

EFFECTIVE COUNTING IN SPHERE PACKINGS

ALEX KONTOROVICH AND CHRISTOPHER LUTSKO

ABSTRACT.

Given a Zariski-dense, discrete group, Γ , of isometries acting on $(n + 1)$ -dimensional hyperbolic space, we use spectral methods to obtain a sharp asymptotic formula for the growth rate of certain Γ -orbits. In particular, this allows us to obtain a best-known effective error rate for the Apollonian and (more generally) Kleinian sphere packing counting problems, that is, counting the number of spheres in such with radius bounded by a growing parameter. Our method extends the method of Kontorovich [Kon09], which was itself an extension of the orbit counting method of Lax-Phillips [LP82], in two ways. First, we remove a compactness condition on the discrete subgroups considered via a technical cut-off and smoothing operation. Second, we develop a coordinate system which naturally corresponds to the inversive geometry underlying the sphere counting problem, and give structure theorems on the arising Casimir operator and Haar measure in these coordinates.

1. INTRODUCTION

The purpose of this paper is to give improved error estimates on the counting problem for Kleinian sphere packings (and discrete counting methods more broadly). A packing \mathcal{P} of \mathbb{S}^n (thought of as the boundary of hyperbolic $(n + 1)$ -space, \mathbb{H}^{n+1}) is an infinite collection of round, disjoint balls whose union is dense in \mathbb{S}^n . Following [KK21], such is called *Kleinian* if its residual set (left over when the interiors of the balls are removed) agrees with the limit set of some discrete, geometrically finite subgroup, Γ , of isometries of \mathbb{H}^{n+1} . A familiar example is the classical Apollonian circle packing in $n = 2$, see e.g [Kon13] for more background and see Figure 1 for an example.

Let \mathcal{P} be a given Kleinian sphere packing. For a sphere $S \in \mathcal{P}$, let $b(S)$ denote its bend, that is, its (signed) inverse-radius; this is determined after a choice of coordinates on \mathbb{S}^n , and in particular a choice of a point at ∞ (see §4). The Counting Problem is to estimate the number

$$N_{\mathcal{P}}(T) := \#\{S \in \mathcal{P} : b(S) < T\}$$

of spheres in \mathcal{P} with bend bounded by a parameter $T \rightarrow \infty$. If the packing \mathcal{P} is bounded, that is, the chosen point at infinity is contained in the interior of some ball, then $N_{\mathcal{P}}(T)$ is finite for all T ; otherwise one can count spheres restricted to a bounded region (such as a period, if the packing is periodic). Let $\delta = \dim(\mathcal{P})$ be the Hausdorff dimension of the residual set of \mathcal{P} .

Kontorovich is partially supported by NSF grant DMS-1802119, BSF grant 2020119, and the 2020-2021 Distinguished Visiting Professorship at the National Museum of Mathematics.

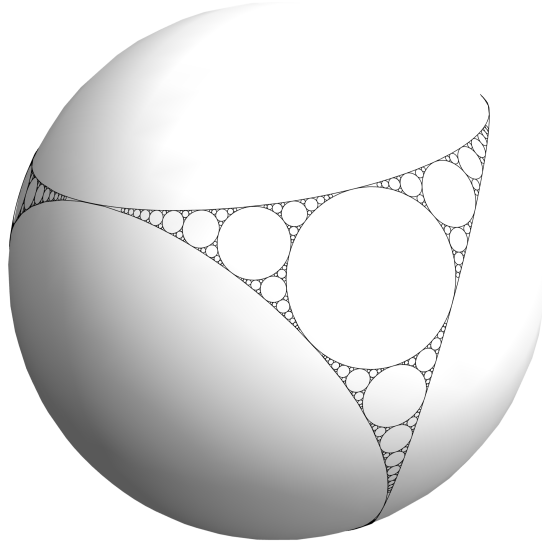


FIGURE 1. A classical Apollonian circle packing in the sphere \mathbb{S}^2 . Note that the union of balls in the packing is dense in \mathbb{S}^2 . (Image by Iván Rasskin.)

For the classical Apollonian packing, it is known that $\delta \approx 1.3$ (in general one has $n - 1 < \delta < n$). In this setting, Boyd [Boy73] showed that $N_{\mathcal{P}}(T) = T^{\delta+o(1)}$, which was improved in Kontorovich-Oh [KO11] to an asymptotic formula, $N_{\mathcal{P}}(T) \sim c_{\mathcal{P}}T^{\delta}$, where $c_{\mathcal{P}}$ is a constant depending on the packing \mathcal{P} . An effective power savings error rate was shown in [Vin12] and [LO13] independently. These tools and results have been generalized by many authors (see, e.g., [Kim15, MO15, Pan17, EO21]). In this paper, we introduce a new method, modifying the approach in [Kon09], to produce a best known error exponent in the Counting Problem (including the classical Apollonian case).

The error exponent involves the spectrum of the hyperbolic Laplacian Δ acting on $L^2(\Gamma \backslash \mathbb{H}^{n+1})$ where Γ is the symmetry group of the packing. From work of Lax-Phillips [LP82], Patterson [Pat76], and Sullivan [Sul84], we have that the Laplace spectrum consists of a discrete isolated bottom eigenvalue $\lambda_0 = \delta(n - \delta)$, then possibly a finite number of further discrete eigenvalues in the “exceptional” range,

$$(1.1) \quad \lambda_0 < \lambda_1 \leq \dots \leq \lambda_k < n^2/4,$$

and purely continuous spectrum above $n^2/4$. (It was conjectured by Sarnak [Sar07] that, in the case of the classical Apollonian packing, $k = 0$, that is, there are no other discrete eigenvalues below $n^2/4$ except the bottom.) Write $\lambda_j = s_j(n - s_j)$ with $n/2 \leq s_j < \delta$.

Theorem 1. *Given a Kleinian packing \mathcal{P} , there exists a constant $c_{\mathcal{P}} > 0$ such that:*

$$(1.2) \quad N_{\mathcal{P}}(T) = c_{\mathcal{P}}T^{\delta} (1 + O(T^{-\eta}))$$

as $T \rightarrow \infty$, where

$$(1.3) \quad \eta = \frac{2}{n+3}(\delta - s_1).$$

If there is no discrete spectrum other than the base eigenvalue, then we have that

$$N_{\mathcal{P}}(T) = c_{\mathcal{P}} T^{\delta} \left(1 + O\left(T^{-\eta} (\log T)^{2/(n+3)}\right)\right),$$

where $\eta = \frac{2}{n+3}(\delta - n/2)$.

For the classical Apollonian packing ($n = 2$), our error exponent is $\eta = \frac{2}{5}(\delta - s_1)$, and if there are no discrete eigenvalues above the base then $\eta = \frac{2}{5}(\delta - 1) \approx 0.12\dots$, whereas the best previously known exponent [LO13, Theorem 1.1] was:

$$(1.4) \quad \eta = \frac{2}{63}(\delta - s_1),$$

and if $s_1 = n/2$, this exponent is $\eta = \frac{2}{63}(\delta - 1) \approx 0.0097$. Hence (1.3) is a significant improvement over (1.4).

Remark. Counting with a smooth cutoff and extracting all of the lower order terms corresponding to eigenvalues other than the base, we obtain the best possible error exponent $\eta = \delta - n/2$ (see Theorem 10), which improves over the smooth error exponent in the Apollonian case $\eta = \frac{2}{7}(\delta - 1)$ in [LO13, Theorem 8.2].

The proof of Theorem 1 introduces two new technical ideas: first, of independent interest, we overcome difficulties in the “abstract spectral theory” method of [Kon09] arising from a non-compactness condition (see the discussion below), and second, we introduce a new decomposition tailored to sphere counting problems and derive structure theorems for the arising Casimir operator and Haar measure.

1.1. Ideas in the Proof.

Previous Approach: To explain the main ideas, we first recall the method introduced in [Kon09], which itself is modeled on [LP82]. Consider the following related but simpler to exposit problem. Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a discrete, Zariski dense, geometrically finite subgroup, with critical exponent $\delta_{\Gamma} > 1/2$, acting on the upper half plane \mathbb{H}^2 , and assume that ∞ is not a *point of approximation* for Γ . This last condition implies that either ∞ is a cusp of Γ with stabilizer Γ_{∞} (in which case the limit set is periodic), or ∞ is not in the limit set of Γ (and hence the limit set is bounded). Either way, consider the problem of counting

$$N_{\Gamma}(T) := \#\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\infty} \backslash \Gamma : c^2 + d^2 < T\right\}.$$

(The Lax-Phillips work counts $a^2 + b^2 + c^2 + d^2 < T$, and new ideas are needed to handle the potential stabilizer when counting $c^2 + d^2$.) The assumption that ∞ is not a point of approximation assures that the set of such $c^2 + d^2$ is discrete, and hence this count is

finite for any T . The main idea, as sketched below, is to use a particular function of the Laplacian to grow balls of size T from balls of bounded size.

In this subsection, write $G = \mathrm{SL}_2(\mathbb{R})$, $N = \left\{ \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} \right\}$, (so that $\Gamma_\infty = \Gamma \cap N$) and $K = \mathrm{SO}(2)$, and let

$$(1.5) \quad \chi_T : G \rightarrow \mathbb{R} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathbb{1}_{\{c^2 + d^2 < T\}}$$

be the indicator function of the region in question in G . Note that χ_T is left- N -invariant and right- K -invariant, and let

$$F_T : \Gamma \backslash G / K \rightarrow \mathbb{R} : g \mapsto \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi_T(\gamma g),$$

so that $N_\Gamma(T) = F_T(e)$. As observed in [Kon09], F_T is in $L^2(\Gamma \backslash G / K) = L^2(\Gamma \backslash \mathbb{H})$ if and only if ∞ is a cusp of Γ . To access the value of F_T at the origin, we let ψ be a smooth bump function about the origin in G , and automorphize it to $\Psi(g) := \sum_{\gamma \in \Gamma} \psi(\gamma g)$. Then, we can write a smooth approximation of $N_\Gamma(T)$ as

$$\widetilde{N}_\Gamma(T) := \langle F_T, \Psi \rangle_\Gamma$$

where the inner-product is with respect to $L^2(\Gamma \backslash \mathbb{H})$. Now suppose we take the inner-product of F_T with an eigenfunction of the Laplacian ϕ , with eigenvalue $\lambda = s(1-s)$. Then, solving a second order ODE, we would have that

$$(1.6) \quad \langle F_T, \phi \rangle_\Gamma = \alpha T^s + \beta T^{1-s},$$

for some α and β depending on ϕ . The key idea of [Kon09] is then to rewrite (1.6) in a way that involves the eigenvalue only, and not the coefficients α , β of the eigenfunction. This is achieved by setting $T = 1$ and $T = b$ (for some $b < 10$, see §3.1) in (1.6) and solving for α, β , to give an expression of the form:

$$\langle F_T, \phi \rangle = K_T(s) \langle F_1, \phi \rangle + L_T(s) \langle F_b, \phi \rangle,$$

for some functions K_T, L_T . This then allows one to prove the “main identity”, which states that, in the sense of L^2 , we have:

$$F_T = K_T(\Delta) F_1 + L_T(\Delta) F_b.$$

This is exactly what we mean by “growing” the ball of radius T from the Laplacian and bounded norm balls. It is this identity that can be proved rigorously even in the absence of explicit Whittaker expansions and spectral decompositions.

New Ideas: Much of this approach fails if Γ does not have a cusp at ∞ , and the main purpose of [KO12] was to bypass this “PDE” approach and replace it with homogeneous dynamics, at the cost of worse error terms. The main new ideas of this paper allow us to recover the PDE approach (and then extend it to the Kleinian setting).

The first issue is that F_T is not in L^2 . The observation which eliminates this issue (made already in [KO12]) is to add a second cut off in the N -direction without losing any of the orbit, since the limit set is anyway bounded; this adds compactness in the x -variable, restricted to some sufficiently large interval $[-X, X]$. Now we again unfold the inner product $\langle F_T, \Psi \rangle_\Gamma$ leading to the following integral:

$$\int_0^\infty \int_{-X}^X \chi_T(z) \tilde{\chi}_X(z) \Psi(z) dx \frac{dy}{y^2}.$$

We proceed with harmonic analysis on $[-X, X] \times (0, \infty)$. However this truncation requires a delicate smoothing procedure to avoid introducing boundary terms in the analysis of the Laplacian. Once this is accomplished, the proof follows in a similar fashion. See §3.1 for the details.

In addition to overcoming the restriction in [Kon09], this $\mathrm{SL}_2(\mathbb{R})$ result is of independent interest, and improves on [KO12, Theorem 1.8] which used methods from homogeneous dynamics to count Pythagorean triples.

In the Kleinian Setting: There are several further modifications and innovations needed to extend the above-described $\mathrm{SL}_2(\mathbb{R})$ approach to the setting of orbits in circle/sphere packings. In the previous setting, the stabilizer of χ_T in (1.5) was left- N and right- K invariant, and so it was natural to use Iwasawa coordinates $\mathrm{SL}_2(\mathbb{R}) = NAK$. In the setting of sphere packings, one counts spheres in the orbit $S_0\Gamma$ with bend less than T ; here S_0 is a fixed $(n-1)$ -sphere in $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\} = \partial\mathbb{H}^{n+1}$ and $\Gamma < G = \mathrm{SO}(n+1, 1)$ is a symmetry group of the packing, acting on the right by Möbius transformations. The analogous function χ_T is given by:

$$\chi_T : G \rightarrow \mathbb{R} : g \mapsto \mathbb{1}_{\{b(S_0g) < T\}},$$

where again $b(S)$ is the bend of a sphere S . This function is left- H invariant, where $H = \mathrm{Stab}_G(S_0) \cong \mathrm{SO}(n, 1)$; it is also right invariant under the group L of affine motions, since translating a sphere does not change its bend. The latter decomposes further as $L = UM$, where U is a one-parameter unipotent group (which controls the co-bend, defined to be the bend of the inversion of a sphere through the unit sphere), and $M \cong \mathrm{SO}(n)$ rotates the sphere about the origin. It turns out that $H \cap M \cong \mathrm{SO}(n-1)$, and set $M_1 := M/(H \cap M) \cong \mathbb{S}^{n-1}$. The subgroup of G which directly controls the bend is also a one-parameter unipotent group we call \overline{U} , leading to the map:

$$H \times \overline{U} \times U \times M_1 \rightarrow G,$$

which is an isomorphism in a neighborhood of the identity; see §4.2 for details. An important feature of this decomposition is the fact that the Haar measure of G in these coordinates decomposes as a product of H -Haar measure on the H component, times the M_1 -Haar measure on the M_1 component, and times something depending only on U and

\bar{U} ; see §4.4. Moreover, in the proof, we only need the Casimir operator restricted to left- H - and right- M_1 -invariant functions, for which we derive a nice, concise explicit expression in any dimension; see §4.5.

