MEAN SQUARE OF EISENSTEIN SERIES.

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Abstract. We study the sup-norm and mean-square-norm problems for Eisenstein series on certain arithmetic hyperbolic orbifolds, producing sharp exponents for the modular surface and Picard 3-fold. The methods involve bounds for Epstein zeta functions, and counting restricted values of indefinite quadratic forms at integer points.

1. Introduction

For a compact Riemannian manifold, X, one can show that if $\phi \in L^2(X)$ has $\|\phi\|_2 = 1$ and is an eigenfunction of the Laplace-Beltrami operator with eigenvalue λ, then $\|\phi\|_\infty \ll \lambda^{\frac{\dim(X)-1}{4}}$, see [SS89, Cor 2.2]. This bound, usually referred to as the convexity bound, is sharp in general. However, when X has negative curvature, it is believed that this can be improved and there are some results of this nature for cusp forms on some arithmetic hyperbolic manifolds. Explicitly, for arithmetic hyperbolic surfaces it is conjectured that $\|\phi\|_\infty \ll_\epsilon \lambda^\epsilon$, and it was shown in [IS95] that $\|\phi\|_\infty \ll_\epsilon \lambda^{5/24+\epsilon}$ for ϕ a Hecke-Maass cusp form. In higher dimensions, the situation is more complicated, as it was shown in the work Rudnick and Sarnak [RS94] and in more detail by Milićević [Mil11], that for any $\epsilon > 0$, there exists a Hecke-Maass form on a given arithmetic hyperbolic 3-manifold for which $\|\phi\|_\infty \gg \lambda^{1/4-\epsilon}$. Nevertheless, a subconvex upper bound of order $\lambda^{5/12+\epsilon}$ was proved in [Koy95, BHM16], so the truth is somewhere in between.

This paper is concerned with the analogous problem where the cusp form is replaced by an Eisenstein series. Explicitly, given a non-uniform lattice, Γ, acting on hyperbolic $n+1$ space $\mathbb{H}^{n+1}$, for each cusp ξ, let $E_{\Gamma,\xi}(s,z)$ denote the Eisenstein series corresponding to this cusp. Then $E_{\Gamma,\xi}(\frac{n}{2}+it, z)$ is an almost-L^2 eigenfunction of the Laplacian with eigenvalue $\lambda = \frac{n^2}{4} + t^2$. Since the Eisenstein series is unbounded as z moves into the cusp, in order to consider the supremum norm, we need to restrict to a compact set. We define the parameter $\nu_\infty = \nu_\infty(\Gamma)$ as the infimum over all $\nu > 0$ such that for any compact set $\Omega \subseteq \mathbb{H}^{n+1}$, any $z \in \Omega$, and any cusp ξ, we have

$$|E_{\Gamma,\xi}(\tfrac{n}{2}+it, z)| \ll_\Omega |t|^\nu.$$

For a general lattice not much is known, but there are known results when Γ is arithmetic. In particular for $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup, it is conjectured that $\nu_\infty(\Gamma) = 0$. Here the convexity bound is $\nu_\infty(\Gamma) \leq \frac{1}{2}$, and the work of Young and consequently Huang [You18, Hua19] using amplification gives the sub-convex bound of $\nu_\infty(\Gamma) \leq \frac{3}{8}$ while the best result to date is due to Blomer [Blo20] who used the approximate functional equation and Burgess' bound to prove that $\nu_\infty \leq \frac{1}{3}$. When Γ is a congruence lattice in $\mathrm{SL}_2(\mathbb{C})$ the

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convexity bound is $\nu_\infty \leq 1$, the conjectured value is $\nu_\infty = \frac{1}{2}$, and the best currently known bound comes from the work of Assing [Ass19] who used amplification to show that $\nu_\infty \leq \frac{7}{8}$ (his result is more general and deals with Eisenstein series for SL_2 defined over general number fields). In higher dimensions the convexity bound $\nu_\infty(\Gamma) \leq \frac{n}{2}$ was proved in [KY22] for a large family of arithmetic lattices acting on hyperbolic $(n+1)$-space. While it may be expected that $\nu_\infty = \frac{n-1}{2}$, there are no known subconvex bounds when $n \geq 3$.

As a proxy to the sup norm bound we may consider the mean square bounds $\nu_2 = \nu_2(\Gamma)$ defined as the infimum of all $\nu > 0$ such that for any cusp ξ, any compact set $\Omega \subseteq \mathbb{H}^{n+1}$, and any $z \in \Omega$, we have

$$\int_{-T}^{T} |E_{\Gamma,\xi}(\tfrac{n}{2}+it, z)|^2 dt \ \ll_\Omega \ T^{1+2\nu}.$$

Then clearly $\nu_2 \leq \nu_\infty$ but one expects that we can give a sharper bound for ν_2. Indeed, the convexity bound $\nu_2(\Gamma) \leq \frac{n}{2}$ is known to hold for a general lattice [CS80, Cor 7.7]. We denote by $\nu_2(\Gamma, z)$ this exponent for a fixed point z and note that for certain arithmetic lattices it is possible to evaluate $E(s, z_0)$ at special points as a product of zeta functions and L-functions (see [KY22]) from which one can conclude that $\nu_2(\Gamma, z_0) = \frac{n-1}{2}$ at these points. This gives a lower bound on what can be expected to hold in general.

Remark 1.1. We note that for a general non-arithmetic lattice acting on $\mathbb{H}^2$, it is likely that the convexity bound $\nu_2(\Gamma) \leq \frac{1}{2}$ is actually sharp. As evidence for this, we show that if $\nu_2(\Gamma, z) < \frac{1}{2}$ at some point $z \in \mathbb{H}^2$, then Γ must have infinitely many cusp forms, in contrast with the Phillips-Sarnak conjecture [PS85], see Theorem 8 below and the discussion in Section 5.

