X_q = \Gamma(q) \backslash \text{SL}(d, \mathbb{R}),$$

where $\Gamma(q) = \{\gamma \in \text{SL}(d, \mathbb{Z}) : \gamma \equiv I \pmod{q}\}$. If $\alpha \in \mathbb{Q}^d$, let q_α be its least positive denominator and put $X_\alpha = X_{q_\alpha}$; for $\alpha \in \mathbb{R}^d \setminus \mathbb{Q}^d$, put $X_\alpha = X$. Write μ_α for Haar probability measure on X_α . The affine lattice, as a point set, represented by $g \in X_\alpha$ is

$$\Lambda_\alpha(g) = \begin{cases} (\mathbb{Z}^d + \alpha)M, & g = \Gamma(q_\alpha)M \in X_{q_\alpha}, \quad \alpha \in q_\alpha^{-1}\mathbb{Z}^d, \\ \mathbb{Z}^d M + \eta, & g = \text{ASL}(d, \mathbb{Z})(M, \eta) \in X, \quad \alpha \notin \mathbb{Q}^d. \end{cases}$$

Write $X_\alpha(y) = \{g : y \in \Lambda_\alpha(g)\}$ and $\nu_y^{(\alpha)}$ for the disintegrated probability measure on this fibre; see [MS10, Section 7].

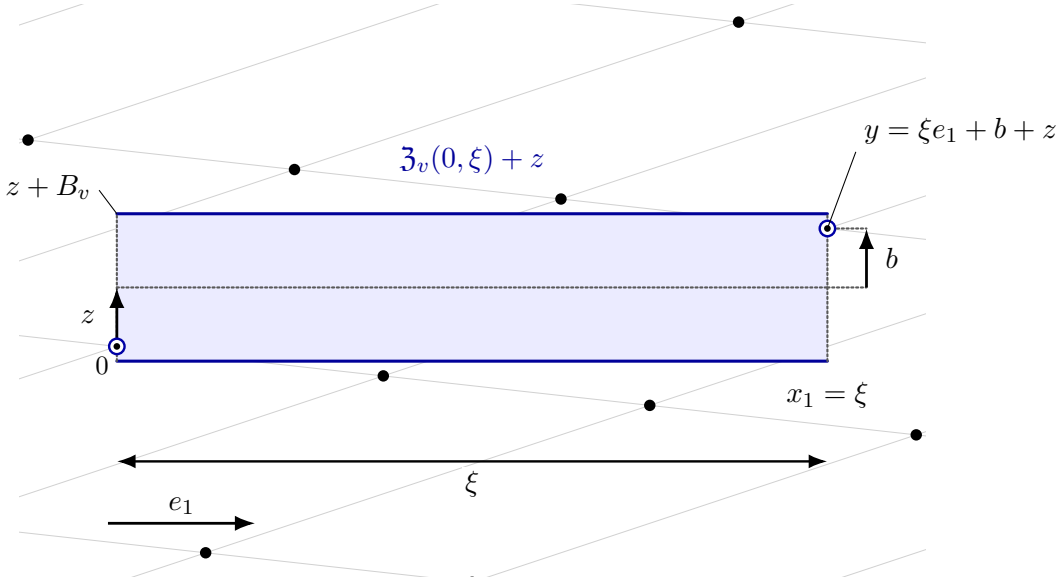


FIGURE 2. A planar schematic of the collision tube. The black dots represent the rotated and stretched lattice. The source centre 0 lies on the left end section and is joined by the vector z to the dotted longitudinal reference line. The terminal centre y lies on the right end section $x_1 = \xi$, with terminal impact parameter b .

Let $B \subset E$ be a bounded open convex set with nonempty interior and Lebesgue-null boundary. For $c_1 < c_2$ write

$$\mathfrak{Z}_B(c_1, c_2) = \{x_1 e_1 + u : c_1 < x_1 < c_2, u \in B\}.$$

For $\xi > 0$, $b \in B$, and $z \in E$, put $y = \xi e_1 + b + z$ and define

$$(3.1) \quad \Phi_{\alpha, B}(\xi, b, z) = \nu_y^{(\alpha)}(\{g \in X_\alpha(y) : \Lambda_\alpha(g) \cap (\mathfrak{Z}_B(0, \xi) + z) = \emptyset\}).$$

If we set $B = B_v$, we obtain the collision tube $\mathfrak{Z}_v(c_1, c_2) := \mathfrak{Z}_{B_v}(c_1, c_2)$ and kernel $\Phi_{\alpha, v}(\xi, b, z) := \Phi_{\alpha, B_v}(\xi, b, z)$. Here z is the departure parameter, b the terminal impact parameter, and y the transformed centre of the terminal scatterer. Figure 2 depicts the collision-to-collision case $\alpha = 0$. The source centre 0 and terminal centre y lie on the two end sections of the tube.

Lemma 1. (i) *Given any $z \in E$, for μ_α -almost every g , the set $\Lambda_\alpha(g) \cap (\mathfrak{Z}_B(0, \infty) + z)$ has a unique point with least positive first coordinate; write it as $y_B(g) = \xi_B(g)e_1 + b_B(g) + z$.*

(ii) *For $z \in E$ and every Borel set $A \subset \mathbb{R}_{>0} \times B$,*

$$(3.2) \quad \mu_\alpha\{g : (\xi_B(g), b_B(g)) \in A\} = \int_A \Phi_{\alpha, B}(\xi, b, z) d\xi db.$$

(iii) *For $z \in E$,*

$$(3.3) \quad \int_0^\infty \int_B \Phi_{\alpha, B}(\xi, b, z) db d\xi = 1.$$

(iv) *The map $(\xi, b, z) \mapsto \Phi_{\alpha, B}(\xi, b, z)$ is Borel measurable.*

Proof. For a bounded Borel set $A \subset \mathbb{R}_{>0} \times B$, bounded away from $\xi = 0$, put

$$N_A^{\text{first}}(g) = \sum_{\substack{y=\xi e_1+b+z \in \Lambda_\alpha(g) \\ (\xi,b) \in A}} \mathbf{1}\{\Lambda_\alpha(g) \cap (\mathfrak{Z}_B(0,\xi) + z) = \emptyset\}.$$

Outside the set of terminal ties and tube boundary points this is the indicator of the event that the first point has coordinates in A . Firstly, there are no tube boundary points almost surely, by Siegel's volume formula. To exclude ties, thicken the terminal hyperplane and apply [MS10, Lemmas 7.12–7.13]; the resulting two-point bound tends to zero with the thickness. The foliation formulas [MS10, Propositions 7.3 and 7.10] give

$$(3.4) \quad \int N_A^{\text{first}}(g) d\mu_\alpha(g) = \int_A \nu_{\xi e_1+b+z}^{(\alpha)} \{g : \Lambda_\alpha(g) \cap (\mathfrak{Z}_B(0,\xi) + z) = \emptyset\} d\xi db.$$

This yields (3.2) for bounded test sets. The translated-cylinder bound of [MS10, Section 8.4], applied to a ball contained in B , shows that the first point is finite almost surely. Hence we obtain (3.2) for general Borel sets and (3.3). Finally, the incidence and avoidance conditions are Borel, as is the kernel $y \mapsto \nu_y^{(\alpha)}$; hence $\Phi_{\alpha,B}$ is Borel measurable. $\square$

Define the *Hausdorff distance* between nonempty compact sets $A, A' \subset E$ by

$$d_H(A, A') = \max \left\{ \sup_{x \in A} \text{dist}(x, A'), \sup_{x' \in A'} \text{dist}(x', A) \right\}.$$

For bounded open convex sets we apply this distance to their closures. The inradius of B is the supremal radius of a ball contained in B , and its outradius the infimal radius of an origin-centred ball containing it.

Lemma 2. (i) *For nonempty bounded open convex sets $B, B' \subset E$, $z, z' \in E$, and $T < \infty$,*

$$(3.5) \quad \begin{aligned} & \mu_\alpha \{g : \Lambda_\alpha(g) \cap [(\mathfrak{Z}_B(0,T) + z) \triangle (\mathfrak{Z}_{B'}(0,T) + z')] \neq \emptyset\} \\ & \leq \text{vol}((\mathfrak{Z}_B(0,T) + z) \triangle (\mathfrak{Z}_{B'}(0,T) + z')). \end{aligned}$$

(ii) *If $B_k \rightarrow B$ in the Hausdorff topology and $z_k \rightarrow z$, the corresponding first-point laws converge against every bounded continuous test function on $\mathbb{R}_{>0} \times E$ supported in $0 < \xi \leq T$.*

(iii) *If B ranges over a Hausdorff-compact convex family whose inradii are uniformly bounded below by a positive constant and whose outradii are uniformly bounded above, and z ranges over a compact subset of E , then the laws are uniformly tight; explicitly,*

$$\lim_{T \rightarrow \infty} \sup_{B,z} \mu_\alpha \{\xi_B > T\} = 0.$$

Proof. Part (i) is Siegel's formula applied to the symmetric difference. Coupling the two first-point constructions on the same lattice gives (ii). Finally, every member of the family contains a ball of uniformly positive radius, so the bound of [MS10, Section 8.4] gives (iii). $\square$

Remark 1. Write $O(E)$ and $SO(E)$ for the orthogonal and orientation-preserving orthogonal groups of the Euclidean space E . For $O \in SO(E)$,

$$(3.6) \quad \Phi_{\alpha,BO}(\xi, bO, zO) = \Phi_{\alpha,B}(\xi, b, z).$$

The same identity holds for $O \in O(E)$ in the affine-Haar case and when $\alpha = 0$. Right multiplication by $\text{diag}(1, O)$ proves the first assertion. When $\det O = -1$, compose on the left with any $J \in GL(d, \mathbb{Z})$ of determinant -1 ; this proves the remaining cases.