Let $\Gamma_1 := \Gamma \cap H$ denote the stabilizer of S_0 in Γ . It follows from the Structure Theorem for Kleinian packings [KK21, Theorem 22] that Γ_1 is a *lattice* in H (that is, it acts with finite covolume); this fact will be used crucially in our analysis. The finiteness of the volume of $\Gamma_1 \backslash H$ is analogous in the $\mathrm{SL}_2(\mathbb{R})$ setting to the finiteness of the volume of K , though the latter is trivial since K is compact. Note that we acted on the left in $\mathrm{SL}_2(\mathbb{R})$ and it is more convenient to act on the right for sphere packings.

In the $\mathrm{SL}_2(\mathbb{R})$ setting, the N direction was unbounded and required a cut-off. Analogously here, the U -direction is unbounded; this can be controlled via a similar truncation procedure by invoking the fact that the limit set is bounded in the U direction, see Lemma 11.

Remark. Note that if the stabilizer of \mathcal{P} contains a full rank unipotent subgroup then the methods of [Kon09] may be applied directly. For the existence of such, see [KN19].

1.2. Organization. In section 2 we collect some preliminaries. In section 3, we warm up to the counting theorem and illustrate the main ideas by proving a result analogous to Theorem 1 in the $\mathrm{SL}_2(\mathbb{R})$ setting. In section 4, we switch to the general $\mathrm{SO}(n+1, 1)$ setting, and derive the Haar measure, and Casimir operator in the above-described coordinate system. Finally, in section 5 we prove Theorem 1.

1.3. Acknowledgements. The authors are grateful to Stephen D. Miller for enlightening conversations and suggestions, and to Iván Rasskin for allowing us to use his image in Figure 1 generated from his `polytopack` Mathematica package (in preparation).

2. PRELIMINARIES

Lie algebras and the Casimir Operator:

We collect here some standard facts about the group $G = \mathrm{SO}^\circ(n+1, 1)$ (where \circ denotes the connected component of the identity), and its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of dimension $d := \dim(\mathfrak{g}) = (n+1)(n+2)/2$.

In general, Casimir operators generate the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In our rank-one setting, we can compute the Casimir operator \mathcal{C} as follows. Let X_1, \dots, X_d be a basis for the Lie algebra \mathfrak{g} , and let X_1^*, \dots, X_d^* be a dual basis with respect to the Killing form:

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)),$$

that is, $B(X_i, X_j^*) = \mathbb{1}_{\{i=j\}}$. Then the Casimir operator can be expressed as

$$\mathcal{C} = \sum_{i=1}^d X_i X_i^*.$$

Since the elements of the Lie algebra act like first order differential operators, the Casimir operator acts as a second order differential operator on smooth functions on G ; see §4.5 for a detailed calculation in our setting.

Let $K \cong SO(n+1) < G$ be a maximal compact subgroup. When restricted to K -invariant smooth functions on G , the Casimir operator \mathcal{C} agrees (up to constant) with the hyperbolic Laplacian Δ under an identification $G/K \cong \mathbb{H}^{n+1}$.

Decomposition of $L^2(\Gamma \backslash G)$ into irreducibles:

Let Γ be a discrete, geometrically finite, Zariski dense subgroup of G . Given Iwasawa coordinates $G = NAK$, let $M \cong SO(n)$ be the centralizer of A in K . The group G acts by the right-regular representation on the Hilbert space $\mathcal{H} := L^2(\Gamma \backslash G)$ of square-integrable Γ -automorphic functions. The space \mathcal{H} decomposes into components as follows:

$$(2.1) \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k \oplus \mathcal{H}^{\text{tempered}} \oplus \mathcal{H}^{\text{non-spherical}}.$$

Here each \mathcal{H}_j is the G -span of the eigenfunction corresponding to the eigenvalue $\lambda_j = s_j(n - s_j)$ in (1.1), each of which is an irreducible spherical complementary series representation with parameter $s_j > n/2$. Moreover, $\mathcal{H}^{\text{tempered}}$ denotes tempered spectrum, and $\mathcal{H}^{\text{non-spherical}}$ consists of non-spherical complementary series representations (which can only occur if $n \geq 3$ [Kna01]). Note that if $v \in \mathcal{H}$ is right M -invariant, then its projection to $\mathcal{H}^{\text{non-spherical}}$ vanishes, since non-spherical complementary series representations have no M -fixed vectors. Moreover, the subspace \mathcal{H}_j^K of K -fixed vectors in \mathcal{H}_j is 1-dimensional, and spanned by the corresponding eigenfunction in $L^2(\Gamma \backslash \mathbb{H}^{n+1}) \cong L^2(\Gamma \backslash G)^K$.

Abstract Spectral Theorem:

We recall the abstract spectral theorem (see for example [Rud73, Ch. 13]) for unbounded self-adjoint operators. Let L be a self-adjoint, positive semidefinite operator on the Hilbert space \mathcal{H} . In our applications \mathcal{H} will be either $L^2(\Gamma \backslash G)$ or a subspace thereof and L will be the Casimir operator.

Theorem 2 (Abstract Spectral Theorem). *There exists a spectral measure ν on \mathbb{R} and a unitary spectral operator $\widehat{\cdot} : \mathcal{H} \rightarrow L^2([0, \infty), d\nu)$ such that:*

$$(2.2) \quad \begin{aligned} & i) \text{ Abstract Parseval's Identity: for } \phi_1, \phi_2 \in \mathcal{H} \\ & \langle \phi_1, \phi_2 \rangle_{\mathcal{H}} = \langle \widehat{\phi}_1, \widehat{\phi}_2 \rangle_{L^2([0, \infty), d\nu)}; \end{aligned}$$

ii) The spectral operator is diagonal with respect to L : for $\phi \in \mathcal{H}$ and $\lambda \geq 0$,

$$(2.3) \quad \widehat{L}\phi(\lambda) = \lambda\widehat{\phi}(\lambda);$$

Moreover, if λ is in the point spectrum of L with associated eigenspace \mathcal{H}_λ , then for any $\psi_1, \psi_2 \in \mathcal{H}$ one has

$$(2.4) \quad \widehat{\psi}_1(\lambda)\widehat{\psi}_2(\lambda) = \langle \text{Proj}_{\mathcal{H}_\lambda} \psi_1, \text{Proj}_{\mathcal{H}_\lambda} \psi_2 \rangle,$$

where Proj refers to the projection to the subspace \mathcal{H}_λ . In the special case that \mathcal{H}_λ is one-dimensional and spanned by the normalized eigenfunction ϕ_λ , we have that

$$(2.5) \quad \widehat{\psi}_1(\lambda)\widehat{\psi}_2(\lambda) = \langle \psi_1, \phi_\lambda \rangle \langle \phi_\lambda, \psi_2 \rangle.$$

3. THE $\text{SL}_2(\mathbb{R})$ CASE

In this section, let $G := \text{SL}_2(\mathbb{R})$ and $\Gamma < G$ be a Zariski dense, finitely generated, discrete subgroup with $\delta_\Gamma > 1/2$. The goal of this section is to prove Theorem 3 below, that is, to improve on [KO12, Theorem 1.11] by extending the proof of [Kon09, Theorem 1.3 (1)] to the setting where Γ_∞ is trivial. This will serve as a model for the method that we will generalize to higher dimensions in the rest of the paper.

Again, from work of Lax-Phillips [LP82] and Patterson [Pat76] we have that the Laplace spectrum below $1/4$ consists of a finite number of discrete eigenvalues

$$(3.1) \quad \lambda_0 = \delta_\Gamma(1 - \delta_\Gamma) < \lambda_1 \leq \dots \leq \lambda_k < 1/4.$$

Write $\lambda_j = s_j(1 - s_j)$ with $s_j \in (\frac{1}{2}, 1)$.

Theorem 3. *Let $G := \text{SL}_2(\mathbb{R})$ and $\Gamma < G$ a Zariski dense, finitely generated, discrete subgroup with $\delta_\Gamma > 1/2$. Assume that ∞ is either a cusp for Γ with stabilizer Γ_∞ or ∞ is not in the limit set of Γ . Then there exist constants $c_0 > 0$, c_1, \dots, c_k , and $\eta > 0$ such that as $T \rightarrow \infty$*

$$(3.2) \quad \begin{aligned} N_\Gamma(T) &:= \#\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma : c^2 + d^2 < T \right\} \\ &= c_0 T^{\delta_\Gamma} + c_1 T^{s_1} + \dots + c_k T^{s_k} + O(T^\eta \log^{1/2} T), \end{aligned}$$

with $\eta = \frac{1}{2}(\delta_\Gamma + \frac{1}{2})$.

Remark. The case where ∞ is a cusp is the content of [Kon09, Theorem 1.3 (1.5)], so we assume below that ∞ is not in the limit set of Γ .

Remark. Note that in this case, since we are counting points in \mathbb{H} (which is K invariant) we can extract all lower order terms corresponding to eigenvalues other than the base.

Remark. The corresponding error term in [KO12, §4.1] is significantly worse (and not even explicitly specified) as compared to Theorem 3, due in part to much worse dependence on Sobolev norms of the corresponding test vectors.

To begin the proof, we proceed as described in the Introduction. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, let

$$(3.3) \quad \chi_T(g) := \begin{cases} 1 & \text{if } c^2 + d^2 < T, \\ 0 & \text{otherwise.} \end{cases}$$

Under the identification $G/K \sim \mathbb{H}$ (where $K = SO(2)$), $g \mapsto z = gi$, χ_T corresponds to the conditions $\Im(z) > 1/T$.

Now average χ_T over the group Γ :

$$(3.4) \quad F_T(g) := \sum_{\gamma \in \Gamma} \chi_T(\gamma g).$$

such that the count in (3.2) can be written as $N_\Gamma(T) = F_T(e)$.

To access the value of F_T at the identity, we follow the standard procedure of smoothing the count. To this end, we fix once and for all a smooth, even, nonnegative, compactly supported bump function $\psi_1 \in C_0^\infty(\mathbb{R})$ with unit total mass, $\int_{\mathbb{R}} \psi_1 dx = 1$. Given $\varepsilon > 0$, we set $\psi_\varepsilon(x) := \frac{1}{\varepsilon} \psi_1(\frac{x}{\varepsilon})$. Write Iwasawa coordinates as $n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N$, $a_y := \text{Diag}(\sqrt{y}, 1/\sqrt{y}) \in A$, and $k \in K = SO(2)$, and by abuse of notation, write $\psi : G \rightarrow \mathbb{R}_+$ as follows:

$$(3.5) \quad \psi(n_x a_y k) := \psi_\varepsilon(x) \psi_\varepsilon(\log y).$$

Clearly ψ is right- K -invariant, and it is easy to compute that $\int_G \psi(n_x a_y k) \frac{dx dy dk}{y^2} = 1 + O(\varepsilon)$.

We automorphise ψ by setting:

$$\Psi(g) := \sum_{\gamma \in \Gamma} \psi(\gamma g),$$

and consider the smoothed count:

$$\tilde{N}_\Gamma(T) := \langle F_T, \Psi \rangle.$$

After unfolding F_T , we see that

$$\tilde{N}_\Gamma(T) := \sum_{\gamma \in \Gamma} w_T(\gamma),$$

where $w_T = w_{T,X,\varepsilon} : \mathbb{H} \rightarrow [0, 1]$ is given by

$$(3.6) \quad w_T(g) := \int_G \chi_T(gh) \psi(h) dh.$$

The following theorem, from which Theorem 3 follows by optimizing error terms, gives an asymptotic expansion for the smoothed count in the $SL_2(\mathbb{R})$ -setting.

Theorem 4. *Assume that $\varepsilon > 0$ is small enough. Then there exist constants $c_{\Gamma,\varepsilon}^{(i)}$ for $i = 0, 1, \dots, k$ such that*

$$(3.7) \quad \tilde{N}_\Gamma(T) = c_{\Gamma,\varepsilon}^{(0)} T^{\delta_\Gamma} + c_{\Gamma,\varepsilon}^{(1)} T^{s_1} + \dots + c_{\Gamma,\varepsilon}^{(k)} T^{s_k} + O\left(\frac{1}{\varepsilon} T^{1/2} \log T\right)$$

with $c_{\Gamma,\varepsilon}^{(0)} > 0$, and the implied constant depending only on Γ . Moreover $c_{\Gamma,\varepsilon}^{(i)} = c^{(i)}(1 + O(\varepsilon))$ for all $i = 0, 1, \dots, k$.

It remains to prove Theorem 4.

3.1. Inserting the Laplacian. The smooth count above is an L^2 inner product of the indicator function F_T with a smooth bump function Ψ . The key idea now is to forget about Ψ and analyze the structure of the inner product of F_T with *any* smooth L^2 function.

Following [Kon09, §3] let

$$(3.8) \quad K_T(s) := \frac{T^s b^{1-s} - T^{1-s} b^s}{b^{1-s} - b^s}, \quad L_T(s) := \frac{T^{1-s} - T^s}{b^{1-s} - b^s},$$

where $b > 1$ is a constant which can be taken to be less than 3 (see [Kon09, (3.7)]). With that, if ϕ were an eigenfunction of the Laplacian then we would have $\langle F_T, \phi \rangle = A_\phi T^s + B_\phi T^{1-s}$. In this case, we could conclude that

$$\langle F_T, \phi \rangle = K_T(s) \langle F_1, \phi \rangle + L_T(s) \langle F_b, \phi \rangle.$$

Thus we would like to show that

$$(3.9) \quad F_T = K_T(\Delta) F_1 + L_T(\Delta) F_b.$$

However, there is a problem created by the fact that F_T is not in $L^2(\Gamma \backslash \mathbb{H})$. To get around this, we will first restrict the support of F_T in the x -direction before applying various smoothing arguments to conclude that a version of (3.9) holds for the modified F_T . Since ∞ is not in the limit set, it is in the free boundary, and hence there exists a fixed $X = X(\Gamma) > 0$ such that the full region $((-\infty, -X] \cup [X, \infty)) \times [0, \infty) \subset \mathbb{H}$ is contained in a single fundamental domain, see Figure 2.

Restricting the Real Direction: Define the following counting function

$$F_{T,X}(z) := \sum_{\gamma \in \Gamma} \chi_T(\gamma z) \tilde{\chi}_X(\gamma z)$$

where

$$\tilde{\chi}_X(x + iy) := \begin{cases} 1 & \text{if } x \in [-X, X], \\ 0 & \text{if } x \notin [-X, X]. \end{cases}$$

Note that by our choice of X , we still have that $N_\Gamma(T) = F_T(i) = F_{T,X}(i)$.