Remark 1.2. In addition to serving as a proxy to the sup-norm bound, such mean-square bounds on Eisenstein series have applications to various counting problems. For example, given a rational quadratic form Q of signature $(n+1, 1)$, the number $N(X)$ of primitive integer points $v \in \mathbb{Z}^{n+2}$ on the light cone $Q = 0$ of norm bounded by X can be estimated precisely in terms of $\nu_2 = \nu_2(\mathrm{SO}_Q(\mathbb{Z}))$, see [KY22, Prop 3.5] showing that

$$N(X) = cX^n + O(X^{n\left(1-\frac{1}{2(\nu_2+1)}\right)}).$$

1.1. **New results for** $\mathrm{SL}_2(\mathbb{Z})$ **and** $\mathrm{SL}_2(\mathbb{Z}[i])$**.** In this work we restrict ourselves to the special cases of the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and the Picard group $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$. In either case, there is only one cusp, and we denote by $E_\Gamma(s, z)$ the corresponding Eisenstein series. In these cases we prove that $\nu_2(\mathrm{SL}_2(\mathbb{Z})) = 0$ and $\nu_2(\mathrm{SL}_2(\mathbb{Z}[i])) = \nu_\infty(\mathrm{SL}_2(\mathbb{Z}[i])) = \frac{1}{2}$ as expected. Explicitly we show the following.

Theorem 1. *For* $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ *and any compact set* $\Omega \subset \Gamma_1\backslash\mathbb{H}$, *there is a constant* $c = c(\Omega)$ *such that for all* $z \in \Omega$ *and all* $T \geq 1$,

$$\int_{-T}^{T} |E_{\Gamma_1}(z, \tfrac{1}{2}+it)|^2 dt \ \leq \ cT\log^4(T). \tag{1.3}$$

For $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z}[i])$, *any compact set* $\Omega \subset \Gamma_2\backslash\mathbb{H}^3$, *and for any* $\epsilon > 0$, *there is a constant* $c = c(\Omega, \epsilon)$ *such that for all* $z \in \Omega$,

$$|E_{\Gamma_2}(z, 1+iT)| \leq \ cT^{1/2+\epsilon}. \tag{1.4}$$

Remark 1.5. We expect that modifications of our techniques would give similar estimates also for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathcal{O}_K)$ with K an imaginary quadratic field.

1.2. **Mean square bounds on Epstein zeta functions.** Both of the above results are derived from the following bounds on Epstein zeta functions. Given a positive-definite quadratic form Q in m variables, let

$$Z_Q(s) := \sum_{v\in\mathbb{Z}^m\setminus 0} Q(v)^{-s}$$

denote the Epstein zeta function, defined originally in some half-plane $\mathfrak{Re}(s) \gg 1$, and having a well-known meromorphic continuation to $s \in \mathbb{C}$ (see [Ter73]) and functional equation relating $Z_Q(s)$ to $Z_{Q_-}(\frac{m}{2} - s)$, where Q_- is the dual form. That is, if Q is given by $Q(x) = x^T Zx$, then $Q_-(x) = x^T Z^{-1}x$.

Theorem 2. *Let Q be a positive-definite quadratic form in m variables, and let the associated Epstein zeta function be Z_Q. If $m = 2$ then*

$$\int_T^{2T} |Z_Q(\frac{1}{2} + it)|^2 dt \ \ll_Q \ T \log^2(T). \tag{1.6}$$

Moreover, if $m \geq 3$, then for any $\varepsilon > 0$,

$$\int_T^{2T} \left|Z_Q\left(\frac{m}{4} + it\right)\right|^2 dt \ \ll_Q \ T^{m/2+\varepsilon}. \tag{1.7}$$

In either case, the implicit constants depend on the form Q, but may be taken uniform as Q varies in a compact set in the space of positive-definite quadratic forms.

For $m \geq 4$, (1.7) follows immediately from Blomer's bounds for the sup norm.

Theorem 3 ([Blo20, Theorem 1])**.** *Let Q be a positive-definite quadratic form in m variables, with $m \geq 4$. Then for any $\varepsilon > 0$,*

$$\left|Z_Q\left(\frac{m}{4} + it\right)\right| \ll_{Q,\varepsilon} T^{(m-2)/4+\varepsilon}.$$

Blomer also gives sup norm bounds for $m = 2$ and $m = 3$, but these are weaker than what is needed for the L^2 bounds in Theorem 2. Thus, it remains to prove the two cases $m = 2$ and $m = 3$ of Theorem 2.

1.3. **Values of indefinite forms at integer points.** To prove Theorem 2, we require a result from the geometry of numbers, namely the following uniform version of [EMM98, Theorem 2.3].

Theorem 4. *For any $n = p + q \geq 3$ with $p \geq q \geq 1$, and any form $Q(v)$ of signature (p, q) having discriminant one, there are constants $c = c(Q)$, A_0, and B_0 such that, for any $A \geq A_0$ and $B \geq B_0$, if $(p, q) \notin \{(2, 1), (2, 2)\}$, then*

$$\#\{v \in \mathbb{Z}^n : \|v\| \leq A,\ Q(v) \in (-B, B)\} \leq cBA^{n-2}.$$

while for $(p, q) \in \{(2, 1), (2, 2)\}$,

$$\#\{v \in \mathbb{Z}^n : \|v\| \leq A,\ Q(v) \in (-B, B)\} \leq cBA^{n-2} \log(A).$$

The constant c can be taken uniform for Q ranging in a compact set.

