POLYHEDRAL BOUNDS ON THE JOINT SPECTRUM AND TEMPEREDNESS OF LOCALLY SYMMETRIC SPACES

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Abstract. Given a real semisimple connected Lie group G and a discrete torsion-free subgroup $\Gamma < G$ we prove a precise connection between growth rates of the group Γ, polyhedral bounds on the joint spectrum of the ring of invariant differential operators, and the decay of matrix coefficients. In particular, this allows us to completely characterize temperedness of $L^2(\Gamma\backslash G)$ in this general setting.

1. Introduction

Consider a locally symmetric space $\Gamma\backslash G/K$, where G is a real connected semisimple non-compact Lie group with finite center, K is a maximal compact subgroup, and $\Gamma < G$ is a discrete torsion-free subgroup. When the group G has rank one, there is an important connection between:

(i) The bottom of the L^2-spectrum of the Laplace-Beltrami operator.
(ii) The exponential growth rate of Γ points in G/K in a ball of growing Riemannian distance (given by the *critical exponent* δ_Γ, see (1.2)).
(iii) The properties of the unitary representation $L^2(\Gamma\backslash G)$, in particular its temperedness.

For $G = \mathrm{SL}_2(\mathbb{R})$ the connection between (i) and (ii) was achieved in the seminal work on the subject by Elstrodt [Els73a, Els73b, Els74] and Patterson [Pat76] (see Subsection 1.1). The relation between (i) and (iii) is a direct consequence of the explicit knowledge of all unitary irreducible $\mathrm{SL}_2(\mathbb{R})$-representations and one deduces that $L^2(\Gamma\backslash G)$ is tempered if and only if $\delta_\Gamma \leq 1/2$. However, the theorem of Elstrodt-Patterson is equally of interest for $\delta_\Gamma > 1/2$ as this ensures an eigenvalue of Δ below $1/4$, often called an exceptional eigenvalue. Such exceptional eigenvalues were the pivotal input in many important works, for example the uniform spectral gap estimates for congruence subgroups and applications to expander graphs obtained by Gamburd [Gam02] and affine sieves by Bourgain, Gamburd, and Sarnak [BGS10] (see also the recent result of Calderón-Magee [CM23]) and the uniform spectral gap estimates for random covers of Magee and Naud [MN20].

The aim of this article is to prove a generalization of the Elstrodt-Patterson theorem for the joint spectrum of invariant differential operators on higher rank locally symmetric spaces and to reproduce the above trichotomy in full generality.

Before stating the main theorem we need to establish some notation. Recall that G admits a Cartan decomposition $G = K\exp(\overline{\mathfrak{a}_+})K$. Hence, for every $g \in G$ there is a $\mu_+(g) \in \overline{\mathfrak{a}_+}$ such that $g \in K\exp(\mu_+(g))K$. $\mu(g)$ can be thought of a higher dimensional distance $d(gK, eK)$. Following the analogy of the rank one case, Quint [Qui02] introduced

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the notion of the growth indicator function $\psi_\Gamma \colon \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$:

$$\psi_\Gamma(H) \coloneqq \|H\| \inf_{H\in\mathcal{C}} \inf \left\{ s \in \mathbb{R} \mid \sum_{\gamma\in\Gamma, \mu_+(\gamma)\in\mathcal{C}} e^{-s\|\mu_+(\gamma)\|} < \infty \right\},$$

where the infimum runs over all open cones $\mathcal{C} \subseteq \mathfrak{a}$ with $H \in \mathcal{C}$.

In higher rank, the role of the Laplacian is played by the full algebra of invariant differential operators on G/K which we denote by $\mathbb{D}(G/K)$. It is convenient to parameterize the joint spectrum of this algebra via the Harish-Chandra isomorphism by a W invariant subset $\widetilde{\sigma} \subseteq \mathfrak{a}^*_\mathbb{C} \cong \mathbb{C}^{\mathrm{rank}(G/K)}$ (see Section 2.3). In general, $\Re\widetilde{\sigma} \subseteq \mathrm{conv}(W\rho)$, where ρ denotes the usual half-sum of restricted roots and $\mathrm{conv}(W\rho)$ is the polyhedron described by the convex hull of the Weyl orbit of ρ.

Furthermore, we introduce the *polyhedral norm* which is the key ingredient to formulate our main theorem: For any homogeneous (not necessarily linear) function $\lambda : \mathfrak{a} \to \mathbb{R}$

$$\|\lambda\|_{poly} = \sup_{w\in W, H\in\overline{\mathfrak{a}_+}} \frac{\lambda(wH)}{\rho(H)}.$$

The terminology polyhedral norm stems from the fact that for linear functionals this is a usual vector space norm on $\mathfrak{a}^*$ whose balls are polyhedra spanned by the Weyl translates of ρ, i.e.

$$\{\lambda \in \mathfrak{a}^*,\ \|\lambda\|_{poly} \leq R\} = R\,\mathrm{conv}(W\rho).$$

Thus, the general bound on the joint spectrum is equivalent to saying that, for arbitrary Γ, $\|\Re\lambda\|_{poly} \leq 1$ for all $\lambda \in \widetilde{\sigma}$ (cf. Figure 1 for a visualisation for $\mathrm{SL}_3(\mathbb{R})$).

As a last ingredient let us introduce the exponential decay rate of matrix coefficients: Let $\theta = \theta(L^2(\Gamma\backslash G)) \in [0, 1]$ denote the infimum θ such that, for all $v \in \overline{\mathfrak{a}_+}$, and $f_1, f_2 \in L^2(\Gamma\backslash G)^K$, one has

$$\left|\langle(\exp v)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}\right| \leq Ce^{(\theta-1)\rho(v)}\|f_1\|\|f_2\|,$$

for some $C > 0$ independent of the choice of v or test functions f_1, f_2. Our main theorem then connects the polyhedral bounds on $\Re\widetilde{\sigma}$ to polyhedral bounds on the growth indicator function ψ_Γ and the exponential decay rate of matrix coefficients of $L^2(\Gamma\backslash G)$.

Theorem 1.1. *Let G be a real semisimple connected non-compact Lie group with finite center and $\Gamma < G$ a discrete and torsion-free subgroup. Then*

$$\sup_{\lambda\in\widetilde{\sigma}} \|\Re\lambda\|_{poly} = \max(0, \|\psi_\Gamma - \rho\|_{poly}) = \theta(L^2(\Gamma\backslash G)). \tag{1.1}$$

We refer to Figure 1 for a visualisation. It is well known that the temperedness of a unitary representation is equivalent to decay properties of its matrix coefficients and we deduce

Corollary 1.2. *$L^2(\Gamma\backslash G)$ is tempered if and only if $\psi_\Gamma \leq \rho$.*

This confirms a conjecture by Hee Oh and generalizes the theorem of Edwards and Oh [EO23, Theorem 1.6]. They prove this result for the case of Γ being Zariski dense and the image of an Anosov representation of a minimal parabolic and their prove is based on mixing results for Anosov subgroups by Edwards, Lee, and Oh [ELO23].

Let us denote by $\sigma(\Delta)$ the spectrum of the Laplace-Beltrami operator on $L^2(\Gamma\backslash G/K)$. In contrast to the rank one case, bounding the bottom of the Laplace spectrum does a priori not suffice in higher rank to obtain a characterization of temperedness and non-temperedness of

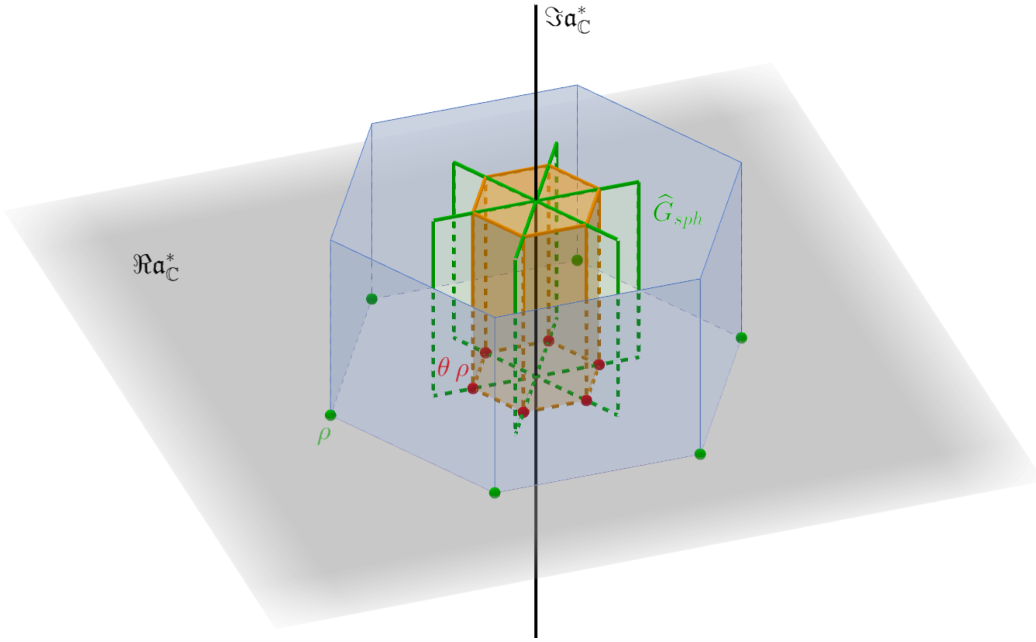

FIGURE 1. Visualization for $G = \mathrm{SL}_3(\mathbb{R})$. The grey plane is the real part of $\mathfrak{a}^*_{\mathbb{C}}$. The two-dimensional imaginary part is depicted as a one dimensional z axis. The green planes together with $W\rho$ is where the joint spectrum can actually occur, i.e. this is $\widehat{G}_{sph}$. The blue hexagonal tube is the region $\{\Re\lambda \in \mathrm{conv}(W\rho)\}$ which is the general bound for the real part of the joint spectrum. The orange tube is the restricted region containing $\widetilde{\sigma}$ by Theorem 1.1. By Theorem 1.1 we know that there is spectrum on the boundary of the orange tube. Proposition 1.5 shows that this occurs actually at $\theta\rho$ (red).