Now, consider the difference operator

$$G_{T,X} := F_{T,X} - K_T(\Delta) F_{1,X} - L_T(\Delta) F_{b,X}.$$

By self-adjointness of Δ , for any $\Psi \in L^2(\Gamma \backslash \mathbb{H})$ we have

$$\langle G_{T,X}, \Psi \rangle_{\Gamma \backslash \mathbb{H}} = \int_{\Gamma \backslash \mathbb{H}} [F_{T,X}(z) \Psi(z) - F_{1,X}(z) (K_T(\Delta) \Psi)(z) - F_{b,X}(z) (L_T(\Delta) \Psi)(z)] dz,$$

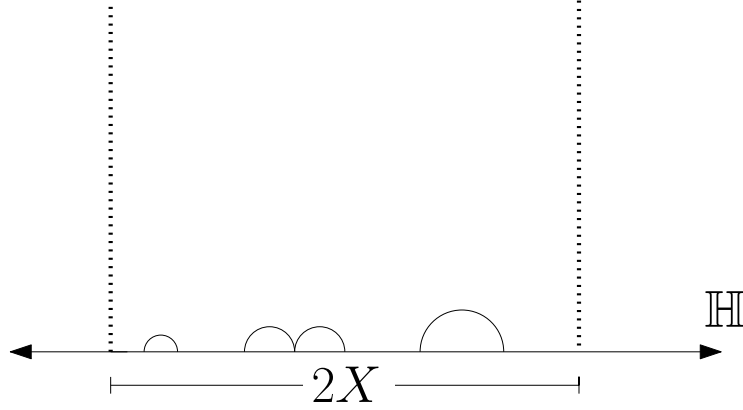


FIGURE 2. Above we sketch a fundamental domain of Γ which extends to infinity in the real direction, and the cut-off due to $\tilde{\chi}_X$ which is large enough such that $[-X, X]$ contains the entire limit set.

which we can unfold to

(3.10)

$$\langle G_{T,X}, \Psi \rangle_{\Gamma \backslash \mathbb{H}} = \int_{\mathbb{H}} (\chi_T(z) \tilde{\chi}_X(z) \Psi(z) - \chi_1(z) \tilde{\chi}_X(z) (K_T(\Delta) \Psi)(z) - \chi_b(z) \tilde{\chi}_X(z) (L_T(\Delta) \Psi)(z)) dz.$$

Our goal is to show, that for any $\Psi \in L^2(\Gamma \backslash \mathbb{H})$ we have $\langle G_{T,X}, \Psi \rangle_{\Gamma} = 0$. This implies the following identity

Proposition 5. *For any values of T and X large enough, we have*

(3.11)

$$F_{T,X} = K_T(\Delta) F_{1,X} + L_T(\Delta) F_{b,X}.$$

where K_T and L_T are the differential operators defined above and b fixed.

Smoothing χ_T : Now we smooth the function χ_T so that we can move the Laplacian over to $\tilde{\chi}_X$ via self-adjointness. Let $\sigma > 0$, and define the following smooth cut-off function

$$\chi_{1,\sigma}(x + iy) := \begin{cases} 1 & \text{if } y > 1, \\ 0 & \text{if } y < (1 - \sigma), \end{cases}$$

and smooth, in between the two bounds. Now let $\chi_{T,\sigma}(x + iy) := \chi_{1,\sigma}(x + iTy)$.

Let $G_{T,X}^\sigma$ be defined similarly to $G_{T,X}$, with χ_T replaced by $\chi_{T,\sigma}$. That is, let

$$\mathcal{I}_X := [-X, X]$$

and let

$$G_{T,X}^\sigma(\Psi) := \int_0^\infty \int_{x \in \mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) \Psi(z) - \chi_{1,\sigma}(z) (K_T(\Delta) \Psi)(z) - \chi_{b,\sigma}(z) (L_T(\Delta) \Psi)(z)) dx \frac{dy}{y^2}.$$

Note that by construction for any fixed t

$$\int_0^\infty \int_{x \in \mathcal{I}_X} |\chi_{t,\sigma}(z) - \chi_t|^2 dx \frac{dy}{y^2} \ll_{t,X} \sigma.$$

Thus, using Cauchy-Schwarz we have that for any $\Psi \in L^2(\Gamma \backslash \mathbb{H})$ we have that

$$\lim_{\sigma \rightarrow 0} G_{T,X}^\sigma(\Psi) = \langle G_{T,X}, \Psi \rangle_\Gamma.$$

Now our goal, is to establish the following lemma, note that we have already fixed T, X large enough, $\sigma > 0$ small enough.

Lemma 6. *For any function $\Psi \in L^2(\Gamma \backslash \mathbb{H})$ we have that $G_{T,X}^\sigma(\Psi) = 0$.*

To prove this lemma, fix $\epsilon > 0$ small enough, we will show that $|G_{T,X}^\sigma(\Psi)| < \epsilon$.

Periodizing and Smoothing Ψ : Now we periodize and smooth Ψ in order to do spectral theory on $\mathcal{I}_X \times (0, \infty)$ which we call the space \mathcal{F}_X . Note that $\mathcal{F}_X \cong \Xi_\infty \backslash \mathbb{H}$ for an elementary group Ξ_∞ , but this coordinate description is more useful when we generalize to higher dimensions. Let $L^2(\mathcal{F}_X)$ denote the L^2 space with measure given by $\frac{dx dy}{y^2}$. Let $\tilde{\Psi}(z)$ denote a function which agrees with Ψ on

$$[-X, X - \eta] \times (0, \infty) \subset \mathbb{H},$$

where $\eta > 0$ is to be chosen later. From $x = X - \eta$ to $x = X - \eta/2$, $\tilde{\Psi}(z)$ smoothly interpolates between the values of $\Psi(z)$ and $\Psi(z - 2X)$, and from $x = X - \eta/2$ up to $x = X$, $\tilde{\Psi}$ is exactly equal to $\Psi(z - 2X)$, ensuring that all derivatives of $\tilde{\Psi}$ at the boundary values $x = X$ and $x = -X$ agree.

Note that the $L^2(\mathcal{I}_X \times [1/2T, \infty))$ cost of moving from Ψ to $\tilde{\Psi}$ is small. Indeed,

$$\begin{aligned} \text{diff}(\Psi) &:= \int_{1/2T}^\infty \int_{X-\eta}^X |\tilde{\Psi}(z) - \Psi(z)|^2 dx \frac{dy}{y^2} \\ &\ll \int_{1/2T}^\infty \int_{X-\eta}^X |\Psi(z - 2X)|^2 + |\Psi(z)|^2 dx \frac{dy}{y^2}. \end{aligned}$$

Since $\Psi \in L^2(\Gamma \backslash \mathbb{H})$, as $\eta \rightarrow 0$, this integral goes to zero, and hence by choosing η small enough, we can make the difference less than ϵ^2 .

Now note that

$$\begin{aligned} &\int_0^\infty \int_{\mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta) \chi_{1,\sigma}(z) - L_{T,\sigma}(\Delta) \chi_{b,\sigma}(z)) \Psi(z) dx \frac{dy}{y^2} \\ &= \int_{1/2T}^\infty \int_{\mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta) \chi_{1,\sigma}(z) - L_T(\Delta) \chi_{b,\sigma}(z)) (\Psi(z) - \tilde{\Psi}(z)) dx \frac{dy}{y^2} \\ &\quad + \int_{1/2T}^\infty \int_{\mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta) \chi_{1,\sigma}(z) - L_T(\Delta) \chi_{b,\sigma}(z)) \tilde{\Psi}(z) dx \frac{dy}{y^2}, \end{aligned}$$

and by Cauchy Schwarz and the above argument

$$\begin{aligned}
& \int_{1/2T}^{\infty} \int_{\mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta)\chi_{1,\sigma}(z) - L_T(\Delta)\chi_{b,\sigma}(z)) (\Psi(z) - \tilde{\Psi}(z)) dx \frac{dy}{y^2} \\
& \ll (\text{diff}(\Psi))^{1/2} \left(\int_0^{\infty} \int_{\mathcal{I}_X} |(\chi_{T,\sigma}(z) - K_T(\Delta)\chi_{1,\sigma}(z) - L_T(\Delta)\chi_{b,\sigma}(z))|^2 dx \frac{dy}{y^2} \right)^{1/2} \\
(3.12) \quad & \ll_{\sigma,T,X} \epsilon.
\end{aligned}$$

Hence it remains to show that

$$(3.13) \quad \int_0^{\infty} \int_{\mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta)\chi_{1,\sigma}(z) - L_T(\Delta)\chi_{b,\sigma}(z)) \tilde{\Psi}(z) dx \frac{dy}{y^2} = 0.$$

Working on \mathcal{F}_X : To prove (3.13) let

$$g_{T,X}^{\sigma}(z) := \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta)\chi_{1,\sigma}(z) - L_T(\Delta)\chi_{b,\sigma}(z)).$$

Thus, we are left with

$$\langle g_{T,X}^{\sigma}, \tilde{\Psi} \rangle_{\mathcal{F}_X}.$$

Lemma 6 will then follow from the following lemma, which shows that, for an arbitrary $\psi \in L^2(\mathcal{F}_X)$ the inner product $\langle g_{T,X}^{\sigma}, \psi \rangle$ is not correlated with any almost eigenfunction:

Lemma 7. *Fix T , X , and σ as above, then for any $\psi \in L^2(\mathcal{F}_X)$ and any $\lambda \geq 0$, we have*

$$(3.14) \quad \langle g_{T,X}^{\sigma}, \psi \rangle_{\mathcal{F}_X} \ll_{\lambda,T,\sigma,X} \|(\Delta - \lambda)\psi\|_{\mathcal{F}_X}.$$

Proof of Lemma 7. Fix $\psi \in L^2(\mathcal{F}_X)$ and consider

$$\langle \chi_{T,\sigma} \tilde{\chi}_X(z), \psi \rangle_{\mathcal{F}_X} = \int_0^{\infty} \chi_{T,\sigma}(y) \int_{\mathcal{I}_X} \tilde{\chi}_X(x) \psi(z) dx \frac{dy}{y^2},$$

and define $f(y) := \int_{\mathcal{I}_X} \tilde{\chi}_X(x) \psi(z) dx$ and $h(y) := \int_{\mathcal{I}_X} (\Delta - \lambda) \tilde{\chi}_X(x) \psi(z) dx$. Then we note that, by periodicity in the x -direction,

$$\begin{aligned}
h(y) &= \int_{\mathcal{I}_X} (\Delta - \lambda) \tilde{\chi}_X(x) \psi(z) dx \\
&= -y^2 \partial_{yy} f(y) - \lambda f(y).
\end{aligned}$$

Thus, [Kon09, Lemma B.1] (which is a simple application of the method of variation of parameters) implies

$$(3.15) \quad f(y) = \alpha y^s + \beta y^{1-s} + y^s u(y) + y^{1-s} v(y),$$

where

$$u(y) := (1 - 2s)^{-1} \int_{(1-\sigma)/T}^y w^{-1-s} h(w) dw, \quad \text{and} \quad v(y) := (1 - 2s)^{-1} \int_{(1-\sigma)/T}^y w^{s-2} h(w) dw,$$

if $\lambda \neq 1/4$. And

$$(3.16) \quad f(y) = \alpha y^{1/2} + \beta y^{1/2} \log y + y^{1/2} u(y) + v(y) y^{1/2} \log y,$$

where

$$u(y) := \int_{(1-\sigma)/T}^y w^{-3/2} \log(w) h(w) dw, \quad \text{and} \quad v(y) := - \int_{(1-\sigma)/T}^y w^{-3/2} h(w) dw.$$

Therefore (assuming $\lambda \neq 1/4$ for simplicity) integrating the y variable gives

$$\langle \chi_{T,\sigma} \tilde{\chi}_X, \psi \rangle_{\Xi_\infty} = \int_0^\infty \chi_{T,\sigma}(y) (\alpha y^s + \beta y^{1-s}) \frac{dy}{y^2} + I + II$$

where $I := \int_0^\infty \chi_{T,\sigma}(y) y^{s-2} u(y) dy$ and $II := \int_0^\infty \chi_{T,\sigma}(y) y^{-1-s} v(y) dy$. Now we can use Cauchy-Schwarz (as in [Kon09, (B.5)]) to establish that $I, II \ll \|(\Delta - \lambda)\psi\|_{\mathcal{F}_X}$. In fact, this is the crucial reason why we needed to work on \mathcal{F}_X which is compact in the x direction, and thus $\chi_{T,\sigma}$ is (square) integrable. Thus, we may conclude

$$\langle \chi_{T,\sigma}, \psi \rangle_{\mathcal{F}_X} = A_\sigma T^s + B_\sigma T^{1-s} + O(\|(\Delta - \lambda)\psi\|_{\mathcal{F}_X})$$

where $A_\sigma = \beta \int_0^\infty \chi_{1,\sigma}(y) y^{-1-s} dy$ and $B_\sigma := \alpha \int_0^\infty \chi_{1,\sigma}(y) y^{s-2} dy$.

Given our choice of K_T and L_T from (3.8) we have that

$$(3.17) \quad K_T(s), L_T(s) \ll \begin{cases} T^s & \text{if } s \in (1/2, 1], \\ T^{1/2} \log T & \text{if } s = 1/2 + it. \end{cases}$$

Then, by the analysis in [Kon09, Proposition 3.5] we have

$$(3.18) \quad \langle g_{T,X}^\sigma, \psi \rangle_{\mathcal{F}_X} \ll_{T,\lambda,X,\sigma} \|(\Delta - \lambda)\psi\|_{\mathcal{F}_X}$$

for any choice of $\psi \in L^2(\mathcal{F}_X)$. □

From there we can choose ψ to be as in [Kon09, Proof of Theorem 3.2] to establish that $g_{T,X}^\sigma$ is almost everywhere 0. Note that for this argument to work one only needs the bound (3.18), a Hilbert space (here $L^2(\mathcal{F}_X)$), an unbounded self-adjoint operator (i.e Δ), and the abstract spectral theorem.