Remark 1.8. The result of [EMM98, Theorem 2.3] gives a similar upper bound to

$$\#\{v \in \mathbb{Z}^n : \|v\| \leq A,\ Q(v) \in (a,b)\}$$

with the bound depending implicitly on the form Q and the target interval (a, b). The novelty of our result is to make the dependence on the target interval explicit.

Outline. In Section 2, we prove Theorem 4. Then in Section 3.1 we first focus on the case $m = 2$ of Theorem 2; and settle the case $m \geq 3$ in Section 3.2. We then show in Section 4 how to derive the bounds on the Eisenstein series from Theorem 2. Finally, in Section 5, we explicate Remark 1.1.

Notation. We use standard Vinogradov notation that $f \ll g$ if there is a constant $C > 0$ so that $f(x) \leq Cg(x)$ for all x. When the implied constant depends on more than the lattice Γ or Q or z varying in a compact set Ω, which we think of as fixed, we denote this with a subscript.

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2. Counting estimate

In this section we prove Theorem 4. We first recall the main ideas in the proof of [EMM98]. Let Q be a quadratic form of signature (p, q) and let $g \in \mathrm{SL}_n(\mathbb{R})$ such that $Q(v) = Q_0(gv)$ with

$$Q_0(x) = x_1 x_n + \sum_{i=2}^{p} x_i^2 - \sum_{i=p+1}^{n-1} x_i^2.$$

We may assume that $\max\{\|g\|, \|g^{-1}\|\} \leq \beta$ for some fixed β.

Let $H = \mathrm{SO}_{Q_0}(\mathbb{R})$, let $K = H \cap \mathrm{SO}(n)$ be a maximal compact, and let $a_t = \mathrm{diag}(e^{-t}, 1 \ldots, 1, , e^t) \in H$. For any compactly supported function f on $\mathbb{R}^n$, any $r > 0$, and any $\xi \in \mathbb{R}$, define the function

$$J_f(r, \xi) = \frac{1}{r^{n-2}} \int_{\mathbb{R}^{n-2}} f(r, x_2, \ldots, x_{n-1}, \tfrac{\xi - Q_0(0, x_2, \ldots, x_{n-1}, 0)}{2r}) dx_2 \ldots dx_{n-1}.$$

Then by [EMM98, Lemma 3.6], there is a constant $c_{p,q}$ and $T_0 > 1$ such that for every $t \geq \log(T_0)$ and any $v \in \mathbb{R}^n$ with $\|v\| > T_0$, we have that

$$\left| J_f(\|v\| e^{-t}, Q_0(v)) - c_{p,q} e^{(n-2)t} \int_K f(a_t k v) dm(k) \right| \leq 1.$$

In particular, if we choose f so that $J_f(r, \xi) \geq 2$ for all $r \in (1, 2)$ and $\xi \in (a, b)$, then for any $v \in \mathbb{R}^n$ with $e^t \leq \|gv\| \leq 2e^t$ and $Q_0(gv) \in (a, b)$, we have that $J_f(\|gv\| e^{-t}, Q_0(gv))) \geq 2$, whence $c_{p,q} e^{(n-2)t} \int_K f(a_t k v) dm(k) \geq 1$ and

$$\begin{aligned} \#\{v \in \mathbb{Z}^n : Q_0(gv) \in (a,b), \|gv\| \in [e^t, 2e^t]\} &\leq \sum_{v \in \mathbb{Z}^n} c_{p,q} e^{(n-2)t} \int_K f(a_t k g v) dm(k) \\ &= c_{p,q} e^{(n-2)t} \int_K \widehat{f}(a_t k g) dm(k) \qquad (2.1) \end{aligned}$$

where

$$\widehat{f}(g) = \sum_{0\neq v\in\mathbb{Z}^n} f(gv)$$

is the Siegel transform. Next, using [Sch68, Lemma 2] we can bound

$$\widehat{f}(g) \leq c_f \alpha(g\mathbb{Z}^n), \tag{2.2}$$

where the function $\alpha(\Lambda)$ is the function on the space of lattices defined in [EMM98, equation (3.3)], and c_f is a constant depending only on f. Finally, [EMM98, Theorem 3.2 and Theorem 3.3] state that

$$\int_K \alpha(a_t kg\mathbb{Z}^n)dm(k) \leq c_g,$$

is uniformly bounded when $(p,q)\notin\{(2,2),(2,1)\}$ and that

$$\int_K \alpha(a_t kg\mathbb{Z}^n)dm(k) \leq c_g t,$$

when $(p,q)=(2,2)$ or $(p,q)=(2,1)$. Here the constant c_g is uniform when g is taken from a compact set. Using this we get that when $(p,q)\notin\{(2,2),(2,1)\}$

$$\#\{v\in\mathbb{Z}^n : Q_0(gv)\in(a,b), \|gv\|\in[e^t,2e^t]\} \leq c_{p,q}c_f c_g e^{(n-2)t},$$

and that for $(p,q)\in\{(2,2),(2,1)\}$

$$\#\{v\in\mathbb{Z}^n : Q_0(gv)\in(a,b), \|gv\|\in[e^t,2e^t]\} \leq c_{p,q}c_f c_g t e^{(n-2)t}.$$

Summing over $t\leq\log(T)$ in dyadic intervals gives a bound of the form

$$\#\{v\in\mathbb{Z}^n : Q_0(gv)\in(a,b), \|gv\|\leq T\} \leq cT^{n-2},$$

when signature $(p,q)\notin\{(2,2),(2,1)\}$ and

$$\#\{v\in\mathbb{Z}^n : Q_0(gv)\in(a,b), \|gv\|\leq T\} \leq cT^{n-2}\log(T),$$

otherwise. Here the constant c depends on the signature on the group element g, and on the function f (and hence on the interval (a,b).