Apply Lemma 1 with $B = B_v$. Formula (3.1) can be viewed as the polyhedral analogue of [MS10, Eq. (4.14)]. For $\alpha \notin \mathbb{Q}^d$ the kernel is independent of α and z ; we then write $\Phi_v(\xi, b)$. Extended by zero for $b \notin B_v$, the map $(v, \xi, b, z) \mapsto \Phi_{\alpha,v}(\xi, b, z)$ is Borel, as are its restrictions to the face cells. Lemma 2 gives the required integrated continuity in v .

Fix a microscopic base point $q_\star \in \mathbb{R}^d$ and let $q_{\rho,\beta}(v) = q_\star + \rho\beta(v)$, where $\beta : S_1^{d-1} \rightarrow \mathbb{R}^d$ is continuous. When $q_\star \in \mathcal{L}$ assume the admissibility condition

$$(3.7) \quad (\beta(v) + \mathbb{R}_{>0}v) \cap \overline{\mathcal{P}} = \emptyset \quad (v \in S_1^{d-1});$$

this excludes an immediate return to the source scatterer. Let α be the chosen representative of the class $-q_\star M_0^{-1} + \mathbb{Z}^d$, taking $\alpha = 0$ when $q_\star \in \mathcal{L}$, and put

$$z_0(v) = \pi_\perp(\beta(v)K(v)).$$

The family $\{B_v\}$ has Hausdorff-compact closure with the required uniform inradius and outradius bounds, while $\{z_0(v)\}$ has compact closure in E ; hence Lemma 2(iii) applies. If the flight starts on a scatterer, $z_0(v)$ is its departure parameter.

Let $\mathcal{A}_{1,\rho}$ be the set of velocities for which the first collision exists and is regular. On this set define

$$\mathcal{I}_{1,\rho}(v) = (\rho^{d-1}\tau_1(q_{\rho,\beta}(v), v; \rho), (v, W_1^{(\rho)})) \in \mathbb{R}_{>0} \times \mathcal{B}.$$

The following theorem identifies the limiting integrals of this quantity as $\rho \rightarrow 0$; taking $G \equiv 1$ also implies that $\lambda(\mathcal{A}_{1,\rho}^c) \rightarrow 0$.

Theorem 2. *Let λ be a Borel probability measure on S_1^{d-1} absolutely continuous with respect to ω . Then, for every $G \in C_b(\mathbb{R}_{>0} \times \mathcal{B})$,*

$$(3.8) \quad \begin{aligned} & \lim_{\rho \rightarrow 0} \int_{\mathcal{A}_{1,\rho}} G(\mathcal{I}_{1,\rho}(v)) d\lambda(v) \\ &= \sum_{\ell=1}^s \int_{S_1^{d-1}} \int_0^\infty \int_{B_{v,\ell}} G(\xi, (v, (\ell, b))) \Phi_{\alpha,v}(\xi, b, z_0(v)) db d\xi d\lambda(v). \end{aligned}$$

Proof. We follow [MS10, Theorem 4.4]. Write $q_{\rho,\beta}(v) + \tau v = m + \rho x$, where $m \in \mathcal{L}$ is the centre of the first scatterer and $x \in \partial_\star \mathcal{P}$ is the local boundary point. Put $\xi = \rho^{d-1}\tau$, $b = -\pi_\perp(xK(v))$, and $z = \pi_\perp(\beta(v)K(v))$. After applying

$$D(\rho) = \text{diag}(\rho^{d-1}, \rho^{-1}, \dots, \rho^{-1}),$$

the transformed centre set is

$$(\mathcal{L} - q_\star)K(v)D(\rho) = (\mathbb{Z}^d + \alpha)M_0K(v)D(\rho).$$

The terminal centre satisfies the exact identity

$$(3.9) \quad (m - q_\star)K(v)D(\rho) = [\xi + \rho^d(\beta(v) - x) \cdot v]e_1 + b + z.$$

More generally, a transformed lattice point is the centre of an obstacle intersecting the open flight segment precisely when it belongs to the region

$$(3.10) \quad \mathfrak{C}_{\rho,v}(\xi, z) := \left\{ [s + \rho^d(\beta(v) - p) \cdot v]e_1 + z - \pi_\perp(pK(v)) : 0 < s < \xi, p \in \mathcal{P} \right\}.$$

Since β is bounded and $\mathcal{P}$ is bounded, there is a constant C such that, uniformly in v and whenever $\xi > 2C\rho^d$,

$$(3.11) \quad \mathfrak{Z}_v(C\rho^d, \xi - C\rho^d) + z \subseteq \mathfrak{C}_{\rho,v}(\xi, z) \subseteq \mathfrak{Z}_v(-C\rho^d, \xi + C\rho^d) + z.$$

If $q_\star \in \mathcal{L}$, we delete the zero vector; admissibility (3.7) excludes a return to the source scatterer.

Now take G supported in $\varepsilon \leq \xi \leq T$ and away from the face-cell boundaries. The equidistribution and lattice-counting results of [MS10, Sections 5–6 and Theorem 3.9], together with (3.11), give convergence of the transformed affine lattice. The first-point map is continuous off the null set of ties and boundary points, and Lemma 1 identifies its limiting law with $\Phi_{\alpha,v}(\xi, b, z) d\xi db$. The grazing set is ω -null, and the cell boundaries have zero flux measure. The discarded short flights have mass $O(A_P \varepsilon)$ by Siegel's formula, while Lemma 2 controls the long flights. The preceding argument first gives convergence on the regular truncated region. Apply it to continuous cutoffs increasing to one as $\varepsilon \rightarrow 0$, $T \rightarrow \infty$, and the neighbourhoods of the grazing and face-cell-boundary strata shrink. By (3.3), the limiting cutoff integrals tend to one, and hence $\lambda(\mathcal{A}_{1,\rho}^c) \rightarrow 0$. The same estimates remove the cutoffs for bounded continuous G , which proves (3.8). $\square$

We will generalise Theorem 2 to n collisions in Theorem 4.

4. ARITHMETIC NONRESONANCE OF THE FACES

Assume throughout this section that $d \geq 3$. For $A \in \mathrm{GL}(d, \mathbb{R})$, put

$$[A] = \{cA : c \in \mathbb{R}^\times\} \in \mathrm{PGL}(d, \mathbb{R}) = \mathrm{GL}(d, \mathbb{R})/(\mathbb{R}^\times I).$$

Thus $[A] \in \mathrm{PGL}(d, \mathbb{Q})$ if some nonzero scalar multiple of A is rational. Two subgroups are *commensurable* if their intersection has finite index in each. The commensurator of $\Gamma \leq G$ is

$$(4.1) \quad \mathrm{Comm}_G(\Gamma) = \{g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}.$$

For the integer special linear group,

$$(4.2) \quad \mathcal{S}_d = \mathrm{Comm}_{\mathrm{SL}(d, \mathbb{R})}(\mathrm{SL}(d, \mathbb{Z})) = \{cT \in \mathrm{SL}(d, \mathbb{R}) : T \in \mathrm{GL}(d, \mathbb{Q}), c \in \mathbb{R}^\times\}.$$

Thus, for $A \in \mathrm{SL}(d, \mathbb{R})$, $A \in \mathcal{S}_d$ if and only if $[A] \in \mathrm{PGL}(d, \mathbb{Q})$.

Let

$$\mathcal{W}_s = \langle a_1, \dots, a_s \mid a_\ell^2 = e \rangle = \mathbb{Z}_2 * \dots * \mathbb{Z}_2$$

be the free product on the formal face involutions. A word is *reduced* if adjacent letters are distinct. If $w = \ell_1 \dots \ell_r$, put $R_w = R_{\ell_1} \dots R_{\ell_r}$ and $|w| = r$.

The labelled polyhedron $\mathcal{P}$ is called $\mathcal{L}$ -*nonresonant* if

$$(4.3) \quad [M_0 R_w M_0^{-1}] \notin \mathrm{PGL}(d, \mathbb{Q}) \quad \text{for every nonempty reduced } w \in \mathcal{W}_s.$$

Replacing M_0 by γM_0 , $\gamma \in \mathrm{SL}(d, \mathbb{Z})$, conjugates each class in (4.3) by a rational projective class and does not affect the condition. The condition excludes parallel faces, since their two reflections have product I , and makes the $[R_\ell]$ generate the free product.

Let $J = \mathrm{diag}(-1, 1, \dots, 1)$. (In odd dimension one may equivalently use $J = -I$.)

Lemma 3. *For a reduced word u put*

$$M_u^\circ = J^{|u|} M_0 R_u \in \mathrm{SL}(d, \mathbb{R}).$$

Then $\mathbb{Z}^d M_u^\circ = \mathbb{Z}^d M_0 R_u$, so M_u° is an oriented basis of the unfolded lattice associated with u . If $\mathcal{P}$ is $\mathcal{L}$ -nonresonant and $u \neq v$, then the arithmetic lattices

$$\Gamma_u = (M_u^\circ)^{-1} \mathrm{SL}(d, \mathbb{Z}) M_u^\circ, \quad \Gamma_v = (M_v^\circ)^{-1} \mathrm{SL}(d, \mathbb{Z}) M_v^\circ$$

are incommensurable.