From there we conclude that

$$\langle g_{T,X}^\sigma, \tilde{\Psi} \rangle_{\mathcal{F}_X} = 0$$

From there, we conclude (3.13), namely

$$(3.19) \quad \int_0^\infty \int_{\mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta) \chi_{1,\sigma}(z) - L_T(\Delta) \chi_{b,\sigma}(z)) \tilde{\Psi}(z) dx \frac{dy}{y^2} = 0.$$

Then thanks to (3.12) we conclude

$$G_{T,X}^\sigma(\Psi) = \int_0^\infty \int_{\mathcal{I}_X} \tilde{\chi}_X(z) (\chi_{T,\sigma}(z) - K_T(\Delta) \chi_{1,\sigma}(z) - L_{T,\sigma}(\Delta) \chi_{b,\sigma}(z)) \Psi(z) dx \frac{dy}{y^2} \ll_{\sigma,T,X} \epsilon$$

for any value of $\epsilon > 0$. Taking ϵ to 0 establishes Lemma 6. Since we have that for any $\Psi \in L^2(\Gamma \backslash \mathbb{H})$

$$\lim_{\sigma \rightarrow 0} G_{T,X}^\sigma(\Psi) = \langle G_{T,X}, \Psi \rangle_\Gamma$$

we conclude that $\langle G_{T,X}, \Psi \rangle_\Gamma = 0$. Which is exactly Proposition 5. \square

3.2. Proof of Theorem 4. For the proof of Theorem 4, we return to the smoothed count

$$\tilde{N}_\Gamma(T) = \langle F_{T,X}, \Psi \rangle_\Gamma.$$

Now apply the abstract Parseval's identity (2.2)

$$\begin{aligned} \langle F_{T,X}, \Psi \rangle_\Gamma &= \langle \widehat{F_{T,X}}, \widehat{\Psi} \rangle_{\text{Spec}(\Gamma)} \\ (3.20) \quad &= \widehat{F_{T,X}}(\lambda_0) \widehat{\Psi}(\lambda_0) + \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{F_{T,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu(\lambda), \end{aligned}$$

where for ease of exposition, we assume that λ_0 is the only eigenvalue below $1/4$; in general, the other eigenvalues are dealt with similarly.

Addressing the first term in (3.20), we can use the abstract spectral theorem, and the fact that $F_{T,X}$ and Ψ are K -fixed, to say

$$\widehat{F_{T,X}}(\lambda_0) \widehat{\Psi}(\lambda_0) = \langle F_{T,X}, \phi_0 \rangle \langle \Psi, \phi_0 \rangle,$$

where ϕ_0 is the base eigenfunction. Note that, for the first factor we can apply the main identity Proposition 5, and the definition to conclude

$$\widehat{F_{T,X}}(\lambda_0) \widehat{\Psi}(\lambda_0) = T^\delta c \langle \phi_0, \Psi \rangle_\Gamma + O(T^{1/2})$$

for some constant c independent of T . As for the second factor, by the mean value theorem (see [Kon09, (4.17)] for details) we have

$$\langle \phi_0, \Psi \rangle_\Gamma = \phi_0(i) + O(\epsilon).$$

It remains to bound the contribution to (3.20) from the remainder of the spectrum (assuming here that there are no isolated eigenvalues apart from the base). Using Proposition 5 we have that

$$\begin{aligned} \text{Err} &:= \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{F_{T,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu \\ &= \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \left(K_T \widehat{(\Delta)} F_{1,X}(\lambda) + L_T \widehat{(\Delta)} F_{b,X}(\lambda) \right) \widehat{\Psi}(\lambda) d\nu. \end{aligned}$$

By the abstract spectral theorem and (3.17) we have $K_T \widehat{F_{1,X}}(\lambda) \ll T^{1/2} \log T \widehat{F_{1,X}}(\lambda)$. Thus, by Cauchy-Schwarz, positivity, and Parseval's identity

$$\begin{aligned} \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} K_T(\lambda) \widehat{F_{1,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu &\ll T^{1/2} \log T \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{F_{1,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu \\ &\ll T^{1/2} \log T \left(\int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \left| \widehat{F_{1,X}}(\lambda) \right|^2 d\nu \right)^{1/2} \left(\int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \left| \widehat{\Psi}(\lambda) \right|^2 d\nu \right)^{1/2} \\ &\ll T^{1/2} \log T \|F_{1,X}\|_{\Gamma} \|\Psi\|_{\Gamma}, \end{aligned}$$

Since Ψ is an ε -approximation to the identity, and since the term involving L_T can be treated similarly, we thus conclude

$$\text{Err} \ll \frac{1}{\varepsilon} T^{1/2} \log T.$$

This completes the proof of Theorem 4.

4. KLEINIAN SPHERE PACKINGS

Turning now to the sphere packing setting, let \mathcal{P} be a fixed bounded Kleinian sphere packing. Given a sphere $S \in \mathcal{P}$ let $b(S)$ denote the bend of S . Then our aim is to establish the asymptotic, (1.4) for

$$N_{\mathcal{P}}(T) := \#\{S \in \mathcal{P} : b(S) < T\}.$$

4.1. Preliminaries on Inverse Coordinate Systems and the Symmetry Group.

We now give a model of hyperbolic space, and develop an inversive coordinate system for the spheres on the ideal boundary of such; see, e.g., [KK21, §3.1] for background. To begin, we fix a real quadratic form Q of signature $(n+1, 1)$. For concreteness, we can change variables over \mathbb{R} to the “standard” example of $Q = -x_0 x_{n+1} + x_1^2 + \cdots + x_n^2$ which has half-Hessian

$$(4.1) \quad Q = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & I_n & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Then the quadratic space $(V = \mathbb{R}^{n+1,1}, Q)$ with product $v \star w = v \cdot Q \cdot w^t$ contains the cone $V_0 := \{v : Q(v) = 0\}$, the one-sheeted hyperboloid $V_1 = \{v : Q(v) = 1\}$, and the two-sheeted hyperboloid $V_{-1} := \{v : Q(v) = -1\}$; fix either component of the latter for our model of \mathbb{H}^{n+1} . The group $O_Q(\mathbb{R})$ acts on V_{-1} , and its subgroup $G = O_Q^{\circ}(\mathbb{R})$, that is, the connected component of the identity, fixes the components.

There is a 1-1 correspondence between vectors $v \in V_1$ and (oriented) spheres in $\partial\mathbb{H}^{n+1}$, obtained as follows. Given such a v , the orthogonal space $v^{\perp} := \{w \in V : v \star w = 0\}$ intersects the fixed component of V_{-1} at a hyperplane $\cong \mathbb{H}^n$, and the ideal boundary of the latter is the desired sphere.

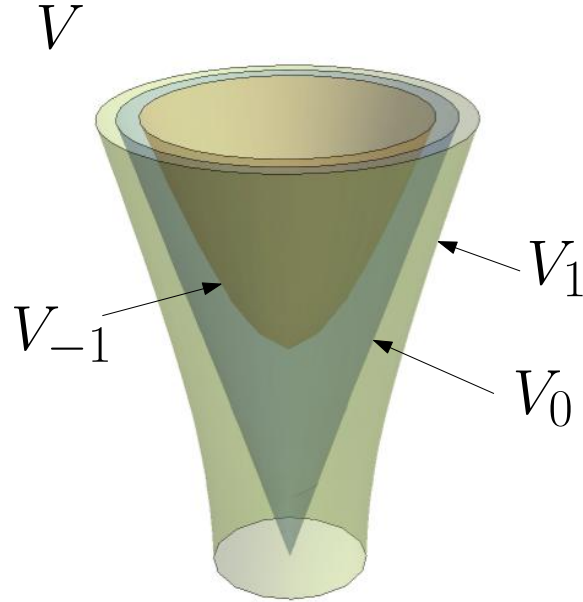


FIGURE 3. In the above figure, we display (half of) the quadratic space V , and its components V_0 , V_1 and V_{-1} .

This geometric correspondence is made algebraic after a choice of (inversive) coordinates on V_1 as follows. Let $V^* := \{v^* : V \rightarrow \mathbb{R}, \text{ linear}\}$ be the dual space to V , and Q^* the dual form, so that $v^* \star w^* = v \star w$. Fix a non-zero null covector $b^* \in V^*$, that is, $Q^*(b^*) = 0$. Also fix a null covector \hat{b}^* , with $b^* \star \hat{b}^* = -2$. For the case of the standard form in (4.1), one can make the choice $b^* = (0, \dots, 0, -2)$ and $\hat{b}^* = (-2, 0, \dots, 0)$. Then one picks an orthonormal system, bx_1^*, \dots, bx_n^* , for the orthogonal complement to the span of b^* and \hat{b}^* . Then the sphere corresponding to the vector $v \in V_1$ has bend (that is, inversive radius)

$$(4.2) \quad b^*(v),$$

and center $\frac{1}{b^*(v)}(bx_1^*(v), \dots, bx_n^*(v))$. When $b^*(v) = 0$, the sphere is a plane (which has no ‘‘center’’), so the expression $(bx_1^*(v), \dots, bx_n^*(v))$ is a unit normal to the plane. The ‘‘co-bend’’ of a sphere is defined as the bend of the image of the sphere on inversion through the unit sphere at the origin. The co-bend of the sphere corresponding to v is given by $\hat{b}^*(v)$. Therefore the tuple $(b^*(v), bx_1^*(v), \dots, bx_n^*(v), \hat{b}^*(v))$ gives a complete inversive coordinate system on V_1 . It is sometimes convenient to isolate the ‘bend-center’, comprising the coordinates not including the first and last; so we define:

$$(4.3) \quad bz^*(v) := (bx_1^*(v), \dots, bx_n^*(v)).$$

A Kleinian sphere packing decomposes into finitely many Γ -orbits (by the Structure Theorem [KK21, Theorem 22]), which in the above coordinate system corresponds simply

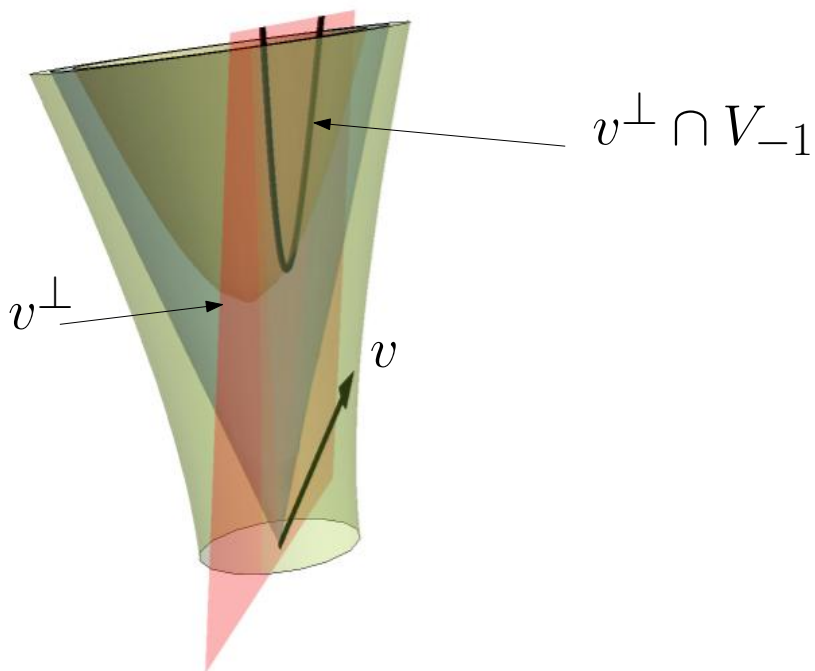


FIGURE 4. Here we display a vector $v \in V_1$, its orthogonal space v^\perp and the intersection this orthogonal space with V_{-1} which corresponds to an ideal sphere.

to orbits $v_0 \cdot \Gamma$, with $v_0 \in V_1$, and the bends of such are measured by $b^*(v)$, for $v \in v_0 \cdot \Gamma$. We count the whole packing by counting one orbit at a time.

4.2. Decomposition of G . Fix v_0 as above. Let $\chi_T(g) = \mathbb{1}\{b^*(v_0g) < T\}$ denote the indicator function of the vector v_0g having bend at most T , where $g \in G$. This function is left-invariant under $H := \text{Stab}_G(v_0) = \{g \in G : v_0g = v_0\}$, and also right-invariant under $L := \text{Stab}_{b^*} = \{g \in G : b^*(vg) = b^*(v), \text{ for all } v \in V\}$.

It will be useful to decompose $G = O_Q^\circ(\mathbb{R})$ (for Q given by (4.1)) and its Lie algebra $\mathfrak{g} := \text{Lie}(G)$ as follows. First we decompose \mathfrak{g} as:

$$\mathfrak{g} = \mathfrak{h} \oplus \bar{\mathfrak{u}} \oplus \mathfrak{u} \oplus \mathfrak{m}_1.$$

Here $\mathfrak{h} = \text{Lie}(H)$, and with

$$M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & O(n) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have that $M \cap H \cong O(n-1)$. Then set $\mathfrak{m} := \text{Lie}(M)$ and $\mathfrak{m}_1 := \mathfrak{m} \ominus \text{Lie}(M \cap H)$. (Note that $\mathfrak{h} \cap \mathfrak{m}$ is trivial when $G \cong O(3,1)$.) Set $M_1 := \exp \mathfrak{m}_1 < G$. The one-parameter

Lie algebras \mathfrak{u} and $\bar{\mathfrak{u}}$ are given by:

$$\mathfrak{u} = \left\{ \begin{pmatrix} 0 & 0_{n-1} & w & 0 \\ 0 & 0 & 0 & 0_{n-1} \\ 0 & 0 & 0 & 2w \\ 0 & 0 & 0 & 0 \end{pmatrix} : w \in \mathbb{R} \right\},$$

and

$$\bar{\mathfrak{u}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0_{n-1} & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0_{n-1} & y/2 & 0 \end{pmatrix} : y \in \mathbb{R} \right\},$$

which exponentiate to the groups

$$U := \left\{ u(w) := \begin{pmatrix} 1 & 0_{n-1} & w & w^2 \\ 0 & I_{n-1} & 0 & 0_{n-1} \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix} : w \in \mathbb{R} \right\},$$

and

$$\bar{U} := \left\{ \bar{u}(y) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0_{n-1} & I_{n-1} & 0 & 0 \\ y & 0 & 1 & 0 \\ y^2/4 & 0_{n-1} & y/2 & 1 \end{pmatrix} : y \in \mathbb{R} \right\},$$

respectively.

This gives the corresponding decomposition:

$$H \times \bar{U} \times U \times M_1 \rightarrow G : (h, \bar{u}, u, m_1) \mapsto h\bar{u}um_1.$$

Geometrically this decomposition is in fact very natural. H is the stabilizer of v_0 , corresponding to the sphere whose orbit we are considering. The action of \bar{U} changes the bend of the sphere, while UM_1 changes the co-bend by moving the center via a polar coordinate description of the plane (U as the radial coordinate and $M_1 \cong O(n)/O(n-1)$ giving a rotation).

Let $d := \dim(\mathfrak{g}) = (n+1)(n+2)/2$. Let $k := \dim(\mathfrak{h}) = n(n+1)/2$, and let $\ell := \dim(\mathfrak{m}_1) = n-1$. Hence $d = k + \ell + 2$.