Up to this point, the proof is identical to the treatment in [EMM98]. It is at this moment where we need one simple extra ingredient to make everything uniform, in the special case of a target interval $(-B,B)$. Our goal is to find a suitable function f so that $J_f(r,\xi)\geq 2$ when $r\in[1,2]$ and $|\xi|\leq B$ such that $c_f\leq cB$. We first note that [Sch68, Lemma 2] implies that that for any $R\geq 1$ and any lattice $\Lambda = g\mathbb{Z}^n$, we have the bound

$$\#\{v\in\Lambda : \|v\|\leq R\} \leq cR^n\alpha(\Lambda),$$

with c and absolute constant depending only on n. We combine this with the following simple observation.

Lemma 5. *For any $x\in\mathbb{R}^n$ and any lattice Λ we can bound*

$$\#\{v\in\Lambda : \|v-x\|\leq R\} \leq \#\{v\in\Lambda : \|v\|\leq 2R\}.$$

Proof. If there is no $v\in\Lambda$ with $\|v-x\|\leq R$, then this is obvious. Otherwise, let $u\in\Lambda$ satisfy $\|u-x\|\leq R$; then $\|v-x\|\leq R$ implies that

$$\|v-u\| = \|v-x+x-u\| \leq 2R.$$

Since $v\in\Lambda$ if and only if $v-u\in\Lambda$, this concludes the proof. □

We now describe our choice of f. Assume that $B \geq (n-2)$ and let f take values in $[0,2]$ and supported on $[0,3]\times[-2,2]^{n-2}\times[-2B,2B]$ such that $f(x)=2$ on $[1,2]\times[-1,1]^{n-2}\times[-B,B]$. Note that for any $r\in[1,2]$ and $\xi\in[-B,B]$ and $(x_2,\ldots,x_{n-2})\in[-1,1]^{n-2}$ we have that $\frac{\xi-Q_0(0,x_2,\ldots,x_{n-1},0)}{2r}\in[-B,B]$ so that

$$f(r,x_2,\ldots,x_{n-1},\tfrac{\xi-Q_0(0,x_2,\ldots,x_{n-1},0)}{2r})=2.$$

We thus get a lower bound $J_f(r,\xi)\geq \frac{2}{2^{n-2}}\int_{[-1,1]^{n-2}}dx_2\ldots dx_{n-1}=2$. On the other hand, we can cover the support of f by $2B+1$ balls of radius $R_n=2\sqrt{n}$ centered at the points $v_j=(1,0,\ldots,0,2j)$ with $-B\leq j\leq B$. From Lemma 5, we can bound

$$\#\{v\in\Lambda:\|v-v_j\|\leq R_n\}\leq\#(\Lambda\cap 2R_n\}\leq c_n\alpha(\Lambda),$$

where the constant depends only on n. Using this, we can bound the Siegel transform $\widehat{f}(g)\leq 2c_nB\alpha(\Lambda)$. Plugging this estimate into (2.2) and (2.1) concludes the proof of Theorem 4.

3. Bounds on the Epstein zeta function

3.1. **Bounds for $m=2$.** In this section we fix $m=2$ and prove (1.6). In [SV05], the authors considered the case of an integral quadratic form Q and proved an approximate functional equation as well as a formula for the mean square. While the formula for the mean square is special for a family of integral quadratic forms, the approximate functional equation [SV05, Theorem 1] holds in general. We record this result in our special case as follows.

Theorem 6. *For Q a positive definite quadratic form of discriminant D with dual form Q_- and sequences a_n, λ_n, b_n, μ_n defined for $\mathfrak{Re}(s)\gg 1$ by:*

$$Z_Q(s)=\sum_{\lambda_n}\frac{a_n}{\lambda_n^s},\qquad \textit{and}\qquad Z_{Q_-}(s)=\sum_{\mu_n}\frac{b_n}{\mu_n^s},$$

we have, for $s=\frac{1}{2}+it$ with $|t|\geq 1$, that:

$$Z_Q(s)=\sum_{\lambda_n\leq X}\frac{a_n}{\lambda_n^s}+\chi(s)\sum_{\mu_n\leq X}\frac{b_n}{\mu_n^{1-s}}+O_D(\log(|t|)),$$

where

$$\chi(s)=(\frac{\sqrt{D}}{\pi})^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)},$$

and $X=X(t):=\frac{|t|\sqrt{D}}{\pi}$.

Noting that $|\chi(s)|=1$ for $s=\frac{1}{2}+it$, we can estimate

$$|Z_Q(\frac{1}{2}+it)|^2\ll F_Q(t)+F_{Q_-}(t)+O(\log^2(t)),\tag{3.1}$$

where we have set

$$F_Q(t):=\left|\sum_{\lambda_n\leq X(t)}\frac{a_n}{\lambda_n^{\frac{1}{2}+it}}\right|^2.$$

Recall again that $X(t)=\frac{|t|\sqrt{D}}{\pi}\asymp|t|$.