$L^2(\Gamma\backslash G)$, because in higher rank there are known examples of non-tempered representations that lead to Laplace eigenvalues bigger then $\|\rho\|^2$. However, based on Theorem 1.1 (see the slightly more detailed version Theorem 5.1) we can prove that temperedness of $L^2(\Gamma\backslash G)$ is nevertheless equivalent to the bottom of the Laplace spectrum being $\|\rho\|^2$ and we obtain a refined version of Corollary 1.2:

Corollary 1.3. *Let G be a real semisimple connected non-compact Lie group with finite center and $\Gamma < G$ a discrete and torsion-free subgroup, then the following statements are equivalent:*

(i) $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$.
(ii) For all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for all $f_1, f_2 \in L^2(\Gamma\backslash G)^K$:

$$|\langle(\exp v)f_1, f_2\rangle| \leq d_\varepsilon e^{\varepsilon\|v\|}e^{-\rho(v)}\|f_1\|\|f_2\|.$$

(iii) $\psi_\Gamma \leq \rho$.
(iv) $L^2(\Gamma\backslash G)$ *is almost* L^2.
(v) $\inf\sigma(\Delta) = \|\rho\|^2$.
(vi) $L^2(\Gamma\backslash G)$ *is tempered.*

Note that, if Γ is a lattice subgroup then, since we have not ruled out the trivial representation, i.e. the constant function in $L^2(\Gamma\backslash G/K)$ which leads to a trivial joint eigenfunction of $\mathbb{D}(G/K)$, the above results are consistent with this case, but do not give anything nontrivial or novel.

1.1. **Related Results.** As discussed above, studying the connections between spectral properties of $\Gamma\backslash G/K$ and the counting of Γ points has a long history. The first instance of this connection is the characterization of the bottom $\inf\sigma(\Delta)$ of the Laplace spectrum for hyperbolic surfaces:

$$\inf\sigma(\Delta) = \begin{cases} 1/4 & : \delta_\Gamma < 1/2 \\ 1/4 - (\delta_\Gamma - 1/2)^2 & : \delta_\Gamma \geq 1/2, \end{cases}$$

where δ_Γ is the critical exponent of the discrete subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$

$$\delta_\Gamma := \inf\left\{ s \in \mathbb{R} \colon \sum_{\gamma\in\Gamma} e^{-sd(\gamma x_0, x_0)} < \infty \right\}, \quad x_0 \in \mathbb{H}. \tag{1.2}$$

This theorem is due to Elstrodt [Els73a, Els73b, Els74] and Patterson [Pat76] and has been extended to real hyperbolic manifolds of arbitrary dimension by Sullivan [Sul87] and then to general locally symmetric spaces of rank one by Corlette [Cor90].

In our higher rank setting, the bottom of the Laplace spectrum was estimated using the same definition of δ_Γ which is defined through $d(\gamma x_0, x_0) = \|\mu_+(x_0^{-1}\gamma x_0)\|$ by Leuzinger [Leu04] and Weber [Web08]. Later, Anker and Zhang [AZ22] (see also [CP04]) proved the exact formula

$$\inf\sigma(\Delta) = \begin{cases} \|\rho\|^2 & : \tilde{\delta}_\Gamma < \|\rho\| \\ \|\rho\|^2 - (\tilde{\delta}_\Gamma - \|\rho\|)^2 & : \tilde{\delta}_\Gamma \geq \|\rho\|, \end{cases}$$

where $\tilde{\delta}_\Gamma$ is the modified critical exponent which is defined through $\|\mu_+(\gamma)\|$ and $\langle\rho, \mu_+(\gamma)\rangle$ and therefore also takes the direction and not only the size of $\mu_+(\gamma)$ into account. However, as mentioned above, such bounds do not lead to temperedness of $L^2(\Gamma\backslash G)$ due to the existence of non-tempered representations with arbitrary high Laplace eigenvalues.

A criterion of temperedness was only achieved in the aforementioned work of Edwards and Oh on quotients by Anosov subgroups. Motivated by this work the latter two named authors [WW23b] obtained bounds on the joint spectrum by counting Γ points in the case where G is a product of rank one groups and $\Gamma < G$ a general discrete, torsion free subgroup. In particular, they extended the aforementioned [EO23, Theorem 1.6] by Edwards and Oh to this case. The methods in [WW23b] however were based on analyzing the resolvent kernels on the individual rank one factors.

In the case where G has no factors locally isomorphic to $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$ it has Kazhdan's Property (T), i.e. the trivial representation is an isolated point in the unitary dual of G. This amounts to a uniform bound on the quantities in (1.1), i.e. an estimate independent of Γ, if Γ has infinite covolume. More precisely, in [LO23, Thm. 7.1] (see also previous work by Quint [Qui03]) it is shown that $\psi_\Gamma \leq 2\rho - \Theta$ for some explicitly given functional Θ. Similarly, in [Oh02, Thm. 1.2] it is shown that

$$\left|\langle(\exp v)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}\right| \leq Ce^{-\Theta(v)}e^{\varepsilon\|v\|}\|f_1\|\|f_2\|$$

for all $v \in \overline{\mathfrak{a}_+}$, and $f_1, f_2 \in L^2(\Gamma\backslash G)^K$ for the same Θ. In [HWW23, Sect. 4A] one can find an analogues statement for the joint spectrum. However, the bounds obtained by Property (T) are not enough to deduce temperedness.

Temperedness in the complementary setting of homogeneous spaces G/H for a closed subgroup H with finitely many connected components has been studied by Benoist and Kobayashi in a series of papers [BK15, BK22, BK21, BK23]. They prove that the regular representation of G on $L^2(G/H)$ is tempered if and only if a growth condition on H is satisfied. They also prove a version similar to Corollary 1.3 (and also Theorem 5.1) where they characterize when $L^2(G/H)$ is almost L^p for $p \in 2\mathbb{N}$.

Let us finally mention two other recent results that concern the spectral theory of higher rank locally symmetric spaces of infinite volume: In [EFLO23] Edwards, Fraczyk, Lee and Oh prove that the bottom of the Laplace spectrum is never an atom, provided that Γ is a Zariski dense subgroup of infinite covolume in a semisimple real algebraic group G with Kazhdan's property (T). They achieve this result by combining previous results on positivity of Laplace eigenvalues [EO23] and the finiteness of Bowen Margulis Sullivan measures [FL23]. In [WW23a] the latter two named authors study the principle joint spectrum (i.e. the part of $\widetilde{\sigma}$ contained in $i\mathfrak{a}^*$) and give a dynamical criterion for the absence of embedded eigenvalues. Combining [WW23a, Theorem 1.1, Proposition 5.1] and Theorem 1.1 we obtain:

Theorem 1.4. *Let G be a real connected semisimple non-compact Lie group of rank ≥ 2 and Γ the image of a P-Anosov representation for an arbitrary parabolic $P \subset G$ with $\psi_\Gamma \leq \rho$, then there exists no joint eigenfunction of the algebra of invariant differential operators $\mathbb{D}(G/K)$ in $L^2(\Gamma\backslash G/K)$.*

1.2. **Shape of the spectrum.** Theorem 1.1 provides a sharp bound on the real part of the joint spectrum in a polyhedral region via the growth indicator function. However, we have a priori no information about the imaginary part. Of course, one could go further, and ask more about the shape of the spectrum. In particular examples we can however say more: For $\mathrm{SL}_3(\mathbb{R})$ we use that the root system of restricted roots is of type A_2 and show

Proposition 1.5. *Let $G = \mathrm{SL}_3(\mathbb{R})$ and $\Gamma < G$ a discrete torsion-free subgroup. Then the supremum $\sup_{\lambda\in\widetilde{\sigma}} \|\Re\lambda\|_{poly} = \theta$ is achieved at $\lambda = \theta\rho$ (see Figure 1).*

We emphasize that $\theta\rho \in \widetilde{\sigma}$ is in general false, as the product case shows (see Section 6.1 for details). Nevertheless, also in the product case there is a real spectral value on the boundary of the polyhedral region. We conjecture that this holds in general:

Conjecture 1.6. *There is a real spectral value on the boundary of the polyhedral region given by Theorem 1.1, i.e.*

$$\widetilde{\sigma} \cap \{\lambda \in \mathfrak{a}^* \mid \|\lambda\|_{poly} = \theta\} \neq \emptyset.$$

1.3. **Outline of the paper.** We start in Section 2 with fixing the notation, introducing the joint spectrum of the algebra of invariant differential operators and recalling some important facts about tempered and almost L^p representations. In Section 3 we then study how the decay of matrix coefficients is related to the joint spectrum. The central step of the paper is done in Section 4 where we derive a precise relation between the decay of matrix coefficients for functions $f_1, f_2 \in C_c(\Gamma\backslash G)$ and the growth indicator function ψ_Γ (Theorem 4.5). In Section 5 we put the obtained results together and also formulate Theorem 5.1 which is a slightly more detailed version of Theorem 1.1. Finally in Section 6 we illustrate the implication of our main theorem for two concrete examples, the case of $G = \mathrm{SL}_3(\mathbb{R})$ and the product case.

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2. Preliminaries

2.1. **Notation.** In this article G is a real semisimple connected non-compact Lie group with finite center and K is a maximal compact subgroup of G, then G/K is a Riemannian symmetric space of noncompact type. We fix an Iwasawa decomposition $G = KAN$, and have $A \cong \mathbb{R}^r$ where r is the ral rank og G or the rank of the symmetric space G/K, respectively. Furthermore, we define M as the centralizer of A in K and $\overline{N}$ to be the nilpotent subgroup such that $KA\overline{N}$ is the opposite Iwasawa decomposition. We denote by $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \mathfrak{m}, \overline{\mathfrak{n}}$ the corresponding Lie algebras. For $g \in G$ let $H(g) \in \mathfrak{a}$ be the logarithm of the A-component in the Iwasawa decomposition. Let $\Sigma \subseteq \mathfrak{a}^*$ be the root system of restricted roots, Σ^+ the positive system corresponding to the Iwasawa decomposition, and W the corresponding Weyl group acting on $\mathfrak{a}^*$. As usual, for $\alpha \in \Sigma$, we denote by m_α the dimension of the root space, and by ρ the half sum of restricted roots counted with multiplicity. Let $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \,\forall \alpha \in \Sigma\}$ the positive Weyl chamber, $\overline{\mathfrak{a}_+}$ its closure, and $\mathfrak{a}_+^*$ the corresponding cone in $\mathfrak{a}^*$ via the identification $\mathfrak{a} \leftrightarrow \mathfrak{a}^*$ through the Killing form $\langle \cdot, \cdot \rangle$. We have the Cartan decomposition $G = K \exp(\overline{\mathfrak{a}_+})K$ and for $g \in G$ there is a unique $\mu_+(g) \in \overline{\mathfrak{a}_+}$ such that $g \in K \exp(\mu_+(g))K$. For the Cartan decomposition the following integral formula holds (see [Hel84, Thm. I.5.8]):

$$\int_G f(g)\, dg = \int_K \int_{\mathfrak{a}_+} \int_K f(k \exp(H) k') \delta(H)\, dk\, dH\, dk' \tag{2.1}$$

where $\delta(H) = \prod_{\alpha \in \Sigma^+} (\sinh(\alpha(H))^{m_\alpha}$. Note that $\delta(H) \leq e^{2\rho(H)}$. We fix a discrete subgroup $\Gamma \leq G$.

