Counting in Lattice Orbits

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Abstract

Given a discrete lattice, $\Gamma < \mathrm{SL}_m(\mathbb{R})$, and a base point $o \in \mathbb{R}^m$, let $N_\Gamma(T)$ denote the number of points in the orbit $o \cdot \Gamma$ whose (Euclidean) length is bounded by a growing parameter, T. We demonstrate an abstract spectral method capable of obtaining strong asymptotic estimates for $N_\Gamma(T)$ without relying on the meromorphic continuation of higher rank Langlands Eisenstein series.

1 Introduction

In this paper, we study the general orbital counting problem in real space, by which we mean the following. Fix $m \geq 2$ and a base point $o \in \mathbb{R}^m$, and let $\Gamma < G := \mathrm{SL}_m(\mathbb{R})$ be a discrete lattice such that the orbit $\mathcal{O} := o \cdot \Gamma \subset \mathbb{R}^m$ is discrete and infinite. Then the orbital counting problem is to obtain sharp asymptotic estimates for

$$N_\Gamma(T) := \#\{z \in \mathcal{O} \ : \ \|z\| \leq T\}, \tag{1.1}$$

where $\|z\|^2 := z_1^2 + \cdots + z_m^2$ (or another archimedean norm).

By standard Tauberian arguments, the asymptotic expansion of $N_\Gamma(T)$ is closely related to the meromorphic continuation of a mirabolic-type Eisenstein (or Poincaré) series:

$$E_\Gamma : \mathbb{R}^m \times \mathbb{C} \ \ni \ (p, s) \ \mapsto \ E_\Gamma(p, s) := \sum_{\gamma \in \Gamma_H \backslash \Gamma} \frac{1}{\|p\gamma\|^s},$$

which converges in some half-plane $\Re(s) \gg 1$. Being a Dirichlet series with non-negative coefficients, this $E_\Gamma(p, \cdot)$ has some abscissa of convergence $\delta \geq 0$, from which it is not difficult to conclude the crude estimate that

$$N_\Gamma(T) = T^{\delta + o(1)},$$

as $T \to \infty$. Since Γ is