Proof. Let w be the reduced representative of uv^{-1} in $\mathcal{W}_s$. Since $u \neq v$, the word w is nonempty. Since J has rational entries,

$$\begin{aligned} [(M_u^\circ)(M_v^\circ)^{-1}] &\in \mathrm{PGL}(d, \mathbb{Q}) \\ \iff [M_0 R_u R_v^{-1} M_0^{-1}] &\in \mathrm{PGL}(d, \mathbb{Q}) \\ \iff [M_0 R_w M_0^{-1}] &\in \mathrm{PGL}(d, \mathbb{Q}). \end{aligned}$$

The last condition is excluded by assumption (4.3). The relative matrix on the first line lies in $\mathrm{SL}(d, \mathbb{R})$, so (4.2) shows that it is not in $\mathcal{S}_d$. By [MS14, Lemma 8], this is equivalent to incommensurability of Γ_u and Γ_v . $\square$

We next justify the use of “generic” for an $\mathcal{L}$ -nonresonant polyhedron. An *open realisation stratum* is a nonempty open subset $\mathcal{U} \subset (\mathbb{RP}^{d-1})^s$ such that, for every $([n_1], \dots, [n_s]) \in \mathcal{U}$, unit representatives n_ℓ and support numbers h_ℓ can be chosen so that

$$\bigcap_{\ell=1}^s \{x \in \mathbb{R}^d : x \cdot n_\ell < h_\ell\}$$

is a bounded full-dimensional polyhedron of the fixed labelled combinatorial type.

Lemma 4. *For every nonempty reduced word $w \in \mathcal{W}_s$, the real-algebraic map*

$$(4.4) \quad \mathcal{R}_w : (\mathbb{RP}^{d-1})^s \longrightarrow \mathrm{PGL}(d, \mathbb{R}), \quad ([n_1], \dots, [n_s]) \longmapsto [R_w]$$

is nonconstant on every nonempty open subset.

Proof. Write $R([n])$ for reflection in the hyperplane orthogonal to $[n]$. For a nonzero row vector n ,

$$[R([n])] = [(n^t n)I - 2^t n n].$$

Hence (4.4) is real algebraic. If it were constant on an open set, the analytic identity principle would make it constant on $(\mathbb{RP}^{d-1})^s$; write its value as $[C]$. Simultaneous rotation gives

$$[Q^{-1} C Q] = [C] \quad (Q \in \mathrm{SO}(d)).$$

Thus $Q^{-1} C Q = c(Q)C$ for a continuous character $c : \mathrm{SO}(d) \rightarrow \mathbb{R}^\times$. The Frobenius norm gives $c(Q) \in \{-1, 1\}$, and connectedness of $\mathrm{SO}(d)$ gives $c \equiv 1$. The commutant of the standard $\mathrm{SO}(d)$ -representation is scalar for $d \geq 3$, so the constant value would be $[I]$.

For $d = 3$, Dekker’s theorem [Dek59, Theorem 2], applied to s cyclic groups of order two, gives rotations $H_1, \dots, H_s \in \mathrm{SO}(3)$, each of order two, for which $a_\ell \mapsto H_\ell$ extends to an injective homomorphism $\mathcal{W}_s \rightarrow \mathrm{SO}(3)$. Since $S_\ell = -H_\ell$ is a reflection, for $w = \ell_1 \cdots \ell_r$ put $H_w = H_{\ell_1} \cdots H_{\ell_r}$ and $S_w = S_{\ell_1} \cdots S_{\ell_r}$. Then

$$S_w = (-I_3)^r H_w.$$

If S_w were scalar, then $H_w = I_3$, contradicting injectivity.

For $d > 3$, put

$$S_\ell^{(d)} = \mathrm{diag}(S_\ell, I_{d-3}).$$

The corresponding word product is $S_w^{(d)} = S_{\ell_1}^{(d)} \cdots S_{\ell_r}^{(d)} = \mathrm{diag}(S_w, I_{d-3})$, which cannot be scalar. Thus $\mathcal{R}_w$ is not identically $[I]$, a contradiction. $\square$

The following proposition shows that an $\mathcal{L}$ -nonresonant polyhedron is generic in both a measure-theoretic and topological sense.

Proposition 1. Fix $d \geq 3$ and $\mathcal{L} = \mathbb{Z}^d M_0$. In every open realisation stratum $\mathcal{U}$ as above, the $\mathcal{L}$ -nonresonant normal configurations form a conull dense G_δ subset of $\mathcal{U}$.

Proof. The exceptional set is

$$\bigcup_{w \in \mathcal{W}_s \setminus \{e\}} \bigcup_{C \in \text{PGL}(d, \mathbb{Q})} \{\mathbf{n} \in \mathcal{U} : [M_0 R_w(\mathbf{n}) M_0^{-1}] = C\}.$$

Both unions are countable. By Lemma 4, each fibre is a proper real-algebraic subset of $\mathcal{U}$, hence closed, null and nowhere dense. The complement is therefore conull and a dense G_δ . Adding support numbers and normalising the volume do not change the conclusion. $\square$

Remark 2. For $d = 3$, Vorobets [Vor05, Theorem 4] proves generic faithfulness of the reflection representation. Together with the analyticity and $\text{SO}(3)$ -equivariance argument above, this gives another proof of Lemma 4.

Remark 3. In $d = 2$, every orientation-reversing orthogonal matrix squares to the identity. If a polygon has parallel faces, their two face reflections already have product I . Otherwise choose three distinct face labels ℓ_1, ℓ_2, ℓ_3 ; then $(R_{\ell_1} R_{\ell_2} R_{\ell_3})^2 = I$, while $\ell_1 \ell_2 \ell_3 \ell_1 \ell_2 \ell_3$ is a nonempty reduced word. This shows that our nonresonance condition (which we require to hold for all reduced words) has no two-dimensional analogue.

5. PRODUCT EQUIDISTRIBUTION AND HIGHER-ORDER COLLISION LAWS

The diagonal flow compatible with the Boltzmann–Grad scaling is

$$A^t = \text{diag}(e^{-(d-1)t}, e^t, \dots, e^t), \quad D(\rho) = A^{\log(1/\rho)}.$$

We use the following velocity-dependent form of the product theorem from [MS14, Theorems 7 and 10]. Given translations $\alpha_0, \dots, \alpha_N$, we use the shorthand $(Y_j, \eta_j) = (X_{\alpha_j}, \mu_{\alpha_j})$, with (X_α, μ_α) defined in Section 3. Furthermore, put

$$\psi_\alpha(M) = \begin{cases} \text{ASL}(d, \mathbb{Z})(I, \alpha)(M, 0), & \alpha \notin \mathbb{Q}^d, \\ \Gamma(q_\alpha)M, & \alpha \in \mathbb{Q}^d. \end{cases}$$

Matrices $M_0, \dots, M_N \in \text{SL}(d, \mathbb{R})$ are called *pairwise incommensurable* when the arithmetic subgroups $M_j^{-1} \text{SL}(d, \mathbb{Z}) M_j$ are pairwise incommensurable in the sense of (4.1).

Theorem 3. Let $N \geq 0$ and let $M_0, \dots, M_N \in \text{SL}(d, \mathbb{R})$ be pairwise incommensurable. If λ is a Borel probability measure absolutely continuous with respect to ω , and $F : S_1^{d-1} \times Y_0 \times \dots \times Y_N \rightarrow \mathbb{R}$ is bounded and continuous, then

$$\begin{aligned} & \lim_{t \rightarrow \infty} \int_{S_1^{d-1}} F(v, \psi_{\alpha_0}(M_0 K(v) A^t), \dots, \psi_{\alpha_N}(M_N K(v) A^t)) d\lambda(v) \\ &= \int_{S_1^{d-1}} \int_{Y_0 \times \dots \times Y_N} F(v, g_0, \dots, g_N) d\eta_0(g_0) \cdots d\eta_N(g_N) d\lambda(v). \end{aligned}$$

Proof. For F independent of v , this is [MS14, Theorems 7 and 10]. The stated form follows by a standard approximation argument; see, e.g., [MS10, Theorem 5.3]. $\square$

Remark 4. Without the incommensurability condition, Shah’s theorem [Sha96, Theorem 1.4] still gives an analogous convergence statement, but the limiting measure is generally not the product measure displayed in Theorem 3.

Fix $\ell = (\ell_1, \dots, \ell_n)$ and put, for $0 \leq j \leq n-1$,

$$P_j = R_{\ell_1} \cdots R_{\ell_j}, \quad v_j = v_0 P_j, \quad K_j(v_0) = P_j^{-1} K(v_0).$$

Then $v_j K_j(v_0) = e_1$. The first transformed point set is $(\mathcal{L} - q_\star)K(v_0)D(\rho)$; after the j th collision, $\mathcal{L} - m_j = \mathcal{L}$, and the next centres form $\mathcal{L} P_j^{-1} K(v_0)D(\rho)$. An $\mathrm{SL}(d, \mathbb{R})$ -representative of this point set is $\widetilde{M}_j K(v_0)A^t$, where

$$\begin{aligned} \widetilde{M}_0 &= M_0, \\ \widetilde{M}_j &= J^j M_0 R_{\ell_j} \cdots R_{\ell_1} = J^j M_0 P_j^{-1} \in \mathrm{SL}(d, \mathbb{R}) \quad (j \geq 1). \end{aligned}$$

Here J ensures the determinant of $\widetilde{M}_j$ is one but does not change the point set. The first factor has translation α , and the later factors have translation 0. Define

$$\widehat{O}_j(v_0) = K_j(v_0)^{-1} K(v_j) = \mathrm{diag}(1, O_j(v_0)), \quad O_j(v_0) \in \mathrm{O}(E).$$

It fixes e_1 , commutes with A^t , and converts the transported frame to $K(v_j)$. Thus $\widehat{O}_0 = I$ and

$$\mathbb{Z}^d \widetilde{M}_j K(v_0) A^t \widehat{O}_j(v_0) = \mathcal{L} K(v_j) A^t, \quad \det O_j = (-1)^j.$$

For $j \geq 1$, Remark 1 applies even when $\det O_j = -1$.