4.3. An Explicit Basis.

Before proceeding, it will help our calculations to fix a basis for \mathfrak{h} and \mathfrak{m}_1 . To ease notation, we will describe a basis in a special case when $n = 4$, so $d = 15$, $k = 10$ and

$\ell = 3$; the general case will be clear by analogy. Take:

$$\begin{aligned}
& \left. X_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right\} \in \mathfrak{h} \cap \mathfrak{m} \\
& \left. X_4 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, X_5 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, X_6 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, \right\} \text{Lower Triangular in } \mathfrak{h} \\
& \left. X_7 := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_8 := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_9 := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right\} \text{Upper Triangular in } \mathfrak{h} \\
& X_{10} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \text{ - Diagonal in } \mathfrak{h} \\
& X_{11} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \in \bar{\mathfrak{u}}, \quad X_{12} := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{u}, \\
& X_{13} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_{14} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_{15} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \in \mathfrak{m}_1.
\end{aligned}$$

To move back to group elements we denote

$$\begin{aligned}
h_j(x_j) &:= \exp(x_j X_j) \in H && \text{for } j \leq k, \\
\bar{u}(y) &:= \exp(y X_{k+1}) \in \bar{U}, \\
u(w) &:= \exp(w X_{k+2}) \in U, \\
m_j(\varphi_j) &:= \exp(\varphi_j X_{j+k+2}) \in M_1 && \text{for } 1 \leq j \leq \ell.
\end{aligned}$$

It will also be convenient to record the following components of H :

$$\begin{aligned}
(4.4) \quad H_M &:= \prod_{X_j \in \mathfrak{h} \cap \mathfrak{m}} \exp(x_j X_j), \quad H_+ := \prod_{X_j \text{ "Upper Triangular"}} \exp(x_j X_j), \\
H_- &:= \prod_{X_j \text{ "Lower Triangular"}} \exp(x_j X_j), \quad \text{and } H_A := \prod_{X_j \text{ "Diagonal"}} \exp(x_j X_j).
\end{aligned}$$

4.4. Calculating the Haar Measure. Given that decomposition of the Lie algebra, we now derive an explicit form of the Haar measure in our chosen coordinate system. Let

$$\mathbf{z} := (x_1, \dots, x_k, y, w, \varphi_1, \dots, \varphi_\ell)$$

denote a set of coordinates and let

$$\mathcal{J} : \mathbb{R}^d \rightarrow G : \mathbf{z} \mapsto h_1(x_1) h_2(x_2) \dots h_k(x_k) \bar{u}(y) u(w) m_1(\varphi_1) \dots m_\ell(\varphi_\ell)$$

map our coordinate space to G . Then, in these new coordinates we have the following decomposition of the Haar measure

Theorem 8. [*Haar Measure Structure Theorem*] *Let $\rho(\mathbf{z})$ denote the density of the Haar measure in \mathbf{z} coordinates. Then*

$$(4.5) \quad \rho(\mathbf{z}) = \rho_H(x_1, \dots, x_k) \rho_{\overline{U}}(y, w) \rho_{M_1}(\varphi_1, \dots, \varphi_\ell),$$

where ρ_H and ρ_{M_1} are (respectively) the densities for the Haar measure on H and M_1 . Moreover we have that $\rho_{\overline{U}} = |1 + wy|^{n-1}$.

Proof. Given a linear differential operator on \mathbf{z} we can represent it as

$$T_{\mathbf{z}} = \eta_1 \partial_{x_1} + \dots + \eta_k \partial_{x_k} + \eta_{k+1} \partial_y + \eta_{k+2} \partial_w + \eta_{k+3} \partial_{\varphi_1} + \dots + \eta_d \partial_{\varphi_\ell}.$$

Now using the correspondence between elements of the Lie algebra and linear differential operators we write

$$T_{\mathbf{z}} = \eta_1 X_1 + \eta_2 X_2 + \dots + \eta_d X_d.$$

To first order, we can write the differential $T_{\mathbf{z}}$ acting on an element of g as

$$\begin{aligned} & h_1(I + \eta_1 X_1) \cdots h_k(I + \eta_k X_k) \bar{u}(I + \eta_{k+1} X_{k+1}) u(I + \eta_{k+2} X_{k+2}) m_1(I + \eta_{k+3}) \cdots m_\ell(I + \eta_d) \\ &= h_1 \cdots h_k \bar{n} n m_1 \cdots m_\ell \left\{ I + \text{ad}((h_2 \cdots h_k \bar{u} u m_1 \cdots m_\ell)^{-1}, \eta_1 X_1) \right. \\ & \quad \left. + \text{ad}((h_3 \cdots h_k \bar{u} u m_1 \cdots m_\ell)^{-1}, \eta_2 X_2) + \dots + \text{ad}(m_\ell^{-1}, \eta_{d-1} X_{d-1}) + \eta_d X_d \right\}. \end{aligned}$$

Giving us the differential operator

$$\begin{aligned} D(\mathcal{J}) : & \eta_1 \partial_{x_1} + \dots + \eta_k \partial_{x_k} + \eta_{k+1} \partial_y + \eta_{k+2} \partial_w + \eta_{k+3} \partial_{\varphi_1} + \dots + \eta_d \partial_{\varphi_\ell} \\ \mapsto & \eta_1 \text{ad}((h_2 \cdots h_k \bar{n} n m_1 \cdots m_\ell)^{-1}, X_1) + \eta_2 \text{ad}((h_3 \cdots h_k \bar{n} n m_1 \cdots m_\ell)^{-1}, X_2) + \dots \\ & + \eta_{k+\ell+1} \text{ad}(m_\ell^{-1}, X_{d-1}) + \eta_d X_d. \end{aligned}$$

Hence, if we want to apply the differential operator X_j on the right to an element g , then we simply solve for η_i on the right hand side of this map. Then the left hand side tells us the action on the coordinates. Let us denote the vector of such η_i 's by $(\eta_{j1}, \eta_{j2}, \dots, \eta_{jd})$. Thus

$$\begin{aligned} X_j = & \eta_{j1} \text{ad}((h_2 \cdots h_k \bar{n} n m_1 \cdots m_\ell)^{-1}, X_1) + \eta_{j2} \text{ad}((h_3 \cdots h_k \bar{n} n m_1 \cdots m_\ell)^{-1}, X_2) + \dots \\ & + \eta_{j(d-1)} \text{ad}(m_\ell^{-1}, X_{d-1}) + \eta_{jd} X_d, \end{aligned}$$

for every $j = 1, \dots, d$.

Now to calculate the Haar measure, we proceed by the following methodology: define the right multiplication operator $R_A(\mathbf{z}) := \mathcal{J}^{-1}(\mathcal{J}(\mathbf{z}) \cdot A)$ and its Jacobian:

$$[R'_A(\mathbf{z})]_{jk} := \frac{\partial^j R_A(\mathbf{z})}{d\mathbf{z}_k}.$$

Then our goal is to find $\rho(\mathbf{z})$ such that:

$$(4.6) \quad \int f(\mathcal{J}(\mathbf{z}))\rho(\mathbf{z})d\mathbf{z} = \int f(\mathcal{J}(R_A(\mathbf{z})))\rho(\mathbf{z})d\mathbf{z}.$$

Changing variables $\mathbf{y} = R_{A^{-1}}(\mathbf{z})$, on the left hand side gives:

$$\int f(\mathcal{J}(\mathbf{z}))\rho(\mathbf{z})d\mathbf{z} = \int f(\mathcal{J}(R_A(\mathbf{y})))\rho(R_A(\mathbf{y}))|\det R'_A(\mathbf{y})|d\mathbf{y}$$

which is equal to the right hand side of (4.6). Hence we want to find ρ such that:

$$\rho(R_A(\mathbf{y})) = \frac{\rho(\mathbf{y})}{|\det R'_A(\mathbf{y})|}$$

for all choices of \mathbf{y} and $A \in G$. In particular, we may choose $\mathbf{y} = 0$, and $A = \mathcal{J}(\mathbf{z})$, which gives

$$(4.7) \quad \rho(\mathbf{z}) = \frac{1}{|\det R'_{\mathcal{J}(\mathbf{z})}(0)|}.$$

Note that since the Haar measure is only unique up to a constant, we set $\rho(0) = 1$ without loss of generality. Using our decomposition of the Lie algebra we can write:

$$[R'_{\mathcal{J}(\mathbf{z})}(0)]_{ij} = \left[\frac{\partial}{\partial t_i} \mathcal{J}(\mathbf{z}) e^{\sum_{i=1}^d t_i X_i} \Big|_{\mathbf{t}=0} \right]_j$$

Now we can linearize the exponential and then find the corresponding coordinate description of the differential operator as we did above. The above argument implies

$$\rho(\mathbf{z})^{-1} = \det |[R'_{\mathcal{J}(\mathbf{z})}(0)]_{ij}| = \det |\eta_{ij}|.$$

Now to simplify matters, we express each adjoint as a linear combination of elements in the basis:

$$\begin{aligned} \text{ad}((h_2 \cdots h_k \bar{u} u m_1 \cdots m_\ell)^{-1}, X_1) &= \sum_{i=1}^d \mu_{1i} X_i \\ \text{ad}((h_3 \cdots h_k \bar{u} u m_1 \cdots m_\ell)^{-1}, X_2) &= \sum_{i=1}^d \mu_{2i} X_i \\ &\dots \\ X_d &= \sum_{i=1}^d \mu_{di} X_i. \end{aligned}$$

Thus, to calculate the Haar measure, by linearity of the adjoint operator, we need to find a $d \times d$ matrix η such that

$$\eta \mu \mathbf{X} = \mathbf{X}$$

where $\mathbf{X} := (X_1, X_2, \dots, X_d)^T$. Hence, since the determinant is multiplicative, we have that $\rho(\mathbf{z}) = \det[\mu]$.

Now suppose we wanted to calculate the Haar measure of H , then we can write

$$\begin{aligned} \text{ad}((h_2 \cdots h_k)^{-1}, X_1) &= \sum_{i=1}^d \nu_{1i} X_i \\ \text{ad}((h_3 \cdots h_k)^{-1}, X_2) &= \sum_{i=1}^d \nu_{2i} X_i \\ &\dots \\ \text{ad}(h_k^{-1}, X_{k-1}) &= \sum_{i=1}^d \nu_{(k-1)i} X_i \\ X_k &= \sum_{i=1}^d \nu_{ki} X_i. \end{aligned}$$

For completeness write $\nu_{ji} = \delta_{i,j}$ for $j > k$. Then the above argument implies that one can write the Haar measure on H as $\rho_H(\mathbf{z}) = \det[\nu]$.

Now the key observation is to write

$$\begin{aligned} \text{ad}((\bar{u}um_1 \cdots m_\ell)^{-1}, X_1) &= \sum_{i=1}^d c_{1i} X_i \\ &\dots \\ \text{ad}((\bar{u}um_1 \cdots m_\ell)^{-1}, X_k) &= \sum_{i=1}^d c_{ki} X_i \\ \text{ad}((um_1 \cdots m_\ell)^{-1}, X_{k+1}) &= \sum_{i=1}^d c_{(k+1)i} X_i \\ &\dots \\ X_d &= \sum_{i=1}^d c_{di} X_i. \end{aligned}$$

By linearity of the adjoint operator we have that $\mu = \nu c$, and hence $\det(\mu) = \det(\nu) \det(c)$. Or rather, $\rho(\mathbf{z}) = \rho_H(x_1, \dots, x_k) f(w, y, \varphi_1, \dots, \varphi_\ell)$ for some function f .

Moreover, since the Haar measure is unimodular, and since M_1 is a group, we can apply the same argument to M_1 on the right, and show that ρ satisfies the product structure from (4.5).

Finally, thanks to this product structure, we can take m_1, \dots, m_l equal to the identity when calculating $\rho_{\bar{U}U}$. Thus, let

$$(4.8) \quad \begin{aligned} \text{ad}((\bar{u}u)^{-1}, X_j) &= \sum_{i=1}^d d_{ji} X_i, & \text{for } j = 1, \dots, k \\ \text{ad}((u)^{-1}, X_{k+1}) &= \sum_{i=1}^d d_{(k+1)i} X_i \\ X_j &= \sum_{i=1}^d d_{ji} X_i, & \text{for } j = k+2, \dots, d. \end{aligned}$$

Hence it remains to find $\det(d)$.

No matter the dimension, our basis elements are each one of seven types (in $\mathfrak{m} \cap \mathfrak{h}$, lower triangular in \mathfrak{h} , upper triangular in \mathfrak{h} , diagonal, in $\bar{\mathfrak{u}}$, in \mathfrak{u} , or in \mathfrak{m}_1). Depending on the type of X_i , we can calculate d_{ij} explicitly independent of dimension via an inductive argument. From which it follows that the d matrix has the following form:

$$(4.9) \quad d = \begin{pmatrix} I_{(n-1)(n-2)/2} & 0 & 0 & 0 & 0 \\ 0 & I_{n-1} & \frac{w^2}{2} I_{n-1} & 0 & w I_{n-1} \\ 0 & \frac{y^2}{2} I_{n-1} & \frac{1}{4}(2+wy)^2 I_{n-1} & 0 & \frac{1}{2}w(2+wy) I_{n-1} \\ 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & I_{n-1} \end{pmatrix}$$

where B is the 3×3 matrix given by

$$B := \begin{pmatrix} 1+wy & -w & \frac{1}{2}(2+wy) \\ -y & 1 & -\frac{y^2}{2} \\ 0 & 0 & 1. \end{pmatrix}$$

Hence, one can determine explicitly thanks to a block matrix decomposition that $\det(d) = |1+wy|^{n-1}$. This completes the proof. \square

4.5. Calculating the Casimir Operator. As in §2, given our basis, X_1, X_2, \dots, X_d , for the Lie algebra in §4.3, there is a corresponding dual basis $X_1^*, X_2^*, \dots, X_d^*$ with respect to the Killing form. Then the Casimir operator can be written as the following second order differential operator:

$$(4.10) \quad \mathcal{C} := \sum_{i=1}^{k+2+l} X_i X_i^*.$$

Using the above argument, in \mathbf{z} coordinates, we can express $X_i = \sum_{j=1}^d \eta_{ij} \partial_j$. Likewise, we can express $X_i^* = \sum_{j=1}^d \eta_{ij}^* \partial_j$.