Integrating (3.1) gives:

$$\int_T^{2T} |Z_Q(\tfrac{1}{2}+it)|^2 dt \ll \int_T^{2T} F_Q(t)dt + \int_T^{2T} F_{Q_-}(t)dt + O(T\log^2(T)). \tag{3.2}$$

These terms can be estimated as:

$$\begin{aligned}\int_T^{2T} F_Q(t)dt &= \int_T^{2T} \sum_{\substack{u,v\in\mathbb{Z}^2\setminus 0\\ Q(u),Q(v)\le X(t)}} \frac{1}{Q(u)^{1/2+it}Q(v)^{1/2-it}}dt \\ &\ll \sum_{\substack{u,v\in\mathbb{Z}^2\setminus 0\\ \|v\|,\|u\|\ll\sqrt{T}}} \frac{1}{\|u\|\|v\|}\left|\int_{\max\{Q(u),Q(v),T\}}^{2T} e^{it\log(\frac{Q(v)}{Q(u)})}dt\right|,\end{aligned}$$

where we used that $Q(u)\asymp \|u\|^2$.

We now break this into different regions depending on the ratio of $\frac{Q(u)}{Q(v)}$. First consider the range when $|\log(\frac{Q(u)}{Q(v)})|\ge 1$. In this range, we can bound the inner integral by 2 to get a bound of

$$\sum_{\substack{u,v\in\mathbb{Z}^2\\ \|v\|,\|u\|\ll\sqrt{T}}} \frac{1}{\|u\|\|v\|} \ll \left(\sum_{\substack{v\in\mathbb{Z}^2\\ \|v\|\ll\sqrt{T}}} \frac{1}{\|v\|}\right)^2 \ll T.$$

For the rest, we have that $Q(u)\asymp Q(v)$ so also $\|u\|\asymp \|v\|$, and we break the sum into dyadic intervals $A\le \|v\|\le 2A$ and $\frac{B}{T}\le |\log(\frac{Q(u)}{Q(v)})|\le \frac{2B}{T}$, with $A\le\sqrt{T}$ and $B\le T$. The contribution of each such dyadic interval is then given by

$$\begin{aligned}N_T(A,B) &= \sum_{\substack{v\in\mathbb{Z}^2\\ A\le\|v\|\le 2A}} \sum_{\substack{u\in\mathbb{Z}^2\\ |\log(\frac{Q(u)}{Q(v)})|\in(\frac{B}{T},\frac{2B}{T})}} \frac{1}{\|u\|\|v\|}\left|\int_{\max\{Q(u),Q(v),T\}}^{2T} e^{it\log(\frac{Q(u)}{Q(v)})}dt\right| \\ &\ll \frac{T}{A^2B}\#\{u,v\in\mathbb{Z}^2 : \|v\|\le 2A,\ |\frac{Q(u)}{Q(v)}-1|\le\frac{2B}{T}\} \\ &\ll \frac{T}{A^2B}\#\{(u,v)\in\mathbb{Z}^4 : \|(u,v)\|\le 2A,\ |Q(u)-Q(v)|\ll\frac{A^2B}{T}\}.\end{aligned}$$

We also have the range where $|\log(\frac{Q(u)}{Q(v)})|\le\frac{1}{T}$ that we can similarly bound by

$$\begin{aligned}N_T(A) &= \sum_{\substack{v\in\mathbb{Z}^2\\ A\le\|v\|\le 2A}} \sum_{\substack{u\in\mathbb{Z}^2\\ |\log(\frac{Q(u)}{Q(v)})|\in[0,\frac{1}{T})}} \frac{1}{\|u\|\|v\|}\left|\int_{\max\{Q(u),Q(v),T\}}^{2T} e^{it\log(\frac{Q(u)}{Q(v)})}dt\right| \\ &\ll \frac{T}{A^2}\#\{(u,v)\in\mathbb{Z}^4 : \|(u,v)\|\le 2A,\ |Q(u)-Q(v)|\ll 1\}.\end{aligned}$$

Note that for $Q(v)$ a positive definite binary quadratic form, the form

$$\tilde{Q}(u,v) = Q(u) - Q(v)$$

has signature $(2,2)$. Applying Theorem 4 gives

$$\#\{v\in\mathbb{Z}^4:\|v\|\le A,\ \tilde{Q}(v)\in(-B,B)\}\le cBA^2\log(A),$$

from which we can bound

$$N_T(A,B)\ll A^2\log(A)\ll A^2\log(T)$$

for $A\le\sqrt{T}$. We also have the bound $N_T(A)\le cT\log(T)$ for all $A\le\sqrt{T}$. Taking A,B to be powers of 2 and summing over $A\le\sqrt{T},B\le T$, we get the bound

$$\int_T^{2T}|Z_Q(\tfrac{1}{2}+it)|^2dt\ll T\log^2(T),$$

as claimed in (1.6).

3.2. **Bounds for** $m\ge 3$**.** Turning now to (1.7) for $m\ge 3$, we follow the same approach, however instead of using the approximate functional equation of [SV05], we instead use the following.

Theorem 7 ([Blo20, (2.2)])**.** *Recalling that Q_- is the dual form to $Q=Q_+$, we have that*

$$Z_Q(\frac{m}{4}+it)\ll 1+|t|^{\varepsilon}\sum_{\pm}\sum_{A}\int_{|w|\le|t|^{\varepsilon}}A^{-m/4}\left|\sum_{v\neq 0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|dw \tag{3.3}$$

where the sum over A ranges over powers of 2 less than $|t|^{1+\varepsilon}$, and V_A is bounded and has compact support in $[A,3A]$.