2.2. **Algebra of invariant differential operators.** As mentioned in the introduction, $\mathbb{D}(G/K)$ denotes the algebra of G-invariant differential operators on G/K. The key result that allows a precise understanding of this algebra is the *Harish-Chandra isomorphism*

$$\chi : \begin{cases} \mathbb{D}(G/K) & \longrightarrow & Poly(\mathfrak{a}_\mathbb{C}^*)^W \\ D & \longmapsto & \chi_\lambda(D) \end{cases}$$

which is an algebra isomorphism between $\mathbb{D}(G/K)$ and the algebra of Weyl group invariant polynomials on $\mathfrak{a}_\mathbb{C}^*$. In particular one deduces that $\mathbb{D}(G/K)$ is abelian and is generated by $\mathrm{rank}(G/K)$ algebraically independent generators.

For any $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we can define the *elementary spherical function*

$$\phi_\lambda(g) := \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\, dk$$

where $H\colon G \to \mathfrak{a}$ is defined by $g \in Ke^{H(g)}N$. This is a bi-K-invariant function and it descends to a left K-invariant function on G/K which is a joint eigenfunction of $\mathbb{D}(G/K)$

fulfilling

$$D\phi_\lambda = \chi_\lambda(D)\phi_\lambda \quad \forall\, D \in \mathbb{D}(G/K).$$

In fact ϕ_λ is the unique such eigenfunction with $\phi_\lambda(e) = 1$ and for $\lambda, \lambda' \in \mathfrak{a}^*_\mathbb{C}$, $\phi_\lambda = \phi_{\lambda'}$ if and only if $\lambda' \in W\lambda$.

Let us next study the action of $\mathbb{D}(G/K)$ on the locally symmetric space $\Gamma\backslash G/K$: Each $D \in \mathbb{D}(G/K)$ is G-invariant and therefore descends to $\Gamma\backslash G/K$. All D are unbounded operators on $L^2(\Gamma\backslash G/K)$ densely defined on $C_c^\infty(\Gamma\backslash G/K)$ and extend to normal operators on $L^2(\Gamma\backslash G/K)$ (we refer to [WW23b, Section 3.2] for more details), thus we can define for any D its $L^2(\Gamma\backslash G/K)$-spectrum and denote it by $\sigma_{L^2}(D) \subset \mathbb{C}$. The spectral theory of $\mathbb{D}(G/K)$ is however, best described by a joint spectrum instead by the individual spectra and it is most convenient to parameterize this spectrum via the Harish-Chandra isomorphism by elements in $\mathfrak{a}^*_\mathbb{C}$:

Definition 2.1. The joint spectrum of $\mathbb{D}(G/K)$ is defined by

$$\widetilde{\sigma} \coloneqq \{\lambda \in \mathfrak{a}^*_\mathbb{C} \mid \chi_\lambda(D) \in \sigma_{L^2}(D) \quad \forall D \in \mathbb{D}(G/K)\} \subset \mathfrak{a}^*_\mathbb{C}.$$

In fact one can also choose a set of generators $D_1, \dots, D_r$ of $\mathbb{D}(G/K)$, show that these are strongly commuting normal operators and consider their joint spectrum in the sense of [Sch12, Chapter 5]. This definition, however, coincides with the technically easier Definition 2.1 as shown in [WW23b, Proposition 3.6].

2.3. **Spherical dual and joint spectrum.** Let us denote with $\widehat{G}$ the unitary dual of G, with $\widehat{G}_{sph} \subset \widehat{G}$ the spherical dual of G, i.e. the set of equivalence classes of irreducible unitary representations containing a non-zero K-invariant vector, and with $\widehat{G}_{tmp}$ the tempered representations, i.e. the support of the Plancherel measure of $L^2(G)$.

In the following we describe how $\widehat{G}_{sph}$ can be parameterized by subset of $\mathfrak{a}^*_\mathbb{C}/W$ (see [Hel84, Thm. IV.3.7]): For $\pi \in \widehat{G}_{sph}$ let v_K be a normalized K-invariant vector. Then the function $\phi\colon G \to \mathbb{C}, \phi(g) = \langle \pi(g)v_K, v_K\rangle$ is bi-K-invariant and positive definite, i.e. the matrix $(\phi(x_i^{-1}x_j))_{ij}$ is positive semidefinite for any choice of finitely many $x_i \in G$. Furthermore, ϕ is an eigenvector for each element in the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K.

Therefore, $\phi = \phi_\lambda$ is an elementary spherical function for $\lambda \in \mathfrak{a}^*_\mathbb{C}$. Recall that $\phi_\lambda = \phi_\mu$ if and only if $W\lambda = W\mu$. It can be shown that the mapping $\pi \mapsto W\lambda$ is a bijection of $\widehat{G}_{sph}$ onto the set $\{\lambda \in \mathfrak{a}^*_\mathbb{C}/W \mid \phi_\lambda \text{ is positive definite}\}$. We identify the two sets and write π_λ for the representation corresponding to $\lambda \in \mathfrak{a}^*_\mathbb{C}/W$ with ϕ_λ positive definite. In particular, for $\lambda \in \widehat{G}_{sph}$ we have $\langle \pi_\lambda(g)v, w\rangle = \phi_\lambda(g)\langle v, w\rangle$ if v, w are both K-invariant.

Every positive definite function on G is bounded by its value at 1 and therefore $\widehat{G}_{sph} \subseteq \mathrm{conv}(W\rho) + i\mathfrak{a}^*$ by [Hel84, Thm. IV.8.1]. Recall from the introduction that $\mathrm{conv}(W\rho)$ is the convex hull of the Weyl orbit $W\rho$ of ρ which can be characterized by

$$\mathrm{conv}(W\rho) = \{\|\lambda\|_{poly} \le 1\} = \{\lambda \in \mathfrak{a}^* \mid \lambda(wH) \le \rho(H)\, \forall H \in \mathfrak{a}_+, w \in W\}.$$

Moreover, every positive definite elementary spherical function ϕ_λ satisfies $\phi_\lambda(g^{-1}) = \overline{\phi_\lambda(g)}$. As $\phi_\lambda(g^{-1}) = \phi_{-\lambda}(g)$ and $\overline{\phi_\lambda(g)} = \phi_{\overline{\lambda}}(g)$, we must have $W(-\lambda) = W\overline{\lambda}$. Hence, $\widehat{G}_{sph} \subseteq \{\lambda \in \mathfrak{a}^*_\mathbb{C} \mid \exists w \in W\colon w\lambda = -\overline{\lambda}\}$.

Let us now explain the relation of the joint spectrum of the invariant differential operators and the spherical dual: Consider the unitary representation R on $L^2(\Gamma\backslash G)$ by right multiplication. By the abstract Plancherel theory, it can be decomposed into a direct integral of irreducible representations

$$(R, L^2(\Gamma\backslash G)) \simeq \int_X^{\oplus} \pi_x \, d\mu(x)$$

where (X, μ) is a measure space and

$$\pi : \begin{cases} X & \longrightarrow & \widehat{G} \\ x & \longmapsto & \pi_x \end{cases}$$

is a measurable map. We should think of X as the Cartesian product of the unitary dual $\widehat{G}$ and a multiplicity space.

The joint spectrum of $\mathbb{D}(G/K)$ on $L^2(\Gamma\backslash G/K)$ can now be expressed as follows:

Proposition 2.2 ([WW23b, Prop. 3.6])**.**

$$\widetilde{\sigma} = \mathrm{supp}(\pi_*\mu) \cap \widehat{G}_{sph} \subseteq \widehat{G}_{sph} \subset \mathfrak{a}_{\mathbb{C}}^*.$$

2.4. **Temperedness and almost L^p.** Recall that a unitary G-representation $(\rho, \mathcal{H})$ with Plancherel decomposition

$$(\rho, \mathcal{H}) \simeq \int_X^{\oplus} \pi_x \, d\mu(x)$$

is called *tempered* if $\mathrm{supp}(\pi_*\mu) \subset \widehat{G}_{tmp} \subset \widehat{G}$. Temperedness of unitary representations has many equivalent characterizations and we want to recall those that are relevant for this paper:

Definition 2.3. Let $p \geq 2$. A unitary representation $(\rho, \mathcal{H})$ of G is called *strongly* $L^{p+\varepsilon}$ or *almost* L^p if there is a dense subset $V \subset \mathcal{H}$ such that for any $v, w \in V$, the matrix coefficient $g \mapsto \langle \rho(g)v, w\rangle$ lies in $L^q(G)$ for all $q > p$.

Note that if π is strongly $L^{p+\varepsilon}$, then π is also strongly $L^{q+\varepsilon}$ for any $q \geq p$ since any matrix coefficients are bounded.

Let us furthermore introduce the Harish-Chandra function $\Xi(g) = \phi_0(g) = \int_K e^{-\rho(H(gk))} dk$. It is well-known that Ξ is a smooth bi-K-invariant function of G with values in $(0, 1]$. Furthermore, there is a constant C such that

$$e^{-\rho(H)} \leq \Xi(e^H) \leq C(1 + |H|)^d e^{-\rho(H)} \tag{2.2}$$

for $H \in \mathfrak{a}_+$. Here d is the number of positive reduced roots. Note that by (2.1) this implies that $\Xi \in L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$.

Proposition 2.4 ([CHH88, Thm. 1 and 2])**.** *Let $(\rho, \mathcal{H})$ be a unitary G-representation then the following are equivalent*

(i) $(\rho, \mathcal{H})$ is tempered.
(ii) $(\rho, \mathcal{H})$ is almost L^2
(iii) For any K-finite unit vectors $v, w \in \mathcal{H}$,

$$|\langle \rho(g)v, w\rangle| \leq (\dim\langle Kv\rangle \dim\langle Kw\rangle)^{1/2} \, \Xi(g)$$

for any $g \in G$, where $\langle Kv\rangle$ denotes the subspace spanned by Kv.