Now define μ^* analogously to how we defined μ , that is

$$\begin{aligned} \text{ad}((h_2 \cdots h_k \bar{u} u m_1 \cdots m_\ell)^{-1}, X_1) &= \sum_{i=1}^d \mu_{1i}^* X_i^* \\ \text{ad}((h_3 \cdots h_k \bar{u} u m_1 \cdots m_\ell)^{-1}, X_2) &= \sum_{i=1}^d \mu_{2i}^* X_i^* \\ &\dots \\ X_d &= \sum_{i=1}^d \mu_{di} X_i^*. \end{aligned}$$

We then have that $\eta^* \cdot \mu^* \mathbf{X}^* = \mathbf{X}^*$, and hence $\eta^* = (\mu^*)^{-1}$. Thus, if we define $\underline{\partial} := (\partial_{x_1}, \dots, \partial_{x_d})^T$, then as a differential operator

$$\mathcal{C} = (\mu^{-1} \underline{\partial}) \cdot ((\mu^*)^{-1} \underline{\partial}).$$

The Casimir operator in all d variables is, of course, rather unwieldy. However, fortunately we shall only need the Casimir operator to act on left H -invariant, and right M_1 -invariant functions. Hence, if we write $\mu^* = \nu^* c^*$, with ν^* and c^* defined analogously to as above, then when acting on such functions, we have

$$\mathcal{C} = (c^{-1} \underline{\partial}) \cdot ((c^*)^{-1} \underline{\partial}).$$

Using this decomposition, we can explicitly calculate the Casimir operator in \mathbf{z} coordinates, when acting on left H -invariant and right M_1 -invariant function.

Theorem 9. [*Structure Theorem for the Casimir Operator*] Let $f : H \backslash G / M_1 \rightarrow \mathbb{C}$. Then in the \mathbf{z} -coordinate system, f is only a function of (y, w) , that is, the \bar{U} and U variables, and the Casimir operator acting on f has the following form:

$$(4.11) \quad \mathcal{C}f(y, w) = \frac{1}{2} \left(y^2 \partial_y^2 + (n+1)y \partial_y + 2\partial_{yw} + \frac{(n-1)w \partial_w}{1+yw} \right) f(y, w).$$

Proof. By definition of c^* we have

$$Y_j := \text{ad}((\bar{u} u m_1 \cdots m_\ell)^{-1}, X_j) = \sum_{i=1}^d c_{ki}^* X_i^* \text{ for } i = 1, \dots, k$$

and

$$\begin{aligned} Y_{k+1} &:= \text{ad}((u m_1 \cdots m_\ell)^{-1}, X_{k+1}) = \sum_{i=1}^d c_{(k+1)i}^* X_i^* \\ &\dots \\ Y_d &:= X_d = \sum_{i=1}^d c_{di} X_i^*. \end{aligned}$$

Thus, $(c^*)^{-1}\mathbf{Y} = \mathbf{X}$. Since our function depends only on w and y , we are only interested in the $(k+1)^{th}$ and $(k+2)^{th}$ columns of $(c^*)^{-1}$. We can explicitly calculate Y_i depending on which type of basis vector is X_i , then we can explicitly solve for $(c^*)_{ij}^{-1}$. Using an inductive argument we then have that $(c^*)^{-1}\underline{\partial}$ has the following explicit form: Let

(4.12)

$$v_M := (\sin(\varphi_1), \cos(\varphi_1) \sin(\varphi_2), \cos(\varphi_1) \cos(\varphi_2) \sin(\varphi_3), \dots, \cos(\varphi_1) \cdots \cos(\varphi_{\ell-1}) \sin(\varphi_\ell))^T$$

then we have (in fact, we only need the fourth and fifth row of the below)

$$(4.13) \quad (c^*)^{-1}\underline{\partial} = \frac{1}{2} \begin{pmatrix} 0_{(n-1)n/2} \\ -v_M \partial_w \\ \frac{1}{2} v_M (w^2 \partial_w - 2(1+wy) \partial_y) \\ (y \partial_y - w \partial_w) \\ \cos(\varphi_1) \cdots \cos(\varphi_\ell) \partial_w \\ \frac{1}{2} \cos(\varphi_1) \cdots \cos(\varphi_\ell) (-w^2 \partial_w + 2(1+wy) \partial_y) \\ 0_{n-2} \end{pmatrix}.$$

Let $(A_1, \dots, A_d) := (c^*)^{-1} \cdot \underline{\partial}$, then our aim is to evaluate

$$\mathcal{C} = \sum_{j=1}^d \sum_{i=1}^d (c^{-1})_{ji} \partial_i A_j.$$

However, since we are considering the Casimir acting on right M_1 invariant functions, we may set the last ℓ coordinates equal to 0, that is

$$\mathcal{C}f(y, w) = \left(\sum_{j=1}^d \sum_{i=1}^d [(c^{-1})_{ji}]_{\varphi=0} [\partial_i A_j]_{\varphi=0} \right) f(y, w),$$

however $[(c^{-1})_{ji}]_{\varphi=0} = (d^{-1})_{ji}$ where d is the matrix of coefficients defined in (4.8).

Since the d matrix is explicitly described in (4.9), and the vector (A_1, \dots, A_d) is given explicitly in (4.13), then (4.11) can be found through direct computation. \square

5. COUNTING

We turn now to the proof of Theorem 1. For simplicity, we assume the packing is bounded; similar methods apply for counting in regions. We also assume that the packing is the orbit of a single sphere S_1 under the action of a symmetry group Γ ; in general, the counting problem reduces to a finite sum of such [KK21, Theorem 22]. Let the sphere S_1 be represented in the inversive coordinate system described in §4 by the vector $v_1 \in V$. We

begin in the same way as we did in the $\mathrm{SL}_2(\mathbb{R})$ case, writing the count as

$$\begin{aligned} N_{\mathcal{P}}(T) &:= \#\{S \in \mathcal{P} \mid b(S) < T\} \\ &= \#\{v \in v_1 \cdot \Gamma \mid b^*(v) < T\} \\ &= \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \chi_T(\gamma), \end{aligned}$$

where b^* is the ‘‘bend’’ covector (as in (4.2)), $\Gamma_1 := \mathrm{Stab}_{v_1}(\Gamma)$, and for $g \in G$

$$(5.1) \quad \chi_T(g) := \begin{cases} 1 & \text{if } b^*(v_1 \cdot g) < T, \\ 0 & \text{else.} \end{cases}$$

We observe that χ_T is a function on $H \backslash G / U M_1$, where H is the stabilizer of v_1 (so that $\Gamma_1 = \Gamma \cap H$).

Now we automorphize χ_T , that is define

$$F_T(g) := \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \chi_T(\gamma g).$$

Hence F_T is left Γ -invariant and $N_{\mathcal{P}}(T) = F_T(e)$. Again, rather than trying to evaluate the discontinuous function F_T at the origin, consider its inner product with a smooth approximation of the identity. However, we want to restrict as few directions of our smooth approximation as possible, to optimize the resulting error terms. To this end, fix an $\varepsilon > 0$, and let

$$(5.2) \quad \psi = \psi_\varepsilon \in L^2(\Gamma_1 \backslash G / M_1)$$

be given as follows. Let \mathcal{F} be a fundamental domain described in \mathbf{z} -coordinates, of $\Gamma_1 \backslash G / M_1$. Then on \mathcal{F} we let ψ be of the form $\psi(\mathbf{z}) = \psi_1(x_1) \cdots \psi_k(x_k) \psi_U(y) \psi_{\bar{U}}(w)$, with all components nonnegative and unit total mass,

$$\int_{\Gamma_1 \backslash G / M_1} \psi dg = 1,$$

and satisfying the following conditions.

For the $n + 1$ variables in the components H_+ , H_A (in the notation of (4.4)) and \bar{U} , we restrict the coefficients to ε -balls around 0. The other variables are restricted only to compact regions of bounded size around 0. We can choose such a ψ have L^2 mass bounded by:

$$\|\psi\|_{L^2(\Gamma_1 \backslash G / M_1)} \ll \varepsilon^{-(n+1)/2}.$$

Now define the function $w_T = w_{T,\varepsilon} : \Gamma_1 \backslash G \rightarrow [0, 1]$ by

$$(5.3) \quad w_T(g) := \int_{\Gamma_1 \backslash G} \chi_T(h) \psi_\varepsilon(gh) dh.$$

Our main counting theorem (Theorem 1) will follow from the following smooth effective count.

Theorem 10. *Let the Laplace eigenvalues of $\Gamma \backslash \mathbb{H}^{n+1}$ be as in (1.1). Then there exist constants $c_{\Gamma, \varepsilon}^i$ for $i = 0, 1, \dots, k$, with $c_{\Gamma, \varepsilon}^0 > 0$, such that*

$$(5.4) \quad \tilde{N}_{\Gamma, \varepsilon}(T) := \sum_{\gamma \in \Gamma_1 \backslash \Gamma} w_T(\gamma) = c_{\Gamma, \varepsilon}^0 T^\delta + c_{\Gamma, \varepsilon}^1 T^{s_1} + \dots + c_{\Gamma, \varepsilon}^k T^{s_k} + O\left(\frac{1}{\varepsilon^{(n+1)/2}} T^{n/2} \log(T)\right)$$

where the implied constant depends only on Γ . Moreover $c_{\Gamma, \varepsilon}^0 = c_\Gamma^0(1 + O(\varepsilon))$, and for $i \geq 1$ (if any exist), we have that $c_{\Gamma, \varepsilon}^i \ll \varepsilon^{-(n+1)/2}$.

Remark. Note that this smoothed counting theorem is optimal in that the error goes all the way to the tempered spectrum.

5.1. Proof of Theorem 1. An explicit calculation shows that the inversive coordinates of a sphere which has been transformed by g are

$$(y, -(1 + wy)v_M^T, (1 + wy) \cos(\varphi_1) \cdots \cos(\varphi_\ell), w(2 + wy))$$

where v_M is the vector defined in (4.12). Thus the ‘bend radius’ (i.e the distance of the center of the sphere to the origin multiplied by the bend) is $(1 + wy)$. Since this distance is necessarily bounded from above (since the packing is bounded) and since y is bounded from below (because it controls the bend, and the packing has a largest sphere), we know that w is bounded from above. In particular we have

Lemma 11. *Let $\gamma \in \Gamma_1 \backslash \Gamma$ then $|w_\gamma|$ is bounded.*

Now, assuming Theorem 10 we present the proof of Theorem 1.

Proof of Theorem 1. Consider

$$\begin{aligned} w_T(\gamma) &= \int_{\Gamma_1 \backslash G} \chi_T(g) \psi_\varepsilon(\gamma^{-1}g) dg \\ &= \int_{\Gamma_1 \backslash G} \chi_T(\gamma g) \psi_\varepsilon(g) dg. \end{aligned}$$

We write $\gamma = \mathcal{J}(x_{1, \gamma}, \dots, x_{k, \gamma}, y_\gamma, w_\gamma, \varphi_{1, \gamma}, \dots, \varphi_{\ell, \gamma})$ and note that χ_T is left H -invariant and right UM_1 invariant. Hence

$$\chi_T(\gamma g) = \chi_T(\bar{u}(y_\gamma)u(w_\gamma)m_1(\boldsymbol{\varphi}_\gamma)h(\mathbf{x})\bar{n}(y)).$$

now we can write h as a product of one dimensional components of H from (4.4) (here, it is convenient to change the order of the decomposition). That is, write

$$h(\mathbf{x}) = h_+(\mathbf{x}_+)h_A(\mathbf{x}_A)h_-(\mathbf{x}_-)h_M(\mathbf{x}_M).$$

Since ψ restricts the \mathbf{x}_- and \mathbf{x}_A coordinates to balls of radius ε , we can apply adjoints to move those factors to the left of $\bar{u}(y_\gamma)$ and the other factors will be perturbed by an ε

error. Then we can use left H -invariance of χ_T . We can do the same for the $\bar{u}(y)$ factor which gets absorbed in the $\bar{u}(y)$ factor. Thus, we have that

$$\begin{aligned}\chi_T(\gamma g) &= \chi_T(\bar{u}(y_\gamma + O(\varepsilon))u(w_\gamma + O(\varepsilon))m_1(\varphi_\gamma + O(\varepsilon))h_+(\mathbf{x}_+ + O(\varepsilon))h_M(\mathbf{x}_M + O(\varepsilon))) \\ &= \chi_T(\bar{u}(y_\gamma + O(\varepsilon))),\end{aligned}$$

where the second line follows from right- M invariance, and right- H_+ invariance.

Thus,

$$\chi_T(\gamma g) = \begin{cases} 1 & \text{if } y_\gamma < \frac{T}{1+c\varepsilon} \\ 0 & \text{if } y_\gamma > \frac{T}{1-c\varepsilon} \end{cases}$$

for some absolute constant c .

Hence, since $\tilde{N}_{\Gamma,\varepsilon}(T) = \sum_{\Gamma_1 \backslash \Gamma} w_T(\gamma)$, then our count satisfies:

$$(5.5) \quad \tilde{N}_{\Gamma,\varepsilon}(T(1 - c\varepsilon)) \leq N^{\mathcal{P}}(T) \leq \tilde{N}_{\Gamma,\varepsilon}(T(1 + c\varepsilon))$$

Now, assuming there are no other eigenvalues, we apply Theorem 10 to find:

$$\tilde{N}_{\Gamma,\varepsilon}(T(1 \pm \varepsilon)) = c^0(1 + O(\varepsilon))T^\delta + O\left(\frac{1}{\varepsilon^{(n+1)/2}}T^{n/2} \log(T)\right).$$

Then choosing $\varepsilon = T^{\frac{2}{n+3}(n/2-\delta)} \log(T)^{2/(n+3)}$ optimizes this inequality. Thus

$$\tilde{N}_{\Gamma,\varepsilon}(T(1 \pm \varepsilon)) = c^0 T^\delta + O(T^{\frac{2}{n+3}(n/2-\delta)+\delta} \log(T)^{\frac{2}{n+3}}).$$

The general case follows similarly. □

5.2. Inserting the Casimir Operator. Once again the smooth count is the inner product of F_T with an L^2 function. Consider the inner product of F_T with a general $\Psi \in L^2(\Gamma \backslash G/M_1)$. That is, given a function ψ on $\Gamma_1 \backslash G/M_1$ we automorphize it

$$\Psi(z) := \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \psi(\gamma g).$$

Let

$$K_T(s) := \frac{T^s b^{n-s} - T^{n-s} b^s}{b^{n-s} - b^s}, \quad L_T(s) := \frac{T^{n-s} - T^s}{b^{n-s} - b^s},$$

while for $s = n/2 + it$ we have

$$(5.6) \quad K_T(s) := T^{n/2} \frac{\sin(t \log T / \log b)}{\sin(t \log b)}, \quad L_T(s) := \left(\frac{T}{b}\right)^{n/2} \frac{\sin(t \log T)}{\sin(t \log b)}.$$

Again, by choosing an appropriate choice of b one can ensure that

$$(5.7) \quad K_T(s), L_T(s) \ll \begin{cases} T^s & \text{if } s \in (n/2, n], \\ T^{n/2} \log T & \text{if } s = n/2 + it. \end{cases}$$

In the $\mathrm{SL}_2(\mathbb{R})$ case, we decomposed the real direction, since $\Gamma \backslash \mathbb{H}$ was not compact in the real direction. Analogously in the current setting, we have the group decomposition

$(\Gamma_1 \backslash H) \overline{U} U M_1$. Since $(\Gamma_1 \backslash H)$ has finite H -Haar measure, and M_1 is compact, and we have imposed a cut-off in the \overline{U} -direction, we are again faced with a one dimensional non-compact direction, the U -direction. To that end, since ∞ lies outside the limit set, from Lemma 11 it follows that there exists an X large enough, such that

$$N_{\mathcal{P}} = F_{T,X}(e)$$

where

$$F_{T,X}(g) := \sum_{\Gamma_1 \backslash \Gamma} \chi_T(\gamma g) \tilde{\chi}_X(\gamma g), \quad \text{and}$$

$$\tilde{\chi}_X(g) := \begin{cases} 1 & \text{if } |bz^*(v_1 \cdot g)| < X, \\ 0 & \text{otherwise.} \end{cases}$$

where bz^* denotes the bend center, as in (4.3). Thus, note that $\tilde{\chi}_X$ is also left Γ_1 invariant, and right M_1 invariant (since it depends only on the distance to the origin of the center, not the polar coordinate angles).