Now consider the L^2 norm and expand the square using Cauchy-Schwarz to get that

$$\int_T^{2T}|Z_Q(\frac{m}{4}+it)|^2dt\ll\int_T^{2T}\left|1+t^{\epsilon}\sum_{\pm}\sum_{A}\int_{|w|\le|t|^{\varepsilon}}A^{-m/4}\left|\sum_{v\neq0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|dw\right|^2dt$$

$$\ll\int_T^{2T}\left(1+t^{\epsilon}\sum_{\pm}\sum_{\substack{A\le t^{1+\epsilon}\\ \text{dyadic}}}\int_{|w|\le t^{\epsilon}}A^{-m/2}\left|\sum_{v\neq0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|^2\right)dt$$

$$\ll T+T^{\epsilon}\sum_{\pm}\sum_{\substack{A\le T^{1+\epsilon}\\ \text{dyadic}}}A^{-m/2}\int_{|w|\le T^{\epsilon}}\int_{\max\{T,A,|w|^{1/\epsilon}\}}^{2T}\left|\sum_{v\neq0}\frac{V_A(Q_\pm(v))}{Q_\pm(v)^{\pm it+iw}}\right|^2dt.$$

For any fixed A and w in this range and for each of the choices of $Q=Q_+$ or $Q=Q_-$, we open the square and estimate the integral

$$\int_{\max\{T,A,|w|^{1/\epsilon}\}}^{2T}\left|\sum_{v\neq0}\frac{V_A(Q(v))}{Q(v)^{it+iw}}\right|^2dt\ll\sum_{v,u\neq0}V_A(Q(v))V_A(Q(u))\left|\int_{\max\{T,A,|w|^{1/\epsilon}\}-w}^{2T-w}e^{it\log\frac{Q(v)}{Q(u)}}dt\right|$$

depending on the range of $\frac{Q(v)}{Q(u)}$.

If $|\log(\frac{Q(v)}{Q(u)})|\ge 1$, the integral is bounded independently on the boundaries of integration. Since $V_A(Q(v))$ is bounded and supported on $Q(u)\le 3A$, the sum over $\|u\|\ll Q(u)^{1/2}\ll A^{1/2}$ and $\|v\|\ll A^{1/2}$ is bounded by $O(A^m)$. In the remaining case we may assume that

$\|v\|\asymp \|u\|\asymp A^{1/2}$. Now further break the summation by taking $\frac{B}{T}\leq |\log(\frac{Q(v)}{Q(u)})|\leq \frac{2B}{T}$, with $B\ll T$ dyadic. For each $B\geq 1$ we get a term of the form

$$N_T(A,B)=\sum_{\substack{v\in\mathbb{Z}^m\\ \sqrt{A}\leq\|v\|\leq\sqrt{3A}}}\sum_{\substack{u\in\mathbb{Z}^m\\ |\log(\frac{Q(v)}{Q(u)})|\in(\frac{B}{T},\frac{2B}{T})}}\left|\int_{\max\{T,A,|w|^{1/\epsilon}\}-w}^{2T-w}e^{it\log(\frac{Q(v)}{Q(u)})}dt\right|$$

$$\ll \frac{T}{B}\#\{(u,v)\in\mathbb{Z}^{2m}:\|(u,v)\|\leq\sqrt{2A},\ |Q(u)-Q(v)|\ll\frac{AB}{T}\}$$

and for $B\leq 1$ we let

$$N_T(A)=\sum_{\substack{v\in\mathbb{Z}^m\\ \sqrt{A}\leq\|v\|\leq\sqrt{3A}}}\sum_{\substack{u\in\mathbb{Z}^m\\ |\log(\frac{Q(v)}{Q(u)})|\in(0,\frac{1}{T})}}\left|\int_{\max\{T,A,|w|^{1/\epsilon}\}-w}^{2T-w}e^{it\log(\frac{Q(v)}{Q(u)})}dt\right|$$

$$\ll T\#\{(u,v)\in\mathbb{Z}^{2m}:\|(u,v)\|\leq\sqrt{2A},\ |Q(u)-Q(v)|\ll 1\}.$$

Again following our approach for $m=2$, we note that $\tilde{Q}(u,v):=Q(u)-Q(v)$ is a form of signature (m,m), and apply Theorem 4. It follows that

$$N_T(A,B)\ll\frac{T}{B}A^{m-1}\frac{AB}{T}=A^m. \tag{3.4}$$

Similarly, we can trivially bound $N_T(A)$ by A^m.

Plugging these bounds back, bounding the integral of w by $O(T^\epsilon)$ and summing over A,B powers of 2 we get the bound

$$\int_T^{2T}|Z_Q(1+it)|^2dt\ll T+T^\varepsilon\sum_{\substack{A\leq T^{1+\epsilon}\\ \text{dyadic}}}A^{-m/2}\left(A^m+N_T(A)+\sum_{\substack{B\ll T\\ \text{dyadic}}}N_T(A,B)\right)\ll T^{m/2+\varepsilon},$$

as claimed in (1.7).

4. Bounds on Eisenstein series: Proof of Theorem 1

To move from bounds on the Epstein zeta function to the Eisenstein series, we use standard arguments. First note that in the hyperbolic plane $\mathbb{H}^2$, we have (for $\mathfrak{Re}s>1$):

$$\zeta(2s)E(s,z)=y^s\sum_{(c,d)\neq(0,0)}\frac{1}{((x^2+y^2)c^2+2xcd+d^2)^s}=y^sZ_Q(s),$$

where $Q(c,d)=Q_z(c,d)=(x^2+y^2)c^2+2xcd+d^2$ and

$$Z_Q(s)=\sum_{v\in\mathbb{Z}^2\backslash 0}Q(v)^{-s}.$$

Applying (1.6) and the well-known bound $\zeta(1+2it)\gg\log(t)^{-1}$, we conclude (1.3).