We next define the collision-record space before constructing the recursive first-point map. For $n \geq 1$, let Ω_n consist of the regular records

$$\varpi = (v_0, \xi_1, b_1, \dots, \xi_n, b_n), \quad \xi_j > 0, \quad b_j \in B_{v_{j-1}},$$

with recursion

$$(5.1) \quad \ell_j = \ell_{v_{j-1}}^-(b_j), \quad W_j = (\ell_j, b_j), \quad v_j = \mathcal{R}(v_{j-1}, b_j), \quad z_j = \mathcal{Z}(v_{j-1}, b_j).$$

Give Ω_n the disjoint-union topology over its face words, and put

$$d\sigma_\lambda^{(n)}(\varpi) = d\lambda(v_0) \prod_{j=1}^n d\xi_j db_j.$$

A face word $\ell = (\ell_1, \dots, \ell_n)$ is *reduced* if $\ell_{j+1} \neq \ell_j$ for $1 \leq j < n$. Let $\Omega_n(\ell)$ be the word component consisting of the records in Ω_n with face word ℓ . Every record in Ω_n has a reduced face word: after a collision on face ℓ_j one has $v_j \cdot n_{\ell_j} > 0$, so the next incoming face cannot again be ℓ_j . For a reduced face word ℓ , let

$$Y_\ell = S_1^{d-1} \times X_\alpha \times X_0^{n-1}$$

and write its elements as $\mathbf{y} = (v_0, g_0, \dots, g_{n-1})$. We construct a subset $D_\ell \subset Y_\ell$ recursively as follows. Put $\alpha_0 = \alpha$ and $\alpha_j = 0$ for $j \geq 1$, and define

$$(5.2) \quad \Lambda_{\alpha_j}^\circ(g) = \begin{cases} \Lambda_0(g) \setminus \{0\}, & \alpha_j = 0, \\ \Lambda_{\alpha_j}(g), & \alpha_j \neq 0. \end{cases}$$

Starting from $z_0 = z_0(v_0)$, for $j = 0, \dots, n-1$ consider the set

$$(\Lambda_{\alpha_j}^\circ(g_j) \widehat{O}_j(v_0)) \cap (\mathfrak{Z}_{v_j}(0, \infty) + z_j).$$

If this set has no unique point with least positive first coordinate, then we cannot complete the construction to obtain a point in D_ℓ . If it does, denote that point by y_{j+1} and set

$$\xi_{j+1} = y_{j+1} \cdot e_1, \quad b_{j+1} = \pi_\perp(y_{j+1}) - z_j.$$

If a grazing velocity, a face-cell boundary point, or a face different from ℓ_{j+1} occurs, the construction again fails. If none of the above occurs, put $z_{j+1} = \mathcal{Z}(v_j, b_{j+1})$ and continue. Let $D_\ell \subset Y_\ell$ be the set of inputs for which this construction succeeds at all n stages, and let

$$\Theta_\ell : D_\ell \longrightarrow \Omega_n(\ell)$$

be the resulting record map. For $G \in C_b(\Omega_n(\ell))$, define on the full input space

$$(5.3) \quad H_\ell^G(\mathbf{y}) = \begin{cases} G(\Theta_\ell(\mathbf{y})), & \mathbf{y} \in D_\ell, \\ 0, & \mathbf{y} \notin D_\ell. \end{cases}$$

Lemma 5. *The set D_ℓ and the map Θ_ℓ are Borel. For every probability measure λ absolutely continuous with respect to ω and every $G \in C_b(\Omega_n(\ell))$, the function H_ℓ^G is continuous outside a set E_ℓ of ν -measure zero, where $\nu = \lambda \otimes \mu_\alpha \otimes \mu_0^{\otimes(n-1)}$.*

Proof. The exceptional configurations are those with a nonzero point on a tube boundary, two points at the least longitudinal coordinate, a terminal point on a face-cell boundary, a grazing velocity, no finite first coordinate, or $v_j = v_s$ at some stage. Tube-boundary points are null by Siegel's first-moment formula, ties by the two-point estimate in the proof of Lemma 1, face-cell boundaries and grazing velocities by the flux formula (2.2), and the absence of a finite first point by Lemma 1(i). The chart-singular inputs are null because λ is absolutely continuous. The distinguished zero vector has already been deleted in (5.2). The defining incidence, avoidance, and cell-membership conditions are Borel. Away from the exceptional configurations, local finiteness and the stability estimate (3.5) show inductively that success or failure of the prescribed word is locally constant and that, on D_ℓ , the extracted point and all subsequent collision data vary continuously. This proves the asserted continuity of H_ℓ^G . $\square$

Proposition 2. *For every bounded Borel function $G : \Omega_n(\ell) \rightarrow \mathbb{R}$,*

$$(5.4) \quad \begin{aligned} & \int_{D_\ell} G(\Theta_\ell(v_0, g_0, \dots, g_{n-1})) d\lambda(v_0) \prod_{j=0}^{n-1} d\mu_{\alpha_j}(g_j) \\ &= \int_{\Omega_n(\ell)} G(\varpi) \Phi_{\alpha, v_0}(\xi_1, b_1, z_0(v_0)) \prod_{j=1}^{n-1} \Phi_{0, v_j}(\xi_{j+1}, b_{j+1}, z_j) d\sigma_\lambda^{(n)}(\varpi). \end{aligned}$$

Proof. Haar invariance and (3.6) reduce each factor to Lemma 1, whose conditional density is

$$\Phi_{\alpha_j, v_j}(\xi_{j+1}, b_{j+1}, z_j) d\xi_{j+1} db_{j+1}.$$

Successive conditioning and (3.4) give (5.4); the bounded Borel case follows by a monotone-class argument. $\square$

Define

$$(5.5) \quad \mathcal{F}_{\alpha, \beta}^{(n)}(\varpi) = \Phi_{\alpha, v_0}(\xi_1, b_1, z_0(v_0)) \prod_{j=1}^{n-1} \Phi_{0, v_j}(\xi_{j+1}, b_{j+1}, z_j).$$

By (3.3), these are probability densities, and they are consistent since

$$\int_0^\infty \int_{B_{v_n}} \mathcal{F}_{\alpha, \beta}^{(n+1)} db_{n+1} d\xi_{n+1} = \mathcal{F}_{\alpha, \beta}^{(n)}.$$

The Ionescu–Tulcea theorem therefore yields the corresponding law on infinite records.

Now put $t = \log(1/\rho)$ and

$$\begin{aligned} g_{j,\rho}(v_0) &= \psi_{\alpha_j}(\widetilde{M}_j K(v_0) A^t), \\ \nu_{\ell,\rho}(v_0) &= (v_0, g_{0,\rho}(v_0), \dots, g_{n-1,\rho}(v_0)) \in Y_\ell. \end{aligned}$$

We construct $D_{\ell,\rho} \subset Y_\ell$ recursively in the same way as D_ℓ above. We now use the point sets $\Lambda_{\alpha_j}^\circ(g_j)\widehat{O}_j(v_0)$ and set $x_0 = \beta(v_0)$. For $j = 0, \dots, n-1$, put $x_j = x_{v_{j-1}}^-(b_j)$ when $j \geq 1$, and in factor j use the collision region

$$(5.6) \quad \begin{aligned} \mathfrak{C}_{\rho,j}(\xi, z_j) &= \{[s + \rho^d(x_j - p) \cdot v_j]e_1 + z_j - \pi_\perp(pK(v_j)) : \\ &\quad 0 < s < \xi, p \in \mathcal{P}\}. \end{aligned}$$

The first physical contact time is

$$\xi_{j+1} = \inf\{\xi > 0 : (\Lambda_{\alpha_j}^\circ(g_j)\widehat{O}_j(v_0)) \cap \mathfrak{C}_{\rho,j}(\xi, z_j) \neq \emptyset\}.$$

If a unique centre y_{j+1} belongs to $\mathfrak{C}_{\rho,j}(\eta, z_j)$ for every $\eta > \xi_{j+1}$ sufficiently close to ξ_{j+1} and gives a regular collision on the prescribed face, put

$$b_{j+1} = \pi_\perp(y_{j+1}) - z_j, \quad x_{j+1} = x_{v_j}^-(b_{j+1}), \quad z_{j+1} = \mathcal{Z}(v_j, b_{j+1}),$$

and continue. The inputs for which all n stages succeed form $D_{\ell,\rho}$. This set and the resulting map are Borel by the same incidence-and-avoidance argument as in Lemma 5. Let

$$\Theta_{\ell,\rho} : D_{\ell,\rho} \longrightarrow \Omega_n(\ell)$$

be the resulting record map.