Moreover, a calculation shows that, in the \mathbf{z} -coordinates, $\tilde{\chi}_X$ can be written as:

$$\tilde{\chi}_X(g) := \begin{cases} 1 & \text{if } \frac{1}{y} - X < w < \frac{1}{y} + X, \\ 0 & \text{otherwise.} \end{cases}$$

The Difference Operator: Again, we will prove an identity in terms of K_T and L_T for $F_{T,X}$. To that end, consider the difference operator

$$G_{T,X} := F_{T,X} - K_T(\mathcal{C})F_{1,X} - L_T(\mathcal{C})F_{b,X}.$$

By self-adjointness of \mathcal{C} , for any $\Psi \in L^2(\Gamma \backslash G/M_1)$ we have

$$\begin{aligned} G_{T,X}(\Psi) &:= \langle G_{T,X}, \Psi \rangle_{\Gamma \backslash G} \\ &= \int_{\Gamma \backslash G} [F_{T,X}(g)\Psi(g) - F_{1,X}(g)(K_T(\mathcal{C})\Psi)(g) - F_{b,X}(g)(L_T(\mathcal{C})\Psi)(g)] dg, \end{aligned}$$

which we can unfold to

$$(5.8) \quad G_{T,X}(\Psi) = \int_{\Gamma_1 \backslash G} \tilde{\chi}_X(g) (\chi_T(g)\Psi(g) - \chi_1(g)(K_T(\mathcal{C})\Psi)(g) - \chi_b(g)(L_T(\mathcal{C})\Psi)(g)) dg.$$

It is more convenient to work using \mathbf{z} -coordinates describing the group G . Now fix a fundamental domain for $\Gamma_1 \backslash G/M_1$, this fundamental domain can be written in coordinates as $\mathcal{F} := P_{\Gamma_1} \times [0, \infty) \times \mathbb{R} \times [-\pi, \pi]^{n-1}$, where P_{Γ_1} is a (finite H -volume) fundamental domain for the action of Γ_1 on H . Given a function $f : G \rightarrow \mathbb{C}$ we abuse notation and write $f(\mathbf{z}) = f(g_{\mathbf{z}})$, thus if we let ρ denote density of the Haar measure, we can write (5.8) as:

$$G_{T,X}(\Psi) = \int_{\mathcal{F}} \tilde{\chi}_X(y, w) (\chi_T(y)\Psi(\mathbf{z}) - \chi_1(y)(K_T(\mathcal{C})\Psi)(\mathbf{z}) - \chi_b(y)(L_T(\mathcal{C})\Psi)(\mathbf{z})) \rho(\mathbf{z}) d\mathbf{z}.$$

Note that by definition $\chi_T(\mathbf{z})$ is a function of y and $\tilde{\chi}_X(\mathbf{z})$ is a function of w (and y), for clarity we ignore the dependence on the other variables.

Let $\mathcal{F}_X := P_{\Gamma_1} \times [0, \infty) \times \mathcal{I}_X \times [-\pi, \pi)^{n-1}$ where $\mathcal{I}_X := [1/y - X, 1/y + X)$, now write

$$G_{T,X}(\Psi) = \int_{\mathcal{F}_X} (\chi_T(y)\Psi(\mathbf{z}) - \chi_1(y)(K_T(\mathcal{C})\Psi)(\mathbf{z}) - \chi_b(y)(L_T(\mathcal{C})\Psi)(\mathbf{z})) \rho(\mathbf{z}) d\mathbf{z}.$$

By showing that $G_{T,X}(\Psi) = 0$ for any choice of Ψ we will prove the following proposition

Proposition 12. *For Γ and $F_{T,X}$ as above we have that*

$$(5.9) \quad F_{T,X} = K_T(\mathcal{C})F_{1,X} + L_T(\mathcal{C})F_{b,X}$$

where K_T and L_T are the differential operators defined in (5.6).

Once again our goal is to work on the fundamental domain of a group (rather than working with discontinuous cut-offs). Thus we will perform the same smoothing as we did in the $\mathrm{SL}_2(\mathbb{R})$ case.

Smoothing χ_T : Let $\sigma > 0$ and let

$$\chi_{1,\sigma}(\mathbf{z}) := \begin{cases} 1 & \text{if } y < 1, \\ 0 & \text{if } y > (1 + \sigma), \end{cases}$$

and let $\chi_{1,\sigma}$ interpolate smoothly for all values in between. Now let $\chi_{T,\sigma}(\mathbf{z}) := \chi_{1,\sigma}(Ty)$.

Let

$$G_{T,X}^\sigma(\Psi) := \int_{\mathcal{F}_X} (\chi_{T,\sigma}(y)\Psi(\mathbf{z}) - \chi_{1,\sigma}(y)(K_T(\mathcal{C})\Psi)(\mathbf{z}) - \chi_{b,\sigma}(y)(L_T(\mathcal{C})\Psi)(\mathbf{z})) \rho(\mathbf{z}) d\mathbf{z}.$$

Now by construction

$$\int_{\mathcal{F}_X} |\chi_{t,\sigma}(y) - \chi_t(y)|^2 \rho(\mathbf{z}) d\mathbf{z} \ll_{t,X} \sigma$$

Thus by Cauchy-Schwarz we have that $\lim_{\sigma \rightarrow 0} G_{T,X}^\sigma(\Psi) = G_{T,X}(\Psi)$. Now our goal is to show that for any fixed $\epsilon > 0$ we have $G_{T,X}^\sigma(\Psi) < \epsilon$.

Periodizing and Smoothing Ψ : Let

$$\mathcal{I}_{X,\eta} := \left[\frac{1}{y} - X + \eta, \frac{1}{y} + X - \eta \right].$$

Let $\tilde{\Psi} : G/M_1 \rightarrow \mathbb{R}$ denote a function which agrees with Ψ on

$$P_{\Gamma_1} \times [0, \infty) \times \mathcal{I}_{X,\eta}$$

for some $\eta > 0$ to be chosen later. When $x_2 \notin \mathcal{I}_{X,\eta}$ we impose the condition $\tilde{\Psi}(1/2y - X) = \tilde{\Psi}(1/2y) = \tilde{\Psi}(1/2y + X)$ for any value of the other variables, and interpolate smoothly in between.

Using the same Cauchy-Schwarz argument as we employed in the $\mathrm{SL}_2(\mathbb{R})$ case, we can choose η such that the $L^2(P_{\Gamma_1} \times [0, 2T] \times \mathcal{I}_X \times [-\pi, \pi)^{n-1})$ cost of moving from Ψ to $\tilde{\Psi}$ is

less than ε . Thus, Proposition 12 follows if we can prove that

$$(5.10) \quad G_{T,X,\sigma}(\tilde{\Psi}) = 0.$$

Working on \mathcal{F}_X : Let

$$g_{T,X}^\sigma(\mathbf{z}) := \tilde{\chi}_X(\mathbf{z})(\chi_{T,\sigma}(\mathbf{z}) - K_T(\mathcal{C})\chi_{1,\sigma}(\mathbf{z}) - L_T(\mathcal{C})\chi_{b,\sigma}(\mathbf{z})).$$

Then (5.10) follows from the following lemma.

Lemma 13. *For any $\psi \in L^2(\mathcal{F}_X)$ be independent of $\varphi \in [-\pi, \pi]^{n-1}$ and any $\lambda \geq 0$ we have*

$$(5.11) \quad \langle g_{T,X}^\sigma, \psi \rangle_{\mathcal{F}_X} \ll_{\lambda, T, \sigma, X} \|(\mathcal{C} - \lambda)\psi\|_{\mathcal{F}_X}$$

where \mathcal{C} denotes the Casimir operator on $L^2(\mathcal{F}_X)$ (i.e the Casimir operator in \mathbf{z} -coordinates).

To prove (5.11), let ψ be an arbitrary function in $L^2(\mathcal{F}_X)$, which is φ invariant. Now consider

$$\int_{\mathcal{F}_X} \chi_{T,\sigma}(y)\psi(\mathbf{z})\rho(\mathbf{z})d\mathbf{z} = \int_0^{2T} \chi_{T,\sigma}(y) \int_{1/y-X}^{1/y+X} \int_{[-\pi, \pi]^{n-1}} \int_{P_{\Gamma_1}} \psi(\mathbf{z})\rho(\mathbf{z})d\mathbf{x}d\varphi dw dy.$$

For convenience, let us change variables $w \mapsto w - 1/y$, thus, the above integral becomes

$$\int_{\mathcal{F}_X} \chi_{T,\sigma}(y)\psi(\mathbf{z})\rho(\mathbf{z})d\mathbf{z} = \int_0^{2T} \chi_{T,\sigma}(y) \int_{-X}^X \int_{[-\pi, \pi]^{n-1}} \int_{P_{\Gamma_1}} \psi(\mathbf{z})\tilde{\rho}(\mathbf{z})d\mathbf{x}d\varphi dw dy,$$

where $\tilde{\rho}(\mathbf{z})$ is the modified Haar measure density after changing variables. Let

$$f(y) := \int_X^{-X} \int_{P_{\Gamma_1}} \int_{[-\pi, \pi]^{n-1}} \psi(\mathbf{z})\tilde{\rho}(\mathbf{z})d\mathbf{x}d\varphi dw,$$

and

$$(5.12) \quad h(y) := \int_{-X}^X \int_{P_{\Gamma_1}} \int_{[-\pi, \pi]^{n-1}} \tilde{\rho}(\mathbf{z})(\mathcal{C} - \lambda)\psi(\mathbf{z})d\mathbf{x}d\varphi dw.$$

Since, as shown in Theorem 8, the density of the Haar measure decomposes into a product of densities depending on \mathbf{x} , one depending on φ and one depending on w and z , we can use this fact to show

$$\int_{P_{\Gamma_1}} \mathcal{C}\psi(\mathbf{x}, y, w)\rho_H(\mathbf{x})d\mathbf{x} = \mathcal{C} \int_{P_{\Gamma_1}} \psi(\mathbf{x}, y, w)\rho_H(\mathbf{x})d\mathbf{x}.$$

This follows from periodicity on the boundary of \mathcal{F}_X .

Recall that the (modified) Haar measure, in \mathbf{z} coordinates is given by $|wy|^{n-1}\rho_H\rho_{M_1}$. If n is odd, then the absolute value plays no role. However if n is even, we need to consider the discontinuity at 0. Because of this, we henceforth assume the harder case, when n is even, the other case is similar but we have fewer boundary terms.

Using the Structure Theorem for the Casimir operator (Theorem 9)

(5.13)

$$\begin{aligned} 2 \int_{\mathcal{F}_X} \tilde{\rho}(\mathbf{z})(\mathcal{C}\psi)(\mathbf{z})d\mathbf{z} \\ = - \int_0^X (wy)^{n-1} \left(y^2 \partial_y^2 + (n+1)y \partial_y + 2\partial_{yw} + \frac{(n-1)(w + \frac{1}{y})\partial_w}{yw} \right) \tilde{\psi}(y, w)dw \\ + \int_{-X}^0 (wy)^{n-1} \left(y^2 \partial_y^2 + (n+1)y \partial_y + 2\partial_{yw} + \frac{(n-1)(w + \frac{1}{y})\partial_w}{yw} \right) \tilde{\psi}(y, w)dw. \end{aligned}$$

Now we use integration by parts to handle the two derivatives in w

$$\begin{aligned} \int_0^X (wy)^{n-1} \left(2\partial_{yw} + \frac{(n-1)(w + 1/y)\partial_w}{yw} \right) \tilde{\psi}(y, w)dw \\ = \left[(wy)^{n-1} \left(2\partial_y + \frac{(n-1)(w + 1/y)}{yw} \right) \tilde{\psi}(y, w) \right]_0^X - \\ \int_0^X (2(n-1)y(wy)^{n-2}\partial_y + (n-1)(wy)^{n-2} + (n-1)(n-2)(wy)^{n-3}(wy+1)) \tilde{\psi}(y, w)dw \end{aligned}$$

Thanks to our assumptions on $\tilde{\psi}$, when added together, we have some cancellation coming from all the terms involving ∂_w . Namely

$$\begin{aligned} - \int_0^X (wy)^{n-1} \left(2\partial_{yw} + \frac{(n-1)(w + 1/y)\partial_w}{yw} \right) \tilde{\psi}(y, w)dw \\ + \int_{-X}^0 (wy)^{n-1} \left(2\partial_{yw} + \frac{(n-1)(w + 1/y)\partial_w}{yw} \right) \tilde{\psi}(y, w)dw \\ = \int_0^X (n-1) (2y(wy)^{n-2}\partial_y + (wy)^{n-2} + (n-2)(wy)^{n-3}(wy+1)) \tilde{\psi}(y, w)dw \\ - \int_{-X}^0 (n-1) (2y(wy)^{n-2}\partial_y + (wy)^{n-2} + (n-2)(wy)^{n-3}(wy+1)) \tilde{\psi}(y, w)dw. \end{aligned}$$

Returning to (5.13) we consider only the term from $[0, 0 + X)$, we thus have

$$\begin{aligned} I := \int_0^X (wy)^{n-1} \left(y^2 \partial_y^2 + (n+1)y \partial_y + 2\partial_{yw} + \frac{(n-1)(w + 1/y)\partial_w}{yw} \right) \tilde{\psi}(y, w)dw \\ = \int_0^X \left\{ (wy)^{n-1} (y^2 \partial_y^2 + (n+1)y \partial_y) - 2(n-1)y(wy)^{n-2}\partial_y \right. \\ \left. - (n-1)(wy)^{n-2} - (n-1)(n-2)(wy)^{n-3}(wy+1) \right\} \tilde{\psi}(y, w)dw. \end{aligned}$$

Now we can use the product rule to pull the derivatives in y to the front of the integral.