In the upper-half-space model of hyperbolic 3-space,

$$\mathbb{H}^3=\{z=x_1+ix_2+jy\ :\ x_j\in\mathbb{R},y>0\},$$

we apply a similar argument. That is, the Eisenstein series is defined (for $\mathfrak{Res} > 2$) by

$$E(s,z) = \sum_{\substack{c,d\in\mathbb{Z}[i] \\ \text{co-prime}}} \frac{y^s}{N(cz+d)^s}.$$

where $N(z) = x_1^2 + x_2^2 + y^2$ denotes the norm on the quaternions. Since the norm is multiplicative, if we write

$$\zeta_{\mathbb{Z}[i]}(s) = \sum_{\alpha\in\mathbb{Z}[i]} \frac{1}{N(\alpha)^s},$$

then we can simplify the Eisenstein series to

$$\frac{1}{4}\zeta_{\mathbb{Z}[i]}(s)E(s,z) = \sum_{(c,d)\neq(0,0)} \frac{y^s}{N(cz+d)^s}.$$

Moreover, we can write $\zeta_{\mathbb{Z}[i]}(s) = 4\zeta(s)L(s,\chi_1)$ with χ_1 the quadratic Dirichlet character modulo 4. Now if we expand the norm on the right hand side, we arrive at

$$\zeta(s)L(s,\chi_1)E(s,z) = y^s \sum_{(c,d)\neq(0,0)} \frac{1}{Q_z(c,d)^s},$$

where

$$Q_z(c,d) = N(z)c_1^2 + N(z)c_2^2 + d_1^2 + d_2^2 + 2(x_1c_1d_1 - x_2c_2d_1 + x_1c_2d_2 + x_2c_1d_2)$$

is a positive definite quaternary quadratic form. Again we have good control on $\zeta(s)$ for $s = 1+it$ and on $L(s,\chi_1)$. Thus the problem reduces to estimating the Epstein zeta function

$$Z_{Q_z}(s) = \sum_{v\in\mathbb{Z}^4\setminus 0} Q_z(v)^{-s},$$

and (1.4) follows easily from Theorem 3. This completes the proof of Theorem 1.

5. Sharpness

We now consider the case of a general non-arithmetic lattice and show that any subconvex bound for $\nu_2(\Gamma)$ implies the existance of infinitely many Maass cusp forms. Explicitly we show the following.

Theorem 8. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a non uniform lattice and assume that there is some z_0 such that $\nu_2(\Gamma, z_0) < 1/2$. Then there are infinitely many Maass cusp forms $\varphi_j \in L^2(\Gamma\backslash\mathbb{H})$ with $\triangle\varphi_j + \lambda_j\varphi_j = 0$. Moreover, they satisfy the local Weyl law: for any test function $h(r)$ with Fourier transform smooth and compactly supported, for all sufficiently large T,*

$$\sum_j h(\frac{r_j}{T})|\varphi_j(z)|^2 \quad = \quad T^2\frac{|\Gamma_z|}{2\pi}\int_0^\infty h(r)rdr + O_{\delta,h}(T^{2-\delta}).$$

where $\Gamma_z = \{\gamma\in\Gamma : \gamma z = z\}$.

Proof. We recall some well known results on the pre-trace formula and refer to [Hej76] for more details. Given a point pair invariant $k(z,w) = k(\sinh^2(d(z,w)))$ with $d(z,w)$ the hyperbolic distance and $k \in C_c^\infty(\mathbb{R}^+)$, its spherical transform is defined as $H(s) = \int_{\mathbb{H}^2} k(z,i)\mathfrak{Im}(z)^s d\mu(z)$. By [Hej76, Proposition 4.1] the point pair invariant can be recovered from $H(s)$ as follows : Let $h(r) = H(\frac{1}{2}+ir)$ and let $g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty h(r)e^{-iru}dr$ denote its Fourier transform, then, defining the auxiliary function $Q \in C_c^\infty(\mathbb{R}^+)$ by $g(u) = Q(\sinh^2(\frac{u}{2}))$ we have that $k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(r)}{\sqrt{r-t}}$. We also recall that $k(0) = \frac{1}{2\pi}\int_0^\infty h(r)r\tanh(\pi r)dr$ (see [Hej76, Proposition 6.4]).

Given any such point pair invariant we have the pre-trace formula

$$\sum_{\gamma\in\Gamma} k(z,\gamma z) = \sum_j h(r_j)|\varphi_j(z)|^2 + \sum_{i=1}^{\kappa}\frac{1}{2\pi}\int_{\mathbb{R}} h(r)|E_{\Gamma,\xi_i}(\tfrac{1}{2}+ir,z)|^2dr,$$

where $\xi_1,\ldots,\xi_\kappa$ are the cusps of Γ and $\{\varphi_j\}_{j\in\mathbb{N}}$ are all all Maass forms in $L^2(\Gamma\backslash\mathbb{H})$ with eigenvalue parametrized by $\lambda_j = \frac{1}{4}+r_j^2$.