For fixed n , let $\mathcal{A}_{n,\rho}$ be the event that the first n collisions exist and are regular. On this event define the physical record

$$\varpi_n^{(\rho)} = (v_0, \xi_1^{(\rho)}, b_1^{(\rho)}, \dots, \xi_n^{(\rho)}, b_n^{(\rho)}) \in \Omega_n.$$

The following theorem identifies the limit distribution of this record; taking $G \equiv 1$ also implies that $\lambda(\mathcal{A}_{n,\rho}^c) \rightarrow 0$.

Theorem 4. *Let $d \geq 3$, let $\mathcal{P}$ be $\mathcal{L}$ -nonresonant, and let $q_{\rho,\beta}(v)$ and λ be as in Theorem 2. Then, for every $G \in C_b(\Omega_n)$,*

$$(5.7) \quad \lim_{\rho \rightarrow 0} \int_{\mathcal{A}_{n,\rho}} G(\varpi_n^{(\rho)}) d\lambda(v_0) = \int_{\Omega_n} G(\varpi) \mathcal{F}_{\alpha,\beta}^{(n)}(\varpi) d\sigma_\lambda^{(n)}(\varpi).$$

Proof. Fix a reduced face word $\ell = (\ell_1, \dots, \ell_n)$, which records the faces in chronological order. Set $u_0 = e$; after j collisions, the corresponding unfolded group word is $u_j = \ell_j \cdots \ell_1$ for $1 \leq j \leq n-1$. Since ℓ is reduced, the words $u_0, \dots, u_{n-1}$ are distinct, and u_j is reduced for every $1 \leq j \leq n-1$. Since $\widetilde{M}_j = M_{u_j}^\circ$, Lemma 3 says that the $\widetilde{M}_j$ are pairwise incommensurable. Let $\nu = \lambda \otimes \mu_\alpha \otimes \mu_0^{\otimes(n-1)}$ and $\nu_{\ell,\rho} = (\nu_{\ell,\rho})_* \lambda$. By Theorem 3, we have the weak convergence

$$(5.8) \quad \nu_{\ell,\rho} \Rightarrow \nu \quad \text{on } Y_\ell.$$

For $j = 0$, (5.6) is (3.10). For $j \geq 1$, we have

$$(m_{j+1} - m_j)K(v_j)D(\rho) = [\xi_{j+1} + \rho^d(x_j - x_{j+1}) \cdot v_j]e_1 + b_{j+1} + z_j.$$

Moreover, $\nu_{\ell,\rho}(v_0) \in D_{\ell,\rho}$ precisely when the first n collisions exist, are regular, and have face word ℓ ; on this set $\Theta_{\ell,\rho}(\nu_{\ell,\rho}(v_0))$ is the physical record for each $\rho > 0$.

For $G \in C_b(\Omega_n(\ell))$, define

$$H_{\ell,\rho}^G(\mathbf{y}) = \begin{cases} G(\Theta_{\ell,\rho}(\mathbf{y})), & \mathbf{y} \in D_{\ell,\rho}, \\ 0, & \mathbf{y} \notin D_{\ell,\rho}. \end{cases}$$

The tube sandwich (3.11), first-point stability (3.5), and the $O(\rho^d)$ correction in (5.6) show inductively that, if $\rho_k \rightarrow 0$, $\mathbf{y}_k \rightarrow \mathbf{y}$, and $\mathbf{y} \notin E_\ell$ (where E_ℓ is the ν -null set defined in Lemma 5), then $\mathbf{1}_{D_{\ell,\rho_k}}(\mathbf{y}_k) = \mathbf{1}_{D_\ell}(\mathbf{y})$ for all sufficiently large k , and, when $\mathbf{y} \in D_\ell$,

$$\Theta_{\ell,\rho_k}(\mathbf{y}_k) \rightarrow \Theta_\ell(\mathbf{y}).$$

In particular, this implies

$$(5.9) \quad H_{\ell,\rho_k}^G(\mathbf{y}_k) \rightarrow H_\ell^G(\mathbf{y}).$$

Since $\|H_{\ell,\rho}^G\|_\infty \leq \|G\|_\infty$, the extended continuous mapping theorem for varying functions, applied to (5.8) and (5.9), now gives

$$(5.10) \quad \begin{aligned} \lim_{\rho \rightarrow 0} \int H_{\ell,\rho}^G(\iota_{\ell,\rho}(v_0)) d\lambda(v_0) &= \int_{Y_\ell} H_\ell^G(\mathbf{y}) d\nu(\mathbf{y}) \\ &= \int_{\Omega_n(\ell)} G(\varpi) \mathcal{F}_{\alpha,\beta}^{(n)}(\varpi) d\sigma_\lambda^{(n)}(\varpi), \end{aligned}$$

where the last identity is Proposition 2.

First take $G \equiv 1$ in (5.10) and sum over the finitely many reduced face words. The successful finite- ρ events associated with these words partition $\mathcal{A}_{n,\rho}$, while (3.3) makes the limiting word probabilities sum to one. Hence $\lambda(\mathcal{A}_{n,\rho}^c) \rightarrow 0$. For general $G \in C_b(\Omega_n)$, apply (5.10) to the restriction of G to each word component $\Omega_n(\ell)$ and sum over the finitely many reduced face words. This gives (5.7). $\square$

Remark 5. In principle, it should be possible to prove an analogue of Theorem 4 for polyhedra that are not $\mathcal{L}$ -nonresonant, in any dimension $d \geq 2$. As noted in Remark 4, the limiting measure would then no longer be a product measure, leading to persistent memory effects in the limiting random flight process. In the related setting of periodic slit barriers, this was carried out in [BKMP11, Theorems 3(1) and 9].

6. VELOCITY AND SEGMENT LAWS

For $z \in E$, define

$$a_{\alpha,\ell}(v, z; \xi) = \int_{B_{v,\ell}} \Phi_{\alpha,v}(\xi, b, z) db.$$

In view of (2.2),

$$a_{\alpha,\ell}(v, z; \xi) = \begin{cases} \int_{F_\ell} \Phi_{\alpha,v}(\xi, -\pi_\perp(xK(v)), z) (-v \cdot n_\ell) dA_\ell(x), & v \cdot n_\ell < 0, \\ 0, & v \cdot n_\ell \geq 0. \end{cases}$$

The one-flight velocity kernel is the probability kernel

$$(6.1) \quad \mathbf{p}_{\alpha,z}(v; d\xi, du) = d\xi \sum_{\ell: v \cdot n_\ell < 0} a_{\alpha,\ell}(v, z; \xi) \delta_{vR_\ell}(du).$$

In distribution notation,

$$p_{\alpha,z}(v, \xi, u) = \sum_{\ell: v \cdot n_\ell < 0} a_{\alpha,\ell}(v, z; \xi) \delta_{vR_\ell}(u).$$

Write $(a)_+ = \max\{a, 0\}$ and put $|F_\ell| = \int_{F_\ell} dA_\ell$. The geometric differential scattering cross section is the finite measure

$$\Sigma_v(du) = \sum_{\ell=1}^s |F_\ell| (-v \cdot n_\ell)_+ \delta_{vR_\ell}(du).$$

Boundary strata and grazing faces are flux-null.

Let

$$\begin{aligned} \mathcal{V}_n : \Omega_n &\longrightarrow S_1^{d-1} \times (\mathbb{R}_{>0} \times S_1^{d-1})^n, \\ \mathcal{V}_n(\varpi) &= (v_0, \xi_1, v_1, \dots, \xi_n, v_n), \end{aligned}$$

where the velocities are generated by (5.1). The flight-length/velocity limit is the Borel probability measure

$$(6.2) \quad \widehat{\mathbb{P}}_{\alpha,\beta}^{(n)} = (\mathcal{V}_n)_* (\mathcal{F}_{\alpha,\beta}^{(n)} \sigma_\lambda^{(n)}).$$

Define

$$\mathcal{S}_n : \Omega_n \longrightarrow (\mathbb{R}^d \setminus \{0\})^n, \quad \mathcal{S}_n(\varpi) = (\xi_1 v_0, \xi_2 v_1, \dots, \xi_n v_{n-1}),$$

and the limiting segment law is

$$(6.3) \quad \mathbb{P}_{\alpha,\beta}^{(n)} = (\mathcal{S}_n)_* (\mathcal{F}_{\alpha,\beta}^{(n)} \sigma_\lambda^{(n)}).$$

Thus, for a bounded continuous test function G ,

$$\int G d\widehat{\mathbb{P}}_{\alpha,\beta}^{(n)} = \int_{\Omega_n} G(\xi_1 v_0, \dots, \xi_n v_{n-1}) \mathcal{F}_{\alpha,\beta}^{(n)} d\sigma_\lambda^{(n)}.$$

On $\mathcal{A}_{n,\rho}$, the images $\mathcal{V}_n(\varpi_n^{(\rho)})$ and $\mathcal{S}_n(\varpi_n^{(\rho)})$ are precisely the physical flight-length/velocity and rescaled-segment tuples.