Let

$$\tilde{f} := \int_0^X (wy)^{n-1} \tilde{\psi}(y, w)dw.$$

Then we have that

$$y^2 \partial_{yy} \tilde{f} = w^2 y^2 (n-1)(n-2)(wy)^{n-3} + 2(w+1/y)y^2(n-1)(wy)^{n-2} \partial_y + y^2 (wy)^{n-1} \partial_{yy}.$$

Rearranging and inserting into I gives

$$\begin{aligned} I = & y^2 \partial_{yy} \tilde{f} + \int \left[(wy)^{n-1} (n+1)y \partial_y - 2(n-1)y (wy)^{n-2} \partial_y \right. \\ & - (n-1)(wy)^{n-2} - (n-1)(n-2)(wy+1)(wy)^{n-3} \\ & \left. - (wy+1)^2 (n-1)(n-2)(wy)^{n-3} - 2y(wy+1)(n-1)(wy)^{n-2} \partial_y \right] \tilde{\psi} dw. \end{aligned}$$

Now we can collect together some of the like terms

$$\begin{aligned} I = & y^2 \partial_{yy} \tilde{f} \\ & - \int \left[(n-3)y (wy)^{n-1} \partial_y + (n-1)(wy)^{n-1} + (n-1)(n-3)(wy+1)(wy)^{n-2} \right] \tilde{\psi} dw. \end{aligned}$$

Again, rearranging the product rule implies that we can write

$$I = y^2 \partial_{yy} \tilde{f} - (n-3)y \partial_y \tilde{f} - (n-1)\tilde{f}.$$

Putting together the two integrals, we conclude that

$$(5.14) \quad -y^2 \partial_{yy} f + (n-3)y \partial_y f + (n-1)f = h(y).$$

If we write $\lambda = s(n-s)$, then

$$-y^2 \partial_{yy} f + (n-3)y \partial_y f + (n-1)f = \lambda f.$$

has solutions

$$f(y) = \frac{y^s}{y}, \quad \text{and} \quad f(y) = \frac{y^{n-s}}{y}$$

As a consequence of (5.14), using the method of variation of parameters (we omit the proof which is the same as the $\mathrm{SL}_2(\mathbb{R})$ -case (see (3.15)), we have the following corollary

Theorem 14. *For any $s > 0$, with $s \neq n/2$, there exist constants c_1 and c_2 such that:*

$$(5.15) \quad f(y) = c_1 \frac{y^s}{y} + c_2 \frac{y^{n-s}}{y} + \frac{y^s}{y} u(y) + \frac{y^{n-s}}{y} v(y)$$

where

$$u(y) := \int_y^T \frac{t^{2-s}}{2s-n} h(t) dt, \quad v(y) := \int_y^T \frac{t^{s-n+2}}{n-2s} h(t) dt,$$

If $s = n/2$, then

$$(5.16) \quad f(y) = c_1 \frac{y^{n/2}}{y} + c_2 \frac{y^{n/2}}{y} \log y + u(y) + v(y) \log y,$$

where the inhomogeneous terms are given by

$$u(y) := \int_y^T \frac{-t^{2-n/2} \log(t) h(t)}{(n-2) \log(t) + 1} dt, \quad v(y) := \int_y^T \frac{t^{2-n/2} h(t)}{(n-2) \log(t) + 1} dt.$$

With Theorem 14 at hand, we may prove Lemma 13.

Proof of Lemma 13. We will prove the bound for the term depending on u as the same proof applies for the term depending on v . Thus, consider

$$\int_0^T \frac{y^s}{y} u(y) dy = \int_0^T y^{s-1} \int_y^T \frac{t^{2-s}}{2s-n} h(t) dt dy.$$

Integrating by parts, and applying the fundamental theorem of calculus then gives

$$\int_0^T \frac{y^s}{y} u(y) dy = c [y^s u(y)]_{y=0}^T - c' \int_0^T y^2 h(y) dy.$$

Now note that $u(T) = 0$, thus:

$$\int_0^T \frac{y^s}{y} u(y) dy = \left(\lim_{y \rightarrow 0} y^s \int_y^T \frac{t^{2-s}}{2s-n} h(t) dt \right) - c' \int_0^T y^2 h(y) dy.$$

Recall $h(y) := \int_{-X}^X \int_{P_{\Gamma_1}} \int_{[-\pi, \pi]^{n-1}} \rho(\mathbf{z}) (\mathcal{C} - \lambda) \psi(\mathbf{z}) d\mathbf{x} d\boldsymbol{\varphi} dw$, thus the first term behaves like $\lim_{y \rightarrow 0} y^3 = 0$. Thus

$$\int_0^T \frac{y^s}{y} u(y) dy \ll \int_0^T \int_{-X}^X \int_{P_{\Gamma_1}} \int_{[-\pi, \pi]^{n-1}} y^2 \rho(\mathbf{z}) (\mathcal{C} - \lambda) \psi(\mathbf{z}) d\mathbf{x} d\boldsymbol{\varphi} dw dy.$$

Applying Cauchy-Schwarz:

$$\begin{aligned} \int_0^T \frac{y^s}{y} u(y) dy &\ll \left(\int_0^T \int_{-X}^X \int_{P_{\Gamma_1}} \int_{[-\pi, \pi]^{n-1}} \rho(\mathbf{z}) |(\mathcal{C} - \lambda) \psi(\mathbf{z})|^2 d\mathbf{x} d\boldsymbol{\varphi} dw dy \right)^{1/2} \\ &\quad \cdot \left(\int_0^T \int_{-X}^X \int_{P_{\Gamma_1}} \int_{[-\pi, \pi]^{n-1}} y^4 \rho(\mathbf{z}) d\mathbf{x} d\boldsymbol{\varphi} dw dy \right)^{1/2} \\ &\ll_{T, X, \lambda} \|(\mathcal{C} - \lambda) \psi(\mathbf{z})\|_{\mathcal{F}_X}. \end{aligned}$$

Note that, here, it is crucial that we imposed the cut-off in the w -direction. This ensures that the second factor on the right hand side is finite. The proof for the v -term is identical. \square

Thus, working our way back up, with the same argument as in the $SL_2(\mathbb{R})$ setting, we conclude the proof of Proposition 12.

5.3. Proof of Theorem 10. As for the $SL_2(\mathbb{R})$ case, we now return to the smooth count with U -cutoff:

$$\tilde{N}_{\Gamma}(T) = \langle F_{T, X}, \Psi \rangle_{\Gamma}.$$

Now apply the abstract Parseval's identity (2.2)

$$\begin{aligned}
\langle F_{T,X}, \Psi \rangle_\Gamma &= \langle \widehat{F_{T,X}}, \widehat{\Psi} \rangle_{\text{Spec}(\Gamma)} \\
(5.17) \qquad &= \widehat{F_{T,X}}(\lambda_0) \widehat{\Psi}(\lambda_0) + \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{F_{T,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu(\lambda).
\end{aligned}$$

As with the $\text{SL}_2(\mathbb{R})$ case, we can apply spectral methods to extract the T and ε -dependence. Applying the abstract spectral theorem gives

$$\widehat{F_{T,X}}(\lambda_0) \widehat{\Psi}(\lambda_0) = \langle \text{Proj}_{\mathcal{H}_0}(F_{T,X}), \text{Proj}_{\mathcal{H}_0}(\Psi) \rangle.$$

Then by linearity and our main identity Proposition 12 we conclude

$$\begin{aligned}
\widehat{F_{T,X}}(\lambda_0) \widehat{\Psi}_\varepsilon(\lambda_0) &= K_T(\lambda_0) \langle \text{Proj}_{\mathcal{H}_0}(F_{1,X}), \text{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon) \rangle + L_T(\lambda_0) \langle \text{Proj}_{\mathcal{H}_0}(F_{b,X}), \text{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon) \rangle \\
&= T^\delta \langle H, \text{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon) \rangle + O(T^{n/2}),
\end{aligned}$$

where $H := c_1 \text{Proj}_{\mathcal{H}_0}(F_{1,X}) + c_b \text{Proj}_{\mathcal{H}_0}(F_{b,X})$ for some constants c_1, c_b .

The problem remains to determine the ε dependence of the projection operator $\text{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon)$. In general, this projection can be realized in a number of ways, either as a Burger-Roblin-type measure of Ψ_ε (see [MO15, p. 861]), or using representation-theoretic decompositions as in [BKS10, Vin12]. We will give a soft argument that avoids either.

We know from (5.5) that

$$N^{\mathcal{P}} \left(\frac{T}{1 + c\varepsilon} \right) \leq \tilde{N}_{\Gamma, \varepsilon}(T) \leq N^{\mathcal{P}} \left(\frac{T}{1 - c\varepsilon} \right),$$

for any value of ε and any value of T . However we also know *a priori* (e.g., using [Kim15]) that

$$N^{\mathcal{P}} \left(\frac{T}{1 \pm c\varepsilon} \right) = c_{\mathcal{P}} \left(\frac{T}{1 \pm c\varepsilon} \right)^\delta (1 + o(1)),$$

as $T \rightarrow \infty$. Dividing by T^δ then gives

$$c_{\mathcal{P}} \left(\frac{1}{1 + c\varepsilon} \right)^\delta - o(1) \leq \langle H, \text{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon) \rangle \leq c_{\mathcal{P}} \left(\frac{1}{1 - c\varepsilon} \right)^\delta + o(1).$$

Now send $T \rightarrow \infty$ and Taylor expand $\frac{1}{(1 \pm c\varepsilon)^\delta}$ in ε , giving:

$$\langle H, \text{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon) \rangle = C + O(\varepsilon).$$

Hence

$$(5.18) \qquad \widehat{F_{T,X}}(\lambda_0) \widehat{\Psi}_\varepsilon(\lambda_0) = T^\delta c(1 + O(\varepsilon)) + O(T^{n/2})$$

for some constant c independent of ε . (Note that this positivity argument does not apply to the other eigenvalues. Hence with sharp cutoffs, as in Theorem 1, we do not extract lower order terms.)

Turning now to the remainder, after extracting the main term corresponding to λ_0 we are left with

$$\begin{aligned} \text{Err} &:= \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{F_{T,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu \\ &= \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \left(K_T(\lambda) \widehat{F_{1,X}}(\lambda) + L_T(\lambda) \widehat{F_{b,X}}(\lambda) \right) \widehat{\Psi}(\lambda) d\nu. \end{aligned}$$

Assume for simplicity that there are no other discrete eigenvalues above the base. Now apply the abstract spectral theorem and the bounds from (5.7) to conclude that

$$\int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} K_T(\lambda) \widehat{F_{1,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu \ll T^{n/2} \log T \int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{F_{1,X}}(\lambda) \widehat{\Psi}(\lambda) d\nu.$$

Now apply Cauchy-Schwarz and Parseval to get

$$\begin{aligned} &\ll T^{n/2} \log T \left(\int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{F_{1,X}}(\lambda)^2 d\nu \right)^{1/2} \left(\int_{\text{Spec}(\Gamma) \setminus \{\lambda_0\}} \widehat{\Psi}(\lambda)^2 d\nu \right)^{1/2} \\ &\leq T^{n/2} \log T \left(\int_{\text{Spec}(\Gamma)} \widehat{F_{1,X}}(\lambda)^2 d\nu \right)^{1/2} \left(\int_{\text{Spec}(\Gamma)} \widehat{\Psi}(\lambda)^2 d\nu \right)^{1/2} \\ &= T^{n/2} \log T \|F_{1,X}\|_{\Gamma} \|\Psi\|_{\Gamma}. \end{aligned}$$

Finally, note that since ψ_ε is normalized to have unit L^1 -mass, we have that $\|\Psi\|_{\Gamma} \ll \varepsilon^{-(n+1)/2}$. In the case of other eigenvalues, we replace the bound $T^{n/2} \log T$ above with T^{s_1} . This completes the proof of Theorem 10. \square

REFERENCES

- [BKS10] J. Bourgain, A. Kontorovich, and P. Sarnak. Sector estimates for hyperbolic isometries. *Geometric and Functional Analysis*, 20(5):1175–1200, Nov 2010.
- [Boy73] D. Boyd. The residual set dimension of the Apollonian packing. *Mathematika*, 20:170–174, 1973.
- [EO21] S. Edwards and H. Oh. Spectral gap and exponential mixing on geometrically finite hyperbolic manifolds. *Duke Math. J.*, 170(15):3417–3458, 2021.
- [Kim15] I. Kim. Counting, mixing and equidistribution of horospheres in geometrically finite rank one locally symmetric manifolds. *J. Reine Angew. Math.*, 704:85–133, 2015.
- [KK21] M. Kapovich and A. Kontorovich. On superintegral kleinian sphere packings, bugs, and arithmetic groups. *arXiv:2104.13838 [math.NT]*, 2021.
- [KN19] A. Kontorovich and K. Nakamura. Geometry and arithmetic of crystallographic sphere packings. *Proc. Natl. Acad. Sci. USA*, 116(2):436–441, 2019.
- [Kna01] A. Knapp. *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.
- [KO11] A. Kontorovich and H. Oh. Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 24(3):603–648, 2011. With an appendix by Oh and Nimish Shah.
- [KO12] A. Kontorovich and H. Oh. Almost prime Pythagorean triples in thin orbits. *Journal für die Reine und Angewandte Mathematik*, 667:89–131, 2012.

- [Kon09] A. Kontorovich. The hyperbolic lattice point count in infinite volume with applications to sieves. *Duke Math. J.*, 149(1):1–36, 2009.
- [Kon13] A. Kontorovich. From Apollonius to Zaremba: local-global phenomena in thin orbits. *Bull. Amer. Math. Soc. (N.S.)*, 50(2):187–228, 2013.
- [LO13] M. Lee and H. Oh. Effective circle count for Apollonian packings and closed horospheres. *Geom. Funct. Anal.*, 23(2):580–621, 2013.
- [LP82] P. Lax and R. Phillips. The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. *J. Functional Analysis*, 46(3):280–350, 1982.
- [MO15] A. Mohammadi and H. Oh. Matrix coefficients, counting and primes for orbits of geometrically finite groups. *Journal of the EMS*, 17:837–897, 2015.
- [Pan17] W. Pan. Effective equidistribution of circles in the limit sets of Kleinian groups. *J. Mod. Dyn.*, 11:189–217, 2017.
- [Pat76] S. Patterson. The limit set of a Fuchsian group. *Acta Mathematica*, 136:241–273, 1976.
- [Rud73] W. Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.
- [Sar07] P. Sarnak. Letter to J. Lagarias, 2007. <http://web.math.princeton.edu/sarnak/AppolonianPackings.pdf>.
- [Sul84] D. Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Mathematica*, 153:259–277, 1984.
- [Vin12] I. Vinogradov. *Effective bisector estimate with application to Apollonian circle packings*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)—Princeton University.

Email address: alex.kontorovich@rutgers.edu

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ

Email address: chris.lutsko@rutgers.edu

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