Note that in [CHH88] the group G is assumed to be a semisimple algebraic group over a local field. However, as observed in [Sun09] the same holds without any modification of the proof as soon as G admits an Iwasawa decomposition. The same applies to Proposition 2.5 below.

Since we're not only interested in temperedness, being strongly $L^{p+\varepsilon}$ gives us a measure for the extent of the non-tempered part. However, the connection to uniform pointwise bounds seems to be established only for $p \in 2\mathbb{N}$:

Proposition 2.5 ([CHH88, Cor. on p. 108]). *If π is a unitary representation without a non-zero invariant vector that is strongly $L^{2k+\varepsilon}$, $k \in \mathbb{N}$, then for any K-finite unit vectors v and w,*

$$|\langle \pi(g)v, w\rangle| \leq (\dim\langle Kv\rangle \dim\langle Kw\rangle)^{1/2}\, \Xi^{1/k}(g).$$

Clearly, since $\Xi \in L^{2+\varepsilon}(G)$ the opposite implication holds as well.

3. Decay of coefficients and the joint spectrum

The aim of this section is to work out how the decay of matrix coefficients is linked to the joint spectrum. We will in particular show that $L^2(\Gamma\backslash G)$ is tempered if and only if $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$ and that there is a relation between polyhedral bounds on $\Re(\widetilde{\sigma})$ and the decay of matrix coefficients of $L^2(\Gamma\backslash G)$. As tools we use standard representation theory and asymptotics of spherical functions. Although we assume these relations to be known to experts we want to include the statements and proof in order to make the article self consistent.

We first prove that bounds on the real part of the joint spectrum lead to decay estimates for the matrix coefficients.

Lemma 3.1. *For all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for all $f, g \in L^2(\Gamma\backslash G)^K$ we have*

$$|\langle R(\exp v)f, g\rangle| \leq d_\varepsilon e^{\sup_{\lambda\in\widetilde{\sigma}}(\Re\lambda-\rho)(v)} e^{\varepsilon\|v\|}\|f\|_2\|g\|_2.$$

Proof. We decompose $f, g \in L^2(\Gamma\backslash G)^K$ into $\int_X^{\oplus} f_x\, d\mu(x)$ and $\int_X^{\oplus} g_x\, d\mu(x)$, respectively, according to the decomposition $L^2(\Gamma\backslash G) \simeq \int_X^{\oplus} \pi_x\, d\mu(x)$. Since f and g are K-invariant f_x and g_x are contained in π_x^K for μ-almost every $x \in X$ and hence they vanish for almost every $x \in X$ with $\pi_x \notin \widehat{G}_{sph}$. We thus get

$$\langle R(\exp v)f, g\rangle = \int_X \langle \pi_x(\exp v)f_x, g_x\rangle\, d\mu(x) = \int_{\pi^{-1}(\widehat{G}_{sph})} \langle \pi_x(\exp v)f_x, g_x\rangle\, d\mu(x).$$

We recall that if $\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W$ corresponds to $\pi_\lambda \in \widehat{G}_{sph}$ we have

$$\langle \pi_\lambda(g)v_K, v_K\rangle = \phi_\lambda(g)\langle v_K, v_K\rangle$$

for $v_K \in \pi_\lambda^K$. Therefore,

$$\langle R(\exp v)f, g\rangle = \int_{\pi^{-1}(\widehat{G}_{sph})} \phi_{\lambda_x}(\exp v)\langle f_x, g_x\rangle\, d\mu(x).$$

Hence we can estimate

$$\begin{aligned}|\langle R(\exp v)f, g\rangle| &\leq \int_{\pi^{-1}(\widehat{G}_{sph})} |\phi_{\lambda_x}(\exp v)|\|f_x\|\|g_x\|\, d\mu(x)\\ &\leq \operatorname{esssup}_{\pi_*\mu|_{\widehat{G}_{sph}}} |\phi_{\lambda_x}(\exp v)|\|f\|_2\|g\|_2\\ &\leq \sup_{\lambda\in\widetilde{\sigma}} |\phi_\lambda(\exp v)|\|f\|_2\|g\|_2.\end{aligned}$$

For the elementary spherical function we have the well-known estimates

$$|\phi_\lambda(\exp v)| \leq e^{\Re\lambda(v)}\Xi(\exp v) \leq d_\varepsilon e^{\Re\lambda(v)} e^{-\rho(v)} e^{\varepsilon\|v\|}$$

for $\Re\lambda \in \overline{\mathfrak{a}_+^*}$ and any $\varepsilon > 0$. This completes the proof. □

We also prove an inverse statement that shows that decay of matrix coefficients in $L^2(\Gamma\backslash G)$ imply obstructions on the joint spectrum.

Lemma 3.2. *Suppose that there exists a homogeneous function $\theta\colon \mathfrak{a}_+ \to \mathbb{R}$ such that for all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for any K-invariant functions $f, g \in L^2(\Gamma\backslash G)$ and any $v \in \mathfrak{a}_+$*

$$|\langle R(\exp v)f, g\rangle| \leq d_\varepsilon e^{-\theta(v)} e^{\varepsilon\|v\|}\|f\|_2\|g\|_2.$$

Then this implies that

$$\Re\lambda \leq \rho - \theta$$

for all $\lambda \in \widetilde{\sigma}$.

Proof. Let $\tilde{\varepsilon} > 0$, $X_{sph} = \pi^{-1}(\widehat{G}_{sph})$, $\lambda_0 \in \widetilde{\sigma}$, and $A_{\tilde{\varepsilon}} := \{x \in X_{sph} \mid |\lambda_x - \lambda_0| < \varepsilon\}$. Then $\mu(A_{\tilde{\varepsilon}}) > 0$ by Proposition 2.2. Put $f_{\tilde{\varepsilon}} = \mu(A_{\tilde{\varepsilon}})^{-1/2} \int_X^{\oplus} \mathbb{1}_{A_{\tilde{\varepsilon}}}(x) w_x^K \, d\mu(x)$ where $w_x^K \in \pi_x^K$ is normalized. By definition $f_{\tilde{\varepsilon}} \in L^2(\Gamma\backslash G)^K$ is normalized and $\langle R(\exp v)f_{\tilde{\varepsilon}}, f_{\tilde{\varepsilon}}\rangle = \mu(A_\varepsilon)^{-1}\int_{A_\varepsilon} \phi_{\lambda_x}(\exp v)\, d\mu(x)$. We infer that $\phi_{\lambda_0}(\exp v) = \lim_{\tilde{\varepsilon}\to 0}\langle R(\exp v)f_{\tilde{\varepsilon}}, f_{\tilde{\varepsilon}}\rangle$ and therefore by the assumed bound on the matrix coefficients we get $|\phi_{\lambda_0}(\exp v)| \leq d_\varepsilon e^{-\theta(v)}e^{\varepsilon\|v\|}$ for any $\varepsilon > 0$. Without loss of generality assume $\Re\lambda_0 \in \overline{\mathfrak{a}_+^*}$. From [vdBS87, Thm. 3.5 and proof of Thm. 10.1] follows that there is a polynomial $p(t)$ such that

$$\phi_{\lambda_0}(\exp tv)p(t)^{-1}e^{-t(\lambda_0-\rho)(v)} \to 1 \quad \text{as} \quad t \to \infty.$$

Hence,

$$1 \leq \limsup_{t\to\infty} d_\varepsilon |p(t)|^{-1} e^{t(-\theta(v)+\varepsilon\|v\|-\Re\lambda_0(v)+\rho(v))}$$

for any $\varepsilon > 0$. We conclude

$$-\theta(v) + \varepsilon\|v\| - \Re\lambda_0(v) + \rho(v) > 0$$

and

$$\Re\lambda_0 \leq \rho - \theta.$$

This completes the proof. □

In the next Proposition we state how the polyhedral bounds on the spectrum are related to almost L^p properties for $L^2(\Gamma\backslash G)$. We also obtain the equality of Theorem 1.1 between the polyhedral norm of the spectrum and $\theta(L^2(\Gamma\backslash G))$.

Proposition 3.3.

(i) $L^2(\Gamma\backslash G)$ is tempered if and only if $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$.

(ii)

$$\sup_{\lambda\in\widetilde{\sigma}} \|\Re\lambda\|_{poly} = \theta(L^2(\Gamma\backslash G)).$$

(iii) $L^2(\Gamma\backslash G)$ is almost L^p for $p = \frac{2}{1-\theta(L^2(\Gamma\backslash G))}$.

(iv) If $L^2(\Gamma\backslash G)$ is almost L^{2k} for some $k \in \mathbb{N}$ then $\sup_{\lambda\in\widetilde{\sigma}} \|\Re(\lambda)\|_{poly} \leq (1 - \frac{1}{k})$.