Corollary 1. *Under the assumptions of Theorem 4, for every*

$$G \in C_b(S_1^{d-1} \times (\mathbb{R}_{>0} \times S_1^{d-1})^n), \quad H \in C_b((\mathbb{R}^d \setminus \{0\})^n),$$

we have

$$\begin{aligned} \lim_{\rho \rightarrow 0} \int_{\mathcal{A}_{n,\rho}} G(\mathcal{V}_n(\varpi_n^{(\rho)})) d\lambda(v_0) &= \int G d\widehat{\mathbb{P}}_{\alpha,\beta}^{(n)}, \\ \lim_{\rho \rightarrow 0} \int_{\mathcal{A}_{n,\rho}} H(\mathcal{S}_n(\varpi_n^{(\rho)})) d\lambda(v_0) &= \int H d\mathbb{P}_{\alpha,\beta}^{(n)}. \end{aligned}$$

Proof. The maps $\mathcal{V}_n$ and $\mathcal{S}_n$ are continuous in the disjoint-word topology of Ω_n . Hence $G \circ \mathcal{V}_n$ and $H \circ \mathcal{S}_n$ belong to $C_b(\Omega_n)$, and the assertions follow from Theorem 4 and the definitions (6.2) and (6.3). $\square$

For a fixed face word and $n \geq 2$, the segment map factors through the embedding

$$(v_0, \xi_1, \dots, \xi_n) \longmapsto (\xi_1 v_0, \dots, \xi_n v_{n-1})$$

of a $(d-1) + n$ -dimensional manifold into an nd -dimensional space: the first segment recovers (v_0, ξ_1) , and the word determines the later velocities. Its image has codimension $(n-1)(d-1)$ and is Lebesgue null. Since there are finitely many words, (6.3) is singular for $n \geq 2$.

Since the next-flight kernel generally depends on the departure parameter z_j as well as on v_j , the velocity projection alone need not be Markov, although (6.2) still determines all its finite-dimensional distributions.

7. MACROSCOPIC LIMIT AND CONTINUOUS-TIME FLIGHT

Put $\varepsilon = \rho^{d-1}$ and

$$\mathcal{Y}_\rho = T^1(\varepsilon \mathcal{K}_\rho),$$

where T^1 denotes the completed billiard phase space defined in Section 2. Away from its boundary it is $(\varepsilon \mathcal{K}_\rho^c) \times S_1^{d-1}$. For $(Q, V) \in \mathcal{Y}_\rho$ whose microscopic trajectory is defined up to time $\rho^{-(d-1)}t$, set

$$(7.1) \quad F_{t,\rho}(Q, V) = \left(\rho^{d-1} q(\rho^{-(d-1)}t), v(\rho^{-(d-1)}t) \right).$$

Here $(q(s), v(s))$ is the microscopic billiard trajectory with $q(0) = \rho^{-(d-1)}Q$ and $v(0) = V$. Extend $F_{t,\rho}$ in any fixed measurable fashion on the Liouville-null set where that trajectory has terminated at a nonregular collision before this time. Let

$$(7.2) \quad \mu_{\text{in}}(dQ, dV) = f_{\text{in}}(Q, V) dQ d\omega(V)$$

be an absolutely continuous probability measure. Since the obstacle volume fraction is $\rho^d \text{vol}(\mathcal{P})$, we have $\mu_{\text{in}}(\mathcal{Y}_\rho^c) \rightarrow 0$. Averaging over Q puts the first affine-lattice factor in the universal regime. Define the macroscopic impact-resolved measure on $\mathbb{R}^d \times \Omega_n$ by the density below, writing Q_0 for the initial macroscopic position:

$$(7.3) \quad f_{\text{in}}(Q_0, v_0) \Phi_{v_0}(\xi_1, b_1) \prod_{j=1}^{n-1} \Phi_{0,v_j}(\xi_{j+1}, b_{j+1}, z_j)$$

with respect to

$$dQ_0 d\omega(v_0) \prod_{j=1}^n d\xi_j db_j.$$

For fixed n , let $\mathcal{A}_{n,\rho}^{\text{mac}} \subset \mathcal{Y}_\rho$ be the set of initial points for which the first n collisions exist and are regular. On this set, let $\mathcal{R}_{n,\rho}(Q, V)$ be the impact-resolved tuple

$$(V, \xi_1^{(\rho)}, b_1^{(\rho)}, \dots, \xi_n^{(\rho)}, b_n^{(\rho)}) \in \Omega_n.$$

We will see below that $\mu_{\text{in}}(\mathcal{Y}_\rho \setminus \mathcal{A}_{n,\rho}^{\text{mac}}) = 0$ for every sufficiently small $\rho > 0$.

Theorem 5. *Let $d \geq 3$, let $\mathcal{P}$ be $\mathcal{L}$ -nonresonant, and let μ_{in} be as in (7.2). Then, for every $G \in C_b(\mathbb{R}^d \times \Omega_n)$,*

$$(7.4) \quad \lim_{\rho \rightarrow 0} \int_{\mathcal{A}_{n,\rho}^{\text{mac}}} G(Q, \mathcal{R}_{n,\rho}(Q, V)) f_{\text{in}}(Q, V) dQ d\omega(V) \\ = \int_{\mathbb{R}^d \times \Omega_n} G(Q_0, \varpi) f_{\text{in}}(Q_0, v_0) \Phi_{v_0}(\xi_1, b_1)$$

$$(7.5) \quad \times \prod_{j=1}^{n-1} \Phi_{0,v_j}(\xi_{j+1}, b_{j+1}, z_j) dQ_0 d\omega(v_0) \prod_{j=1}^n d\xi_j db_j.$$

Note that the limit is independent of M_0 except through the nonresonance hypothesis.

Proof. We use the fundamental-domain reduction of [MS11, Section 4.5]. Let $\mathcal{F}$ be a bounded fundamental cell of $\mathcal{L}$ and write the microscopic initial point as $m + r$, with $m \in \mathcal{L}$ and $r \in \mathcal{F}$. First take f_{in} continuous and compactly supported and insert an auxiliary compact cutoff in the full record variable ϖ . The spatial weight is

$$(7.6) \quad \varepsilon^d \sum_{m \in \mathcal{L}} G(\varepsilon(m + r), \varpi) f_{\text{in}}(\varepsilon(m + r), v_0).$$

As $\varepsilon \rightarrow 0$, this Riemann sum converges uniformly in r , v_0 , and ϖ on compacta, to

$$\int_{\mathbb{R}^d} G(Q, \varpi) f_{\text{in}}(Q, v_0) dQ.$$

The collision record depends on (r, v_0) but not on m , and the excluded part of $\mathcal{F}$ has volume $O(\rho^d)$.

At fixed ρ , grazing contacts and contacts with lower-dimensional face strata form a null set for billiard flux measure. Infinite flights are also flux-null: unless the velocity is orthogonal to a nonzero vector of the dual lattice, its forward linear orbit modulo $\mathcal{L}$ is dense and hence enters the open obstacle. The exceptional velocities form a countable union of great subspheres. The regular collision map preserves flux measure, while Liouville measure in free-flight coordinates is flux measure times flight time. Finite iteration therefore gives $\mu_{\text{in}}(\mathcal{Y}_\rho \setminus \mathcal{A}_{n,\rho}^{\text{mac}}) = 0$, since μ_{in} is absolutely continuous with respect to $dQ d\omega$.

For almost every r , $-rM_0^{-1} \notin \mathbb{Q}^d$. Apply Theorem 4 with $\beta = 0$ and dominated convergence in r . The first factor is Φ_{v_0} and the later, lattice-centred factors are Φ_{0,v_j} . Equation (7.6) now gives (7.5). Tightness and continuity-set approximation remove the auxiliary record cutoff; L^1 approximation then yields the claim for general f_{in} . $\square$

Let $T_0 = 0$ and $T_n = \xi_1 + \dots + \xi_n$. For $T_j \leq t < T_{j+1}$ define

$$\Xi(t) = \left(Q_0 + \sum_{k=1}^j \xi_k v_{k-1} + (t - T_j) v_j, v_j \right).$$

The impact-extended process will be defined in the next section.

Let $\mathbb{P}_{\text{in}}$ denote the path law determined by (7.3). Since $0 \leq \Phi_{\alpha,v} \leq 1$, integration over the simplex $\xi_1 + \dots + \xi_n \leq T$ gives the uniform collision-count bound

$$(7.7) \quad \mathbb{P}_{\text{in}}(T_n \leq T) \leq \frac{(A_P T)^n}{n!}.$$

In particular, the limiting flight almost surely has only finitely many collisions in every bounded time interval.

Corollary 2. *Let $d \geq 3$, let $\mathcal{P}$ be $\mathcal{L}$ -nonresonant, and let μ_{in} be as in (7.2). For any $t_1, \dots, t_M \in \mathbb{R}_{\geq 0}$ and any Borel sets $D_1, \dots, D_M \subset \mathbb{R}^d \times S_1^{d-1}$ with $(dQ d\omega)(\partial D_j) = 0$, $j = 1, \dots, M$, we have*

$$\begin{aligned} \lim_{\rho \rightarrow 0} \mu_{\text{in}} \left\{ (Q, V) \in \mathcal{Y}_\rho : F_{t_j, \rho}(Q, V) \in D_j, j = 1, \dots, M \right\} \\ = \mathbb{P}_{\text{in}}(\Xi(t_j) \in D_j, j = 1, \dots, M). \end{aligned}$$

The convergence is uniform for $(t_1, \dots, t_M)$ in compact subsets of $\mathbb{R}_{\geq 0}^M$.