Now, fix a smooth compactly supported function $g(u) \in C_c^\infty((-1,1))$ and for any $T \geq 1$ let $g_T(u) = Tg(Tu)$ so that $h_T(r) = h(\frac{r}{T})$ and $k_T(z,w)$ the corresponding point pair invariant. Since $g_T(u)$ is supported on $(-\frac{1}{T},\frac{1}{T})$ the point pair invariant $k(z,w)$ is supported on the set $\{(z,w)|d(z,w) \leq \frac{1}{T}\}$ with $d(z,w)$ the hyperbolic distance. Since Γ acts properly discontinuously on $\mathbb{H}^2$ for any fixed z there is $\delta = \delta(z)$ such that $d(z,\gamma z) \geq \delta$ for any $\gamma \in \Gamma$ with $\gamma z \neq z$. In particular taking $T_0 \geq \frac{1}{\delta(z)}$ for any $T \geq T_0$ we have that $k_T(z,\gamma z) = 0$ if $\gamma z \neq z$. Hence for any $T \geq T_0$ we have

$$\sum_j h(\frac{r_j}{T})|\varphi_j(z)|^2 + \sum_{i=1}^{\kappa}\frac{1}{2\pi}\int_{\mathbb{R}} h(\frac{r}{T})|E_{\Gamma,\xi_i}(\tfrac{1}{2}+ir,z)|^2dr \quad = \quad |\Gamma_z|k(0)$$

with

$$\Gamma_z = \{\gamma \in \Gamma : \gamma z = z\}.$$

Denote by $\nu_2 = \nu_2(\Gamma,z)$ and note that for any $\ell \geq 0$ we can bound $h(t) \ll_{\ell,h} |r|^{-\ell}$ We can thus bound the contribution of the integrals over Eisenstein series by

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} h(\frac{r}{T})|E_{\Gamma,\xi_i}(\tfrac{1}{2}+ir,z_0)|^2dr &= \sum_{k\in\mathbb{Z}}\int_{kT}^{(k+1)T} h(\frac{r}{T})|E_{\Gamma,\xi_i}(\tfrac{1}{2}+ir,z)|^2dr \\ &\ll_h T^{2\nu_2+1}.\end{aligned}$$

On the other hand, since the right hand side is

$$|\Gamma_z|k_T(0) = \frac{|\Gamma_z|}{2\pi}\int_0^\infty h(\frac{r}{T})r\tanh(\pi r)dr = T^2\frac{|\Gamma_z|}{2\pi}\int_0^\infty h(r)rdr + O_h(1),$$

we can conclude that if $\nu_2 < \frac{1}{2}$, then for any $\delta \in (0,1-2\nu_2)$ and any $T \geq T_0$

$$\sum_j h(\frac{r_j}{T})|\varphi_j(z)|^2 \quad = \quad T^2\frac{|\Gamma_z|}{2\pi}\int_0^\infty h(r)rdr + O_{\delta,h}(T^{2-\delta}).$$

This completes the proof.

□

Reiterating Remark 1.1, if one believes the Phillips-Sarnak conjecture [PS85], then the convexity L^2 bounds on Eisenstein series should be sharp for generic lattices. So (1.3) really is relying heavily on the arithmeticity of $\mathrm{SL}_2(\mathbb{Z})$.

References

[Ass19] E. Assing. On sup-norm bounds part II: GL(2) Eisenstein series. *Forum Math.*, 31(4):971–1006, 2019. 2

[BHM16] V. Blomer, G. Harcos, and D. Milićević. Bounds for eigenforms on arithmetic hyperbolic 3-manifolds. *Duke Math. J.*, 165(4):625–659, 2016. 1

[Blo20] V. Blomer. Epstein zeta-functions, subconvexity, and the purity conjecture. *J. Inst. Math. Jussieu*, 19(2):581–596, 2020. 1, 3, 8

[CS80] P. Cohen and P. Sarnak. Note on the Selberg trace formula. 1980. 2

[EMM98] A. Eskin, G. A. Margulis, and S. Mozes. Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture. *Ann. of Math. (2)*, 147(1):93–141, 1998. 3, 4, 5

[Hej76] D. Hejhal. *The Selberg trace formula for* $\mathrm{PSL}(2,R)$. *Vol. I.* Lecture Notes in Mathematics, Vol. 548. Springer-Verlag, Berlin-New York, 1976. 11

[Hua19] B. Huang. Sup-norm and nodal domains of dihedral Maass forms. *Comm. Math. Phys.*, 371(3):1261–1282, 2019. 1

[IS95] H. Iwaniec and P. Sarnak. L^∞ norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2)*, 141(2):301–320, 1995. 1

[Koy95] S. Koyama. L^∞-norms on eigenfunctions for arithmetic hyperbolic 3-manifolds. *Duke Math. J.*, 77(3):799–817, 1995. 1

[KY22] D. Kelmer and S. Yu. Fourier expansion of light-cone Eisenstein series. *arXiv preprint, arXiv:2209.06696*, 2022. 2

[Mil11] D. Milićević. Large values of eigenfunctions on arithmetic hyperbolic 3-manifolds. *Geom. Funct. Anal.*, 21(6):1375–1418, 2011. 1

[PS85] R. S. Phillips and P. Sarnak. On cusp forms for co-finite subgroups of $\mathrm{PSL}(2,\mathbf{R})$. *Invent. Math.*, 80(2):339–364, 1985. 2, 12

[RS94] Z. Rudnick and P. Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994. 1

[Sch68] W. Schmidt. Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height. *Duke Math. J.*, 35:327–339, 1968. 5

[SS89] A. Seeger and C. D. Sogge. Bounds for eigenfunctions of differential operators. *Indiana Univ. Math. J.*, 38(3):669–682, 1989. 1

[SV05] O. V. Savastru and P. D. Varbanets. On the mean square of the Epstein zeta-function. *Algebra Discrete Math.*, (1):105–121, 2005. 6, 8

[Ter73] A. Terras, A. Bessel series expansions of the Epstein zeta function and the functional equation. *Trans. Amer. Math. Soc.*, 183:477–486, 1973. 3

[You18] M. P. Young. A note on the sup norm of Eisenstein series. *Q. J. Math.*, 69(4):1151–1161, 2018. 1

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