Proof. We start with proving (ii): By definition of $\theta = \theta(L^2(\Gamma\backslash G))$ we have

$$|\langle R(\exp v)f_1, f_2\rangle| \leq d_\varepsilon e^{(\theta+\varepsilon-1)\rho(v)}\|f_1\|\|f_2\|$$

for all $\varepsilon > 0$, $v \in \overline{\mathfrak{a}_+}$, and $f_1, f_2 \in L^2(\Gamma\backslash G)^K$. By Lemma 3.2 this implies

$$\Re\lambda(v) \leq \theta\rho(v)$$

for every $v \in \overline{\mathfrak{a}_+}$ and $\lambda \in \widetilde{\sigma}$, i.e. $\|\Re\lambda\|_{poly} \leq \theta$. On the other hand we have

$$|\langle R(\exp v)f_1, f_2\rangle| \leq d_\varepsilon e^{\sup_{\lambda\in\widetilde{\sigma}}(\Re\lambda-\rho)(v)}e^{\varepsilon\|v\|}\|f\|_2\|f_2\|_2$$

by Lemma 3.1. It follows that if $\Re\lambda(v) \leq \theta'\rho(v)$ for every $\lambda \in \widetilde{\sigma}$ and $v \in \overline{\mathfrak{a}_+}$ for some $\theta' \in [0,1]$ then $\theta \leq \theta'$. We conclude

$$\theta = \inf\{\theta' \in [0,1] \mid \Re\lambda(v) \leq \theta'\rho(v) \,\forall\, v \in \overline{\mathfrak{a}_+}, \lambda \in \widetilde{\sigma}\} = \sup_{\lambda\in\widetilde{\sigma}} \|\Re\lambda\|_{poly}.$$

We continue with proving (iii): Let $q > 2/(1-\theta)$ and consider $f_1, f_2 \in C_c(\Gamma\backslash G) \subset L^2(\Gamma\backslash G)$ which is a dense subspace. Then by setting $\tilde{f}_i(g) \coloneqq \max_{k\in K} |f_i(gk)|$ we get right K-invariant functions and compute

$$\begin{aligned}\int_G |\langle R(g)f_1, f_2\rangle|^q dg &\leq \int_G |\langle R(g)\tilde{f}_1, \tilde{f}_2\rangle|^q dg \\ &\leq \int_{\mathfrak{a}_+} |\langle R(\exp(H))\tilde{f}_1, \tilde{f}_2\rangle|^q e^{2\rho(H)} dH\end{aligned}$$

As for (ii) we use the definition of $\theta = \theta(L^2(\Gamma\backslash G))$ to obtain

$$\int_G |\langle R(g)f_1, f_2\rangle|^q dg \leq d_\varepsilon \|\tilde{f}_1\|\|\tilde{f}_2\| \int_{\mathfrak{a}_+} e^{(q(\theta-1)+2)\rho(H)-q\varepsilon\|H\|}) dH.$$

By our choice of q this is integrable for ε sufficiently small.

We next consider (iv): If $L^2(\Gamma\backslash G)$ is almost L^{2k} then by Proposition 2.5 we get for any $f_1, f_2 \in L^2(\Gamma\backslash G)^K$

$$|\langle R(g)f_1, f_2\rangle| \leq \|f_1\|\|f_2\|(\Xi(g))^{\frac{1}{k}}$$

and thus by (2.2) for any $\varepsilon > 0$

$$|\langle R(\exp(v))f_1, f_2\rangle| \leq d_\varepsilon e^{-\frac{1}{k}\rho(v)} e^{\varepsilon\|v\|} \|f_1\|\|f_2\|.$$

Consequently, $\theta(L^2(\Gamma\backslash G)) \leq 1 - 1/k$.

Finally (i) follows from (ii), (iii), and (iv) because temperedness is equivalent to being almost L^2. □

4. Decay of matrix coefficients and the growth indicator function

In this section we want to study the connection between the decay of matrix coefficients. We start with a slight modification of [LO23, Prop. 7.3].

Lemma 4.1. *Suppose that there exists a homogeneous function $\theta\colon \mathfrak{a}_+ \to \mathbb{R}$ such that for any $\varepsilon > 0$ there is $d_\varepsilon > 0$ such that for any K-invariant functions $f, g \in L^2(\Gamma\backslash G)^K$, any $v \in \mathfrak{a}_+$,*

$$|\langle R(\exp v)f, g\rangle| \leq d_\varepsilon e^{-\theta(v)} e^{\varepsilon\|v\|} \|f\|_2 \|g\|_2. \tag{4.1}$$

Then this implies

$$\psi_\Gamma \leq 2\rho - \theta.$$

Note that by continuity of both sides in f, g it is enough to verify (4.1) for $f, g \in C_c(\Gamma\backslash G)^K$ since C_c is dense in L^2.

Proof. One fixes a unit vector $u \in \mathfrak{a}^+$ and a cone $\mathcal{C}$ containing u and consider the cutoff cone $\mathcal{C}_T \coloneqq \{v \in \mathcal{C}, \|v\| \leq T\}$. Then with the same arguments as [LO23, Prop. 7.3] one gets

$$\#(\mu(\Gamma) \cap \mathcal{C}_T) \leq C e^{(T+\varepsilon)((2\rho-\theta)u+\varepsilon\|u\|)+2(T+\varepsilon)\eta_{\mathcal{C}}},$$

with

$$\eta_{\mathcal{C}} := \sup\{|\rho(u) - \rho(v)|, v \in \mathcal{C}, \|v\| = 1\}$$

Therefore,

$$\limsup_{T\to\infty} \frac{1}{T} \log \#(\mu(\Gamma) \cap \mathcal{C}_T) \le (2\rho - \theta)(u) + \varepsilon\|u\| + 2\eta_{\mathcal{C}}.$$

This implies $\psi_\Gamma(u) \le (2\rho - \theta)(u) + \varepsilon\|u\|$ and the lemma by letting $\varepsilon \to 0$ and $\mathcal{C} \to \mathbb{R}_+ u$. $\square$

As a direct consequence of Lemma 4.1 and 3.1 we get the following proposition.

Proposition 4.2.

$$\psi_\Gamma(v) \le \sup_{\lambda \in \widetilde{\sigma}} \Re\lambda(v) + \rho(v).$$

Note that this bound on the counting function is even a little bit more precise compared to the bounds stated in the main theorem, because the right hand side is not simply a dilation of ρ but might be a more precise functional.

For the converse we will prove

Proposition 4.3. *For all $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that for all $f_1, f_2 \in L^2(\Gamma\backslash G)^K$ we have*

$$|\langle R(\exp v) f_1, f_2\rangle| \le d_\varepsilon e^{\varepsilon\|v\|} e^{(\max(0, \|\psi_\Gamma - \rho\|_{poly}) - 1)\rho(v)} \|f_1\|\|f_2\|.$$

In order to prove this theorem let us first state the following lemma of Cowling.

Lemma 4.4 ([Cow23, Lemma 3.5])**.** *Let $t \in [0,1]$ and $(\pi, \mathcal{H})$ a unitary representation of G. Then the following statements are equivalent:*

(i) For all ξ and η in a dense subspace $\mathcal{H}^0$ of $\mathcal{H}$, there is a constant $C(\xi, \eta)$ such that

$$\left(\int_K \int_K |\langle \pi(kxk')\xi, \eta\rangle|^2 \, dk \, dk'\right)^{1/2} \le C(\xi, \eta)\phi_{t\rho}(x) \quad \forall x \in G;$$

(ii) For all ξ and η in $\mathcal{H}$,

$$\left(\int_K \int_K |\langle \pi(kxk')\xi, \eta\rangle|^2 \, dk \, dk'\right)^{1/2} \le \|\xi\|_{\mathcal{H}} \|\eta\|_{\mathcal{H}} \phi_{t\rho}(x) \quad \forall x \in G.$$

The further key ingredient is the following decay of matrix coefficient for compactly supported functions.

Theorem 4.5. *Let $f_1, f_2 \in C_c(\Gamma\backslash G)$, $H_0 \in \overline{\mathfrak{a}_+}$ normalized, and $s > \psi_\Gamma(H_0)$, $s \ge 0$. Then there exists $\delta > 0$ and $C > 0$ such that*

$$|\langle R(\exp tH) f_1, f_2\rangle_{L^2(\Gamma\backslash G)}| \le C e^{t(s - 2\rho(H))}$$

for all $t \ge 0$ and $H \in B_\delta(H_0)$ normalized.

Remark. If H_0 is not in the limit cone and therefore $\psi_\Gamma(H_0) = -\infty$, then choose $\delta > 0$ s.t. $B_\delta(H_0) \cap C_\Gamma = \emptyset$ and we get by Definition of the limit cone

$$\langle R(\exp(tH)) f_1, f_2\rangle = 0$$

for $H \in B_\delta(H_0)$ and t large enough.

Remark. It should further be noted, that if $\psi_\Gamma \le \rho$, then the exponent in Theorem 4.5 is smaller then in Proposition 4.3, where the decay is studied for L^2 functions. This is a well known phenomenon for example for decay estimates for geodesic flows on convex co-compact hyperbolic surfaces with $\delta_\Gamma < \frac{1}{2}$.

Before proving Theorem 4.5 let us first show how it implies Proposition 4.3.

Proof of Proposition 4.3. Let us fix an arbitrary $\varepsilon > 0$. Based on Theorem 4.5 we will show that the matrix coefficients for functions in $C_c(\Gamma\backslash G)$ satisfy (i) of Lemma 4.4. Let $f_1, f_2 \in C_c(\Gamma\backslash G)$. Since

$$\left|\int_{\Gamma\backslash G} f_1(\Gamma gh)f_2(\Gamma g)\, d\Gamma g\right| \leq \int_{\Gamma\backslash G} \max_{k\in K}|f_1(ghk)| \max_{k\in K}|f_2(gk)|\, dg \tag{4.2}$$

we can assume that f_i is non-negative and right K-invariant. Therefore,

$$\left(\int_K\int_K |\langle R(kxk')f_1, f_2\rangle|^2\, dk\, dk'\right)^{1/2} = |\langle R(x)f_1, f_2\rangle| = |\langle R(\exp v)f_1, f_2\rangle|$$

for $x \in K(\exp v)K$ with $v \in \overline{\mathfrak{a}_+}$.

Let $\theta' := \max(0, \|\psi_\Gamma - \rho\|_{poly})$ so that $\psi_\Gamma \leq (1+\theta')\rho$ and $\theta' \geq 0$. For any $H_0 \in \overline{\mathfrak{a}_+}$, we can find an $s_{H_0} \geq 0$ such that $\psi_\Gamma(H_0) < s_{H_0} < (1+\theta'+\varepsilon)\rho(H_0)$. Then by Theorem 4.5 for any $H_0 \in \overline{\mathfrak{a}_+}$ normalized, there is $\delta > 0$ and $C > 0$ such that

$$|\langle R(\exp tH)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}| \leq Ce^{t(s_{H_0}-2\rho(H))}$$

for all $t \geq 0$ and $H \in B_\delta(H_0)$. By shrinking δ we can assume that $s_{H_0} < (1+\theta'+\varepsilon)\rho(H)$ for any $H \in \overline{B}_\delta(H_0)$. By compactness of the unit sphere in $\mathfrak{a}$ we only need finitely many H_0^i in order to have

$$\overline{\mathfrak{a}_+} \subseteq \bigcup_i \mathbb{R}_+\cdot\tilde{B}_i \text{ where } \tilde{B}_i := B_{\delta_i}(H_0^i)\cap\{H\in\mathfrak{a}, \|H\|=1\}.$$

Therefore,

$$|\langle R(\exp tH)f_1, f_2\rangle_{L^2(\Gamma\backslash G)}| \leq C\max_i e^{t\left(s_{H_0^i}-2\rho(H)\right)} \leq Ce^{(\theta'-\varepsilon-1)\rho(tH)}$$

for $t \geq 0$ and $H \in \overline{\mathfrak{a}_+}$ normalized. Hence

$$|\langle R(\exp v)f_1, f_2\rangle| \leq C\phi_{(\theta'+\varepsilon)\rho}(\exp v) = C\phi_{(\theta'+\varepsilon)\rho}(x)$$

(see e.g. [Cow23, Thm. 2.5]). We now apply Lemma 4.4 to obtain

$$|\langle R(\exp v)f_1, f_2\rangle| \leq \phi_{(\theta'+\varepsilon)\rho}(\exp v)\|f_1\|\|f_2\|$$

for all $f_1, f_2 \in L^2(\Gamma\backslash G)^K$. Since $\phi_{(\theta'+\varepsilon)\rho}(\exp v) \leq Ce^{(\theta'+\varepsilon)\rho(v)}$ as $\theta' \geq 0$ (see again [Cow23, Thm. 2.5]) and $\rho(v) \leq \varepsilon'\|v\|$ uniformly in $v \in \overline{\mathfrak{a}_+}$ the theorem follows. □

Before proving Theorem 4.5 let us prove the following lemma that is certainly known to experts but might still be of independent interest.