Proof. This is the counterpart of [MS11, Theorem 1.4]. We follow [MS11, Sections 5.1, 5.2 and 5.4], with Theorem 5 in place of the segment-limit theorem used there. Let n_j be the number of collisions before time t_j and put $N = \max_j n_j$; the first $N + 1$ flights determine

the observed states. The hypersurfaces $\{T_k = t_j\}$, the impact-cell boundaries, the grazing records, and the exceptional chart direction are null.

For each fixed face word and fixed flight variables, the map from (Q_0, v_0) to an observed position–velocity pair has absolute Jacobian one: the position is a translation of Q_0 , while the observed velocity is v_0 multiplied by a fixed orthogonal product of reflection matrices. Thus every one-time marginal is absolutely continuous with respect to $dQ d\omega$.

The collision-count truncation and time bracketing in [MS11] now apply, with geometric factors bounded by $A_{\mathcal{P}}$. For each fixed N and T , Theorem 5, applied to the continuity event $T_N \leq T$, transfers the finite- ρ collision-count tail to the limiting one. Equation (7.7) then bounds this tail uniformly for T in compact intervals. Letting $N \rightarrow \infty$, followed by the standard tightness and L^1 approximations, proves the claim. $\square$

8. A CONTINUOUS-TIME MARKOV PROCESS

Following [MS11, Section 6], we enlarge the state space by the remaining time and the data of the next collision:

$$\mathcal{X}_{\mathcal{P}} = \{(Q, V, \xi, W) : Q \in \mathbb{R}^d, V \in S_1^{d-1}, \xi \geq 0, W \in \mathcal{B}_V\}.$$

Here $W = (\ell, b)$ is the next labelled impact parameter. Write $\ell(W) = \ell$ and $b(W) = b$, and set

$$m_V(dW) = \sum_{\ell=1}^s \mathbf{1}_{B_{V,\ell}}(b) \delta_{\ell}(d\ell) db, \quad W = (\ell, b),$$

so that the reference measure is

$$d\sigma_1 = dQ d\omega(V) d\xi m_V(dW).$$

Let $T_0 = 0$ and $T_n = \xi_1 + \dots + \xi_n$. If $T_j \leq t < T_{j+1}$, define

$$(8.1) \quad \widehat{\Xi}(t) = \left(Q_0 + \sum_{k=1}^j \xi_k v_{k-1} + (t - T_j) v_j, v_j, T_{j+1} - t, W_{j+1} \right).$$

At the end of a flight with state (V, W) , $W = (\ell, b)$, put

$$V^+ = \mathcal{R}(V, b) = V R_{\ell}, \quad z^+ = \mathcal{Z}(V, b).$$

The new velocity, remaining time and labelled impact parameter have transition law

$$(8.2) \quad J((V, W); dV' d\xi' dW') = \delta_{V^+}(dV') \Phi_{0,V^+}(\xi', b(W'), z^+) d\xi' m_{V^+}(dW').$$

The Borel property of the incidence and avoidance events in (3.1), together with disintegration, shows that J is a Borel probability kernel. Hence $\widehat{\Xi}$ is a time-homogeneous Markov process.

More explicitly, from $x = (Q, V, \xi, W)$ the process follows $(Q + tV, V, \xi - t, W)$ for $0 \leq t < \xi$ and, at $t = \xi$, jumps at the collision position $Q + \xi V$ to a state $(Q + \xi V, V', \xi', W')$ distributed according to $J((V, W); dV' d\xi' dW')$. If $\xi = 0$, we keep the initial state at time zero and apply the jump before every positive time. Iteration defines the process from every state. The same simplex estimate as in (7.7), applied after the first jump and uniformly in the initial state, shows that, almost surely, there are at most finitely many collisions in every bounded time window.

Let $\mathcal{G}_{\rho} \subset \mathcal{Y}_{\rho}$ be the full-measure Borel set on which the next microscopic collision from $(\rho^{-(d-1)}Q, V)$ exists and is regular. If ℓ_1 and $x_1^{(\rho)}$ are the face label and local boundary point

of that collision, put

$$W_1^{(\rho)}(Q, V) = \left(\ell_1, -\pi_\perp(x_1^{(\rho)} K(V)) \right) \in \mathcal{B}_V.$$

Define

$$(8.3) \quad \mathcal{E}_\rho(Q, V) = \left(Q, V, \rho^{d-1} \tau_1(\rho^{-(d-1)} Q, V; \rho), W_1^{(\rho)}(Q, V) \right)$$

for $(Q, V) \in \mathcal{G}_\rho$ and extend to a measurable map to all of $\mathcal{Y}_\rho$.

Set $\widehat{F}_{t,\rho} = \mathcal{E}_\rho \circ F_{t,\rho}$. For the initial density in Section 7, define

$$f_{\text{lift}}(Q, V, \xi, (\ell, b)) = f_{\text{in}}(Q, V) \Phi_V(\xi, b), \quad b \in B_{V,\ell}.$$

This is a probability density with respect to σ_1 . Let $\mathbb{P}_{f_{\text{lift}}}$ denote the law of (8.1) with this initial density.

Theorem 6. *Let $d \geq 3$, let $\mathcal{P}$ be $\mathcal{L}$ -nonresonant, and let μ_{in} be as in (7.2). For any $t_1, \dots, t_M \in \mathbb{R}_{\geq 0}$ and Borel sets $\mathcal{D}_1, \dots, \mathcal{D}_M \subset \mathcal{X}_{\mathcal{P}}$ with $\sigma_1(\partial \mathcal{D}_j) = 0$, $j = 1, \dots, M$,*

$$\begin{aligned} \lim_{\rho \rightarrow 0} \mu_{\text{in}} \left\{ (Q, V) \in \mathcal{Y}_\rho : \widehat{F}_{t_j, \rho}(Q, V) \in \mathcal{D}_j, \ j = 1, \dots, M \right\} \\ = \mathbb{P}_{f_{\text{lift}}}(\widehat{\Xi}(t_j) \in \mathcal{D}_j, \ j = 1, \dots, M). \end{aligned}$$

The convergence is uniform for $(t_1, \dots, t_M)$ in compact subsets of $\mathbb{R}_{\geq 0}^M$.

Proof. We follow [MS11, Section 6.1], using Theorem 5 as the impact-resolved analogue of [MS11, Eq. (6.12)]. For a fixed collision-count vector one retains one further flight, since (ξ, W) records the flight straddling the last observation time. Relative to [MS11, Theorem 6.1], its future-velocity coordinate is replaced by W and its transition density by (8.2).

By a standard L^1 approximation argument, we may assume without loss of generality that f_{in} is bounded and compactly supported. Propagation is finite, the impact bundle is bounded, the remaining flight is tight by Lemma 2(iii), and (7.7) controls the collision count.

After summing over face words, the flux change of variables and $\text{vol}_E(B_V) \leq A_{\mathcal{P}}$ give the analogue of [MS11, Eqs. (6.13)–(6.14)]: for each bounded Borel test set $\mathcal{D}$, the unweighted n -collision reference-measure contribution is at most $(A_{\mathcal{P}} t)^n \sigma_1(\mathcal{D})/n!$. Hence, since $f_{\text{lift}} = f_{\text{in}} \Phi_V$ and $0 \leq \Phi_V \leq 1$, the f_{lift} -weighted contribution is at most $\|f_{\text{in}}\|_\infty (A_{\mathcal{P}} t)^n \sigma_1(\mathcal{D})/n!$. This and (7.7) give the continuity-set approximation and compact-time uniformity as in [MS11, Section 6.1]. $\square$

Decomposition by collision number and the facewise map

$$(V, b) \mapsto (VR_\ell, \mathcal{Z}(V, b)), \quad b \in B_{V,\ell},$$

which preserves $d\omega(V) db$ by (2.2)–(2.3), show that an initial density remains absolutely continuous with respect to σ_1 . For $f \in L^1_{\text{loc}}(\mathcal{X}_{\mathcal{P}}, \sigma_1)$, define $K_t f$ by

$$(8.4) \quad \int_{\mathcal{D}} K_t f d\sigma_1 = \int_{\mathcal{X}_{\mathcal{P}}} \mathbb{P}_x\{\widehat{\Xi}(t) \in \mathcal{D}\} f(x) d\sigma_1(x)$$

for every bounded Borel set $\mathcal{D}$, where $\mathbb{P}_x$ is the law started at x ; finite propagation makes the right-hand side locally finite. Spatial translation covariance is understood also after taking Q modulo any full-rank lattice $\mathcal{L}_Q$.

Proposition 3. *The locally integrable density*

$$f_*(Q, V, \xi, (\ell, b)) = \Phi_V(\xi, b)$$

satisfies $K_t f_* = f_*$ for all $t \geq 0$. On $\mathbb{R}^d / \mathcal{L}_Q$ it becomes a probability density after multiplication by $\text{vol}(\mathbb{R}^d / \mathcal{L}_Q)^{-1}$.

Proof. It suffices to test the two locally finite measures against bounded Borel continuity sets $\mathcal{D}$ whose Q -projection is contained in a bounded set A . Fix $t \geq 0$, choose a bounded Borel set C with $A + \overline{B(0, t)} \subset C$, and put

$$h(Q, V) = c \mathbf{1}_C(Q), \quad c = [\text{vol}(C) \omega(S_1^{d-1})]^{-1}.$$

Thus $h dQ d\omega$ is a probability measure. Every unit-speed trajectory whose state at time t belongs to $\mathcal{D}$ starts with its position in $A + \overline{B(0, t)}$, where $h = c$; on the time-zero event $\mathcal{E}_\rho^{-1}(\mathcal{D})$ we likewise have $h = c$. On the part of the labelled collision section corresponding to F_ℓ , the map

$$(V, b) \mapsto (VR_\ell, \mathcal{Z}(V, b)), \quad b \in B_{V, \ell},$$

preserves $d\omega(V) db$ by (2.2)–(2.3). Liouville invariance of the microscopic billiard flow therefore gives, for every ρ ,

$$\int_{\mathcal{Y}_\rho} \mathbf{1}_{\mathcal{D}}(\widehat{F}_{t, \rho}(Q, V)) h(Q, V) dQ d\omega(V) = \int_{\mathcal{Y}_\rho} \mathbf{1}_{\mathcal{D}}(\mathcal{E}_\rho(Q, V)) h(Q, V) dQ d\omega(V).$$

Apply Theorem 6 at times t and 0. Its initial lift is hf_* , and finite propagation gives $K_t(hf_*) = cK_t f_*$ on the Q -projection A , while $hf_* = cf_*$ there. Consequently

$$\int_{\mathcal{D}} K_t f_* d\sigma_1 = \int_{\mathcal{D}} f_* d\sigma_1,$$

which yields $K_t f_* = f_*$ in L_{loc}^1 . Finally, (3.3) gives the stated normalisation on the torus. $\square$

Proposition 4. *The family $\{K_t : t \geq 0\}$ forms a semigroup on $L_{\text{loc}}^1(\mathcal{X}_{\mathcal{P}}, \sigma_1)$ and a positive, mass-preserving contraction semigroup on $L^1(\mathcal{X}_{\mathcal{P}}, \sigma_1)$.*

Proof. As in [MS11, Eqs. (6.23)–(6.27)], decompose $K_t = \sum_{n \geq 0} K_t^{(n)}$ according to the number of collisions before time t . Here the transition density is (8.2), and the bound $0 \leq \Phi_{0, V} \leq 1$ gives

$$\|K_t^{(n)}\|_{L^1 \rightarrow L^1} \leq \frac{(A_{\mathcal{P}} t)^{n-1}}{(n-1)!}, \quad n \geq 1.$$

The series therefore converges in operator norm on L^1 , locally uniformly in t . Concatenation of the impact-resolved paths gives $K_{t+s} = K_t K_s$. Positivity, mass preservation and the contraction property follow from (8.4), cf. the remark after [MS11, Eq. (6.16)]. The strong continuity required of a contraction semigroup follows from [MS11, Eqs. (6.32)–(6.34)]. $\square$

Let us now turn to the generalised Boltzmann equation, which is the Kolmogorov forward equation for the Markovian random flight process. It is the polyhedral impact-variable analogue of [CG10, Eq. (2.30)] and [MS11, Section 6.3, especially Eq. (6.37)].

For $V \cdot n_\ell > 0$ and $x \in F_\ell$, put

$$U_\ell(V) = VR_\ell, \quad b_\ell^-(V, x) = -\pi_\perp(xK(U_\ell(V))), \quad z_\ell^+(V, x) = \pi_\perp(xK(V)).$$

Thus $b_\ell^-(V, x) \in B_{U_\ell(V), \ell}$ and $z_\ell^+(V, x) \in C_{V, \ell}$. By (2.2)–(2.3), reflection preserves the facewise flux measure:

$$(8.5) \quad d\omega(U_\ell(V)) db_\ell^-(V, x) = d\omega(V) dz_\ell^+(V, x).$$

Here (8.5) is an equality of pushforward measures: both sides are the image of $\mathbf{1}_{\{V \cdot n_\ell > 0\}}(V \cdot n_\ell) d\omega(V) dA_\ell(x)$ under the corresponding coordinate map.

Let $f_0 \in L^1(\sigma_1)$ be an initial density and put $f(t) = K_t f_0$. Suppose that, locally for $t > 0$, the weak derivatives $\partial_t f$, $V \cdot \nabla_Q f$ and $\partial_\xi f$ are integrable and that f has the characteristic collision value

$$f(t, Q, V, 0, W) := \lim_{s \rightarrow 0} f(t, Q - sV, V, s, W)$$

almost everywhere, with the collision integral in (8.6) locally integrable. Then, for $W = (\ell', b) \in \mathcal{B}_V$, f satisfies in the distributional sense for $t > 0$ and $\xi > 0$, with initial condition $f(0) = f_0$,

$$(8.6) \quad (\partial_t + V \cdot \nabla_Q - \partial_\xi) f(t, Q, V, \xi, W) = [\mathcal{C}_P f](t, Q, V, \xi, W)$$

with the collision operator

$$(8.7) \quad [\mathcal{C}_P f](t, Q, V, \xi, (\ell', b)) = \sum_{\ell: V \cdot n_\ell > 0} \int_{F_\ell} f(t, Q, U_\ell(V), 0, (\ell, b_\ell^-(V, x))) \\ \times \Phi_{0,V}(\xi, b, z_\ell^+(V, x))(V \cdot n_\ell) dA_\ell(x).$$

This follows from the analogous calculation in [MS11, Section 6.3]. Transport between collisions gives the left-hand side. At $\xi = 0$, (8.5) and the conditional density $\Phi_{0,V}(\xi, b, z_\ell^+) d\xi db$ give the right-hand side. The new features are the face-label sum, the impact variable, and the flux factor $V \cdot n_\ell$. No differentiability of $\Phi_{0,V}$ is required, and no separate loss term appears because $-\partial_\xi f$ accounts for passage through $\xi = 0$.

For the physical initial density f_{in} , Theorem 6 identifies the limiting extended density as $K_t f_{\text{lift}}$. Its position–velocity marginal is

$$[L_t f_{\text{in}}](Q, V) = \int_0^\infty \int_{\mathcal{B}_V} [K_t f_{\text{lift}}](Q, V, \xi, W) m_V(dW) d\xi.$$

Note that, as in the case of the periodic Lorentz gas, the family $\{L_t : t \geq 0\}$ does not in general form a semigroup.

REFERENCES

- [BKMP11] P. Bachurin, K. Khanin, J. Marklof, and A. Plakhov. Perfect retroreflectors and billiard dynamics. *Journal of Modern Dynamics*, 5(1):33–48, 2011. <https://doi.org/10.3934/jmd.2011.5.33>.
- [CG10] E. Caglioti and F. Golse. On the Boltzmann–Grad limit for the two dimensional periodic Lorentz gas. *Journal of Statistical Physics*, 141(2):264–317, 2010. <https://doi.org/10.1007/s10955-010-0046-1>.
- [Dek59] T. J. Dekker. On free products of cyclic rotation groups. *Canadian Journal of Mathematics*, 11:67–69, 1959. <https://doi.org/10.4153/CJM-1959-009-7>.
- [EE11] P. Ehrenfest and T. Ehrenfest. Begriffliche Grundlagen der statistischen Auffassung in der Mechanik. In F. Klein and C. Müller, editors, *Encyklopädie der mathematischen Wissenschaften mit Einschluß ihrer Anwendungen, Band IV, 2. Teil, Heft 6*, Art. 32, pp. 3–90. B. G. Teubner, Leipzig, 1911.
- [Gal69] G. Gallavotti. Divergences and the approach to equilibrium in the Lorentz and the wind-tree models. *Physical Review*, 185(1):308–322, 1969. <https://doi.org/10.1103/PhysRev.185.308>.
- [Lor05] H. A. Lorentz. Le mouvement des électrons dans les métaux. *Archives néerlandaises des sciences exactes et naturelles*, series 2, 10:336–371, 1905.
- [LT20] C. Lutsko and B. Tóth. Invariance principle for the random Lorentz gas—beyond the Boltzmann–Grad limit. *Communications in Mathematical Physics*, 379(2):589–632, 2020. <https://doi.org/10.1007/s00220-020-03852-8>.

- [LT21] C. Lutsko and B. Tóth. Invariance principle for the random wind-tree process. *Annales Henri Poincaré*, 22(10):3357–3389, 2021. <https://doi.org/10.1007/s00023-021-01106-4>.
- [MS10] J. Marklof and A. Strömbergsson. The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems. *Annals of Mathematics*, 172(3):1949–2033, 2010. <https://doi.org/10.4007/annals.2010.172.1949>.
- [MS11] J. Marklof and A. Strömbergsson. The Boltzmann–Grad limit of the periodic Lorentz gas. *Annals of Mathematics*, 174(1):225–298, 2011. <https://doi.org/10.4007/annals.2011.174.1.7>.
- [MS14] J. Marklof and A. Strömbergsson. Power-law distributions for the free path length in Lorentz gases. *Journal of Statistical Physics*, 155(6):1072–1086, 2014. <https://doi.org/10.1007/s10955-014-0935-9>.
- [MS24] J. Marklof and A. Strömbergsson. Kinetic theory for the low-density Lorentz gas. *Memoirs of the American Mathematical Society*, 294(1464):v+136, 2024. <https://doi.org/10.1090/memo/1464>.
- [Sha96] N. A. Shah. Limit distributions of expanding translates of certain orbits on homogeneous spaces. *Proceedings of the Indian Academy of Sciences—Mathematical Sciences*, 106(2):105–125, 1996. <https://doi.org/10.1007/BF02837164>.
- [Vor05] Y. Vorobets. On stability of periodic billiard orbits in polyhedra. Unpublished manuscript, available at <https://people.tamu.edu/~yvorobets/Research/Stable.pdf>, 2005.

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