Recall, that by Bruhat's decomposition (see [Hel84, Prop. I.5.21]) that the mapping

$$(\overline{n}, m, a, n) \mapsto \overline{n}man \in G$$

is a bijection of $\overline{N}\times M\times A\times N$ onto an open submanifold of G whose complement has Haar measure 0. Moreover,

$$\int_G f(g)\, dg = \int_{\overline{N}\times M\times A\times N} f(\overline{n}man)e^{2\rho(\log a)}\, d\overline{n}\, dm\, da\, dn.$$

Lemma 4.6. *Let $\varphi_1, \varphi_2 \in C_c(G)$ with $\operatorname{supp}\varphi_i \subseteq \overline{N}MAN$. Then there is a constant $C = C_{\varphi_1,\varphi_2}$ such that for all $h \in A$*

$$\left|\int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg\right| \leq Ce^{-2\rho(\log h)}.$$

Proof. By the triangle inequality we can assume that $\varphi_i \geq 0$. Since $\operatorname{supp}\varphi_i \subseteq \overline{N}MAN$ there exist compact sets $C_{\overline{N}} \subseteq \overline{N}$, $C_A \subseteq A$, and $C_N \subseteq N$ with $\operatorname{supp}\varphi_i \subseteq C_{\overline{N}}MC_AC_N$. We thus have

$$\begin{aligned} c := c_{\varphi_1,\varphi_2,h} &:= \int_G \varphi_1(h^{-1}gh)\varphi_2(g)\,dg \\ &= \int_{C_{\overline{N}}\times M\times C_A\times C_N} \varphi_1(h^{-1}\overline{n}manh)\varphi_2(\overline{n}man)e^{2\rho(\log a)}\,d\overline{n}\,dm\,da\,dn \\ &\leq \|\varphi_2\|_\infty \int_{C_{\overline{N}}\times M\times C_A\times C_N} \varphi_1(h^{-1}\overline{n}manh)e^{2\rho(\log a)}\,d\overline{n}\,dm\,da\,dn. \end{aligned}$$

Since M centralizes A and A is abelian

$$c \leq \|\varphi_2\|_\infty \int_{C_{\overline{N}}\times M\times C_A\times C_N} \varphi_1(h^{-1}\overline{n}hmah^{-1}nh)e^{2\rho(\log a)}\,d\overline{n}\,dm\,da\,dn.$$

Estimating φ_1 by its absolute value and using that A normalizes both N and $\overline{N}$ we get

$$\begin{aligned} c &\leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_M dm \int_{C_A} e^{2\rho(\log a)}\,da \int_{C_{\overline{N}}\cap hC_{\overline{N}}h^{-1}} d\overline{n} \int_{C_N\cap hC_Nh^{-1}} dn \\ &\leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_M dm \int_{C_A} e^{2\rho(\log a)}\,da \int_{C_N} dn \int_{hC_{\overline{N}}h^{-1}} d\overline{n}. \end{aligned}$$

Since the Jacobian factor for the diffeomorphism $\overline{n} \mapsto h^{-1}\overline{n}h$ of $\overline{N}$ is $\det \operatorname{Ad}(h)|_{\overline{\mathfrak{n}}} = e^{-2\rho(\log h)}$ we have

$$\int_{hC_{\overline{N}}h^{-1}} d\overline{n} = \int_{\overline{N}} 1_{C_{\overline{N}}}(h^{-1}\overline{n}h)\,d\overline{n} = \int_{\overline{N}} 1_{C_{\overline{N}}}(\overline{n})e^{-2\rho(\log h)}\,d\overline{n} = \int_{C_{\overline{N}}} d\overline{n}\,e^{-2\rho(\log h)}.$$

We conclude

$$c_{\varphi_1,\varphi_2,h} \leq \|\varphi_1\|_\infty\|\varphi_2\|_\infty \int_M dm \int_{C_A} e^{2\rho(\log a)}\,da \int_{C_N} dn \int_{C_{\overline{N}}} d\overline{n}\,e^{-2\rho(\log h)} = C_{\varphi_1,\varphi_2}e^{-2\rho(\log h)}$$

proving the theorem. □

Let us now prove Theorem 4.5.

Proof of Theorem 4.5. Let $f_1, f_2 \in C_c(\Gamma\backslash G)$. We can find $\tilde{f}_i \in C_c(G)$ such that $f_i(\Gamma g) = \sum_{\gamma\in\Gamma} \tilde{f}_i(\gamma g)$.

We then have

$$\begin{aligned} \langle R(h)f_1, f_2\rangle_{L^2(\Gamma\backslash G)} &= \int_{\Gamma\backslash G} f_1(\Gamma gh)f_2(\Gamma g)\,d\Gamma g = \int_G \tilde{f}_1(gh)f_2(\Gamma g)\,dg \\ &= \sum_{\gamma\in\Gamma}\int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\,dg. \end{aligned} \tag{4.3}$$

For any $g \in G$ there is an open neighborhood U_g of g such that $U_g^{-1}U_g \subseteq \overline{N}MAN$ since $\overline{N}MAN$ is an open neighborhood of the identity element. Since $\operatorname{supp}\tilde{f}_i$ is compact there are finitely many g_k such that $\operatorname{supp}\tilde{f}_i \subseteq \bigcup_k U_{g_k}$. There exists a partition of unity χ_k subordinate to U_{g_k}, i.e. $\chi_k \in C_c(G)$ with $\operatorname{supp}\chi_k \subseteq U_{g_k}$ and $\sum_k \chi_k(x) = 1$ for all $x \in \operatorname{supp}\tilde{f}_i$. We decompose $\tilde{f}_i$ as $\sum_k \chi_k\tilde{f}_i$ in (4.3). This allows us to assume without loss of generality that $\operatorname{supp}\tilde{f}_i$ is contained in some U_g, since we can estimate each of the finite summands individually. In particular, we can assume that $(\operatorname{supp}\tilde{f}_i)^{-1}\operatorname{supp}\tilde{f}_i \subseteq \overline{N}MAN$.

Let $\gamma \in \Gamma$ such that $\int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\, dg \neq 0$. Then there is $g \in G$ with $gh \in \operatorname{supp} \tilde{f}_1$ and $\gamma g \in \operatorname{supp} \tilde{f}_2$. Therefore, $\gamma \in (\operatorname{supp} \tilde{f}_2)g^{-1} \subseteq \operatorname{supp} \tilde{f}_2 h(\operatorname{supp} \tilde{f}_1)^{-1}$. Hence, there are s_1 and s_2 in $\operatorname{supp} \tilde{f}_1$ and $\operatorname{supp} \tilde{f}_2$, respectively, with $\gamma = s_2 h s_1^{-1}$. By change of variables

$$\int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\, dg = \int_G \tilde{f}_1(gh)\tilde{f}_2(s_2 h s_1^{-1} g)\, dg = \int_G \tilde{f}_1((hs_1^{-1})^{-1} gh)\tilde{f}_2(s_2 g)\, dg$$
$$= \int_G \tilde{f}_1(s_1 h^{-1} gh)\tilde{f}_2(s_2 g)\, dg.$$

If we define $\varphi_i(g) := \max_{s \in \operatorname{supp} \tilde{f}_i} |\tilde{f}_i(sg)|$ we can estimate

$$\left| \int_G \tilde{f}_1(gh)\tilde{f}_2(\gamma g)\, dg \right| \leq \int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg.$$

Hence we have

$$|\langle R(h)f_1, f_2\rangle| \leq \#(\Gamma \cap (\operatorname{supp} \tilde{f}_2)h(\operatorname{supp} \tilde{f}_1)^{-1}) \int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg.$$

Note that if $\varphi_i(g) \neq 0$ then there is $s \in \operatorname{supp} \tilde{f}_i$ such that $sg \in \operatorname{supp} \tilde{f}_i$. Hence, $\operatorname{supp} \varphi_i \subseteq (\operatorname{supp} \tilde{f}_i)^{-1} \operatorname{supp} \tilde{f}_i$ is compact and contained in $\overline{N}MAN$. Therefore, by Lemma 4.6

$$\int_G \varphi_1(h^{-1}gh)\varphi_2(g)\, dg \leq Ce^{-2\rho(\log h)}.$$

The theorem now follows from Lemma 4.7 and Lemma 4.8 below. □

Lemma 4.7 (see [Ben96, Prop. 5.1])**.** *For all compact sets $C \subseteq G$ there exists a compact set $L \subseteq \mathfrak{a}$ such that $\mu(CgC) \subseteq \mu(g) + L$.*

Lemma 4.8. *For all $H_0 \in \mathfrak{a}_+$ normalized, all $L \subseteq \mathfrak{a}$ compact, all t large enough, and all $s > \psi_\Gamma(H_0)$ with $s \geq 0$ there exists $\delta > 0$ and $C > 0$ such that*

$$\#\{\gamma \in \Gamma \mid \mu(\gamma) \in tH + L\} \leq Ce^{ts}$$

for $H \in B_\delta(H_0)$ normalized.

Remark. If $\psi_\Gamma(H_0) < 0$ then H_0 is not in the limit cone and $\psi_\Gamma(H_0) = -\infty$. Moreover, there is an open cone containing H_0 that contains only finitely many Γ points. In particular, $\{\gamma \in \Gamma \mid \mu(\gamma) \in tH + L\}$ is empty for t large enough depending on H_0.

Proof. By definition there exists an open cone $\mathcal{C}$ containing H_0 such that

$$\sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-s\|\mu(\gamma)\|} < \infty.$$

Therefore for any $R > 0$, there is $C > 0$ such that

$$\#\{\gamma \mid \mu(\gamma) \in \mathcal{C}, \|\mu(\gamma)\| \in]t - R, t + R]\} \leq Ce^{ts}.$$

Note that for every $\delta > 0$ with $\overline{B_\delta(H_0)} \subseteq \mathcal{C}$ there is $t_0 > 0$ such that $tH + L \subseteq \mathcal{C}$ for every $t \geq t_0$ and $H \in B_\delta(H_0)$. If we take $R > 0$ is such that $L \subseteq B_R(0)$ then we can estimate for all $t \geq t_0$ and $H \in B_\delta(H_0)$, normalized

$$\#\{\gamma \mid \mu(\gamma) \in tH + L\} \leq \#\{\gamma \mid \mu(\gamma) \in \mathcal{C}, \|\mu(\gamma)\| \in]t\|H\| - R, t\|H\| + R]\}$$
$$\leq Ce^{ts}.$$

□

5. Main Theorem

We finally want to prove our main theorem which we formulate in a more detailed form compared to the introduction.

Theorem 5.1. *Let G be a real semisimple connected non-compact Lie group with finite center and $\Gamma < G$ a discrete and torsion-free subgroup. Then for all $\theta' \in [0,1]$ the following statements are equivalent:*

(i) $\Re\widetilde{\sigma} \subseteq \theta' \operatorname{conv}(W\rho)$.

(ii) $\theta(L^2(\Gamma\backslash G)) \leq \theta'$, *i.e. for all* $\varepsilon > 0$, *there is* $d_\varepsilon > 0$ *such that for all* $f_1, f_2 \in L^2(\Gamma\backslash G)^K$*:*

$$|\langle R(\exp v) f_1, f_2\rangle| \leq d_\varepsilon e^{\varepsilon\|v\|} e^{(\theta'-1)\rho(v)} \|f_1\|\|f_2\|.$$

(iii) $\psi_\Gamma \leq (1+\theta')\rho$.

and imply

(iv) $L^2(\Gamma\backslash G)$ *is almost* L^p *for* $p \leq \frac{2}{1-\theta'}$.

(v) $\inf \sigma(\Delta) \geq (1-\theta'^2)\|\rho\|^2$.

Furthermore, (iv) implies (i) if $p \in 2\mathbb{N}$.

Proof. (i) and (ii) are equivalent by Lemma 3.1 and 3.2 (see also Proposition 3.3). (ii) implies (iii) by Lemma 4.1. (iii) implies (ii) by Proposition 4.3. (ii) implies (iv) by Proposition 3.3 (iii). (iii) implies (v) by [WZ23, Cor. 1.4]:

$$\begin{aligned}\inf \sigma(\Delta) &= \|\rho\|^2 - \max\left\{0, \sup_{H\in\overline{\mathfrak{a}_+}} \frac{\psi_\Gamma(H) - \langle\rho, H\rangle}{\|H\|}\right\}^2 \\ &\geq \|\rho\|^2 - \theta'^2 \left(\sup_{H\in\overline{\mathfrak{a}_+}} \frac{\langle\rho, H\rangle}{\|H\|}\right)^2 = (1-\theta'^2)\|\rho\|^2.\end{aligned}$$

(iv) implies (i) for $p \in 2\mathbb{N}$ by Proposition 3.3 (iv). ☐

We obtain Theorem 1.1 by taking the infimum.

Proof of Theorem 1.1. Since $\operatorname{conv}(W\rho) = \{\lambda \in \mathfrak{a}^* \mid \|\lambda\|_{poly} \leq 1\}$,

$$\inf\left\{\theta' \in [0,1] \text{ such that (i) holds}\right\} = \sup_{\lambda\in\widetilde{\sigma}} \|\Re\lambda\|_{poly}.$$

Since $\psi_\Gamma(H) = -\infty$ for $H \notin \overline{\mathfrak{a}_+}$ we infer that $\|\psi_\Gamma - \rho\|_{poly} = \sup_{H\in\overline{\mathfrak{a}_+}} \frac{\psi_\Gamma(H)-\rho(H)}{\rho(H)}$. Therefore,

$$\inf\left\{\theta' \in [0,1] \text{ such that (iii) holds}\right\} = \max\left(0, \|\psi_\Gamma - \rho\|_{poly}\right).$$

Note that we don't need the additional error $e^{\varepsilon\|v\|}$ in (ii) as we take the infimum. Hence Theorem 1.1 follows from Theorem 5.1. ☐

6. Examples of precise descriptions of the spectrum

In this last section we want to consider two concrete examples: The product case $G = G_1 \times G_2$ of two rank one groups and the case $G = \mathrm{SL}_3(\mathbb{R})$. In the product case we also consider the product of two discrete subgroups $\Gamma = \Gamma_1 \times \Gamma_2$, such that the spectral theory of the joint spectrum of invariant differential operators trivially reduces to the rank one case. Nevertheless we think that it is quite instructive to illustrate the main result in this concrete example. In the case of $\mathrm{SL}_3(\mathbb{R})$ we show that using the additional information of the root system A_2 with our main result allows us to deduce some even finer information about the spectrum.

6.1. **Product case.** Let us first consider the product case in which the joint spectrum is explicitly given by the product of the two rank-one spectra and which yields a nice illustration of our result: More precisely, let $G = G_1 \times G_2$ be the product of two rank one groups G_i, $i = 1, 2$. We indicate by the subscript i the corresponding subspaces resp. subgroups of G_i resp. its Lie algebra. Assume that the discrete subgroup Γ is also a product of discrete subgroups Γ_i of G_i. Clearly,

$$\widetilde{\sigma} = \{(\lambda_1, \lambda_2) \in \mathfrak{a}_1/\{\pm 1\} \times \mathfrak{a}_2/\{\pm 1\} \mid |\rho_i|^2 - |\Re \lambda_i|^2 + |\Im \lambda_i|^2 \in \sigma(\Delta_i)\}, \tag{6.1}$$

where Δ_i is the Laplacian of $\Gamma_i \backslash G_i/K_i$ acting on one factor of $\Gamma\backslash G/K$. Recall that $\inf \sigma(\Delta_i) = |\rho_i|^2 - \max(0, \delta_{\Gamma_i} - |\rho_i|)^2$, where δ_{Γ_i} is the critical exponent of Γ_i. Let us assume that $\delta_{\Gamma_i} \geq \rho_i$ for notational simplicity. Then we additionally know that $|\rho_i|^2 - (\delta_{\Gamma_i} - |\rho_i|)^2$ is an L^2 eigenvalue for the Laplacian Δ_i on $\Gamma_i \backslash G_i/K_i$.

One easily checks, that

$$\theta := \|\psi_\Gamma - \rho\|_{poly} = \max\left(\frac{\delta_{\Gamma_1}}{\rho_1} - 1, \frac{\delta_{\Gamma_2}}{\rho_2} - 1\right)$$

and our main theorem implies that the real part of the joint spectrum has to be contained in

$$\Re\widetilde{\sigma} \subseteq [-\theta\rho_1, \theta\rho_1] \times [-\theta\rho_2, \theta\rho_2] = \theta \operatorname{conv}(W(\rho_1, \rho_2)) \tag{6.2}$$

by identifying $(\mathfrak{a}_i)^*_{\mathbb{C}}$ with $\mathbb{C}$ through a choice of a normalized functional.

In fact (6.1) implies that

$$\Re\widetilde{\sigma} \subseteq [-(\delta_{\Gamma_1} - \rho_1), \delta_{\Gamma_1} - \rho_1] \times [-(\delta_{\Gamma_2} - \rho_2), \delta_{\Gamma_2} - \rho_2] \tag{6.3}$$

which is compatible with the polyhedral bound (6.2). Furthermore, as $|\rho_i|^2 - (\delta_{\Gamma_i} - |\rho_i|)^2$ are L^2 eigenvalues on Δ_i on $\Gamma_i\backslash G_i/K_i$ we deduce that $(\pm(\delta_1 - \rho_1), \pm(\delta_2 - \rho_2)) \in \widetilde{\sigma}$ and that they even correspond to joint L^2-eigenvalues of (Δ_1, Δ_2) now acting on the product space $\Gamma\backslash G/K$. These joint eigenvalues all lie on the boundary of $\theta \operatorname{conv}(W(\rho_1, \rho_2))$ and illustrates that the polyhedral bound is sharp. Furthermore we see that this peripheral eigenvalue can occur basically at any position on the polyhedron depending on the ratio of δ_1 and δ_2.

Finally note that while $(\delta_1 - \rho_1, \delta_2 - \rho_2)$ is a discrete joint L^2 eigenvalue there are also continuous spectral families on the boundaries: At least if one assumes that Γ_i are geometrically finite and non-cocompact it is known that $\sigma(\Delta_i)$ contain continuous spectrum $[\rho_i^2, \infty[\subset \sigma(\Delta_i)$. In view of (6.1) this yields that there are continuous families of joint spectra $(\pm(\delta_1 - \rho_1), i\mathbb{R}) \in \widetilde{\sigma}$ and $(i\mathbb{R}, \pm(\delta_2 - \rho_2)) \in \widetilde{\sigma}$ and at least two of these families also lie on the boundary of the polyhedral region.

6.2. $\mathbf{SL_3}(\mathbb{R})$ **case.** In the example $G = \mathrm{SL}_3(\mathbb{R})$ or more generally if the root system of restricted roots is A_2, there are two simple roots α_1, α_2 with an angle of $2\pi/3$. The half sum of positive roots ρ is a multiple of the third positive root $\alpha_3 = \alpha_1 + \alpha_2$. The Weyl group consists of 6 elements, 3 rotations of an angle of $0, 2\pi/3$, $4\pi/3$, as well as the three reflections along the three positive roots. Since $\widehat{G}_{sph} \subseteq \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid -\overline{\lambda} \in W\lambda\}$, we must have $\Re\lambda \in \mathbb{R}\alpha_i$ with $\Im\lambda \in \alpha_i^\perp$ if $\Re\lambda \neq 0$ for some $i = 1, 2, 3$ for every $\lambda \in \widetilde{\sigma}$. In this case if $\lambda \in \widetilde{\sigma}$ with $\Re\lambda = r\rho \in \mathfrak{a}^*_+$, then $\|\Re\lambda\|_{poly} = r \in [0, 1]$.

By Theorem 1.1 there is $\theta = \max(0, \|\psi_\Gamma - \rho\|_{poly}) \in [0, 1]$ such that $\sup_{\lambda\in\widetilde{\sigma}} \|\Re\lambda\|_{poly} = \theta$. However, as mentioned in the introduction there might be exceptional spectrum on the

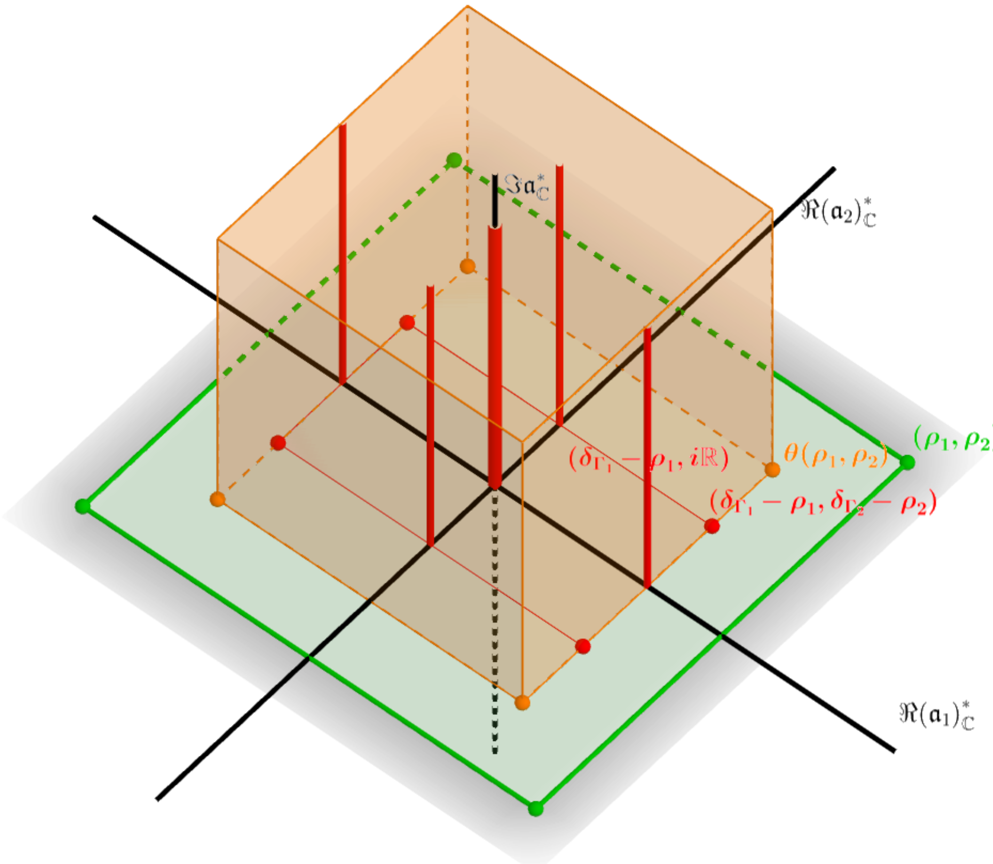

FIGURE 2. Joint spectrum in the product case under the assumption that the single factors have no exceptional eigenvalues besides $\delta_{\Gamma_i} - \rho_i$. There is a joint eigenvalue $(\delta_{\Gamma_1} - \rho_1, \delta_{\Gamma_2} - \rho_2)$ but also continuous spectrum $\pm(\delta_{\Gamma_1} - \rho_1, i\mathbb{R})$ and $\pm(i\mathbb{R}, \delta_{\Gamma_2} - \rho_2)$ as well as $i\mathbb{R} \times i\mathbb{R}$ (red). One observes that the rectangle with corner $\theta(\rho_1, \rho_2)$ (orange) is as small as possible but the peripheral spectral value $(\delta_{\Gamma_1} - \rho_1, \delta_{\Gamma_2} - \rho_2)$ can occur at any place of the boundary.

boundary with $\Re(\lambda) \neq 0$ and $\Im(\lambda) \neq 0$ thus we a priori don't know whether there is $\lambda \in \widetilde{\sigma}$ on the boundary of the described region $\|\cdot\|_{poly} = \theta$ that is real. In this example we will however be able to prove that this has to be the case. By definition if $\theta > 0$ then there is $H_0 \in \overline{\mathfrak{a}_+}$ such that $\psi_\Gamma(H_0) = (1+\theta)\rho(H_0)$ and $\psi_\Gamma \leq (1+\theta)\rho$. However, ψ_Γ and ρ are invariant under the opposition involution

$$\iota(H) \coloneqq -w_0 H, \text{ where } w_0 \in W \text{ is the longest Weyl group element}$$

which is the negative of the reflection along α_3, i.e. the reflection on $\mathbb{R}\alpha_3 = \mathbb{R}\rho$. Therefore, $\frac{1}{2}(H_0 + \iota(H_0)) \in \mathbb{R}\rho$ and $\psi_\Gamma(\frac{1}{2}(H_0 + \iota(H_0)) \geq \frac{1}{2}\psi_\Gamma(H_0) + \frac{1}{2}\psi_\Gamma(\iota(H_0)) = \psi_\Gamma(H_0)$ by concavity and i-invariance of ψ_Γ. Therefore, we can assume without loss of generality that $H_0 \in \mathbb{R}\rho$.

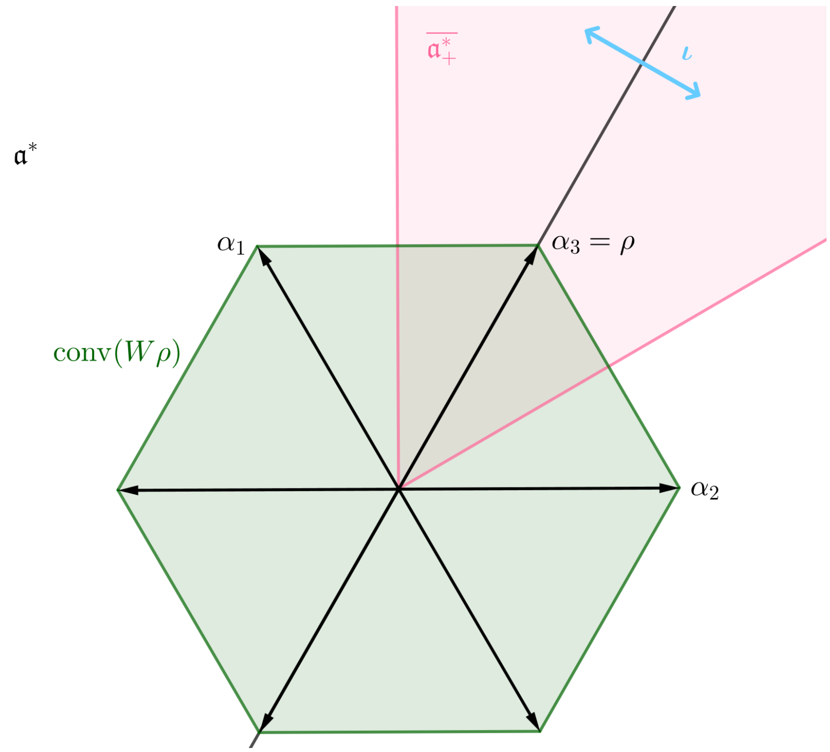

FIGURE 3. The root system of restricted roots for $\mathrm{SL}_3(\mathbb{R})$. The opposition involution ι is the reflection on the line spanned by ρ.

Let us take a look at the bottom $\inf \sigma(\Delta)$ of the Laplace spectrum. By [WZ23, Cor. 1.4]

$$\inf \sigma(\Delta) = |\rho|^2 - \max(0, \sup_{|H|=1} \psi_\Gamma(H) - \rho(H))^2.$$

There exists $\lambda \in \widetilde{\sigma}$ with $\Re\lambda \in \mathfrak{a}_+^* \cup \{0\}$ and $\inf \sigma(\Delta) = \chi_\lambda(\Delta) = |\rho|^2 - |\Re\lambda|^2 + |\Im\lambda|^2$. Hence,

$$|\Re\lambda|^2 = |\Im\lambda|^2 + \max\left(0, \sup_{|H|=1} \psi_\Gamma(H) - \rho(H)\right)^2 \geq |\Im\lambda|^2 + \left(\frac{\theta\rho(H_0)}{|H_0|}\right)^2 = |\Im\lambda|^2 + \theta^2|\rho|^2.$$

Here we used $\psi_\Gamma(H_0) = (1+\theta)\rho(H_0)$ for the inequality and $H_0 \in \mathbb{R}\rho$ for the last equality. On the other hand, since $\|\Re\lambda\|_{poly} \leq \theta$ we have $|\Re\lambda| \leq \theta|\rho|$. We conclude that $\Im\lambda = 0$ and $\Re\lambda = \theta\rho$, i.e. $\theta\rho \in \widetilde{\sigma}$.

The case $\theta = 0$ means that $\psi_\Gamma \leq \rho$ and $\widetilde{\sigma} \subseteq i\mathfrak{a}^*$, as well as $\inf \sigma(\Delta) = |\rho|^2$. Therefore, $\lambda \in \widetilde{\sigma}$ with $\chi_\lambda(\Delta) = |\rho|^2$ has to be 0.

To summarize, in the A_2 case $\sup_{\lambda \in \widetilde{\sigma}} \|\Re\lambda\|_{poly} = \theta$ is achieved at $\theta\rho$. Let us emphasize, that our analysis provides no information whether $\theta\rho$ is an isolated joint L^2-eigenvalue or is part of continuous spectrum. However, as the joint spectral value $\theta\rho$ corresponds to the bottom of the L^2 spectrum the recent work of [EFLO23] implies that (for Zariski dense Γ) $\theta\rho$ cannot correspond to a joint L^2 eigenvalue of $\mathbb{D}(G/K)$ because otherwise the bottom of the spectrum would be a L^2 eigenvalue contradicting their result. We think that studying the properties of the spectrum inside the polyhedral tubes is a highly interesting question which should be addressed in the future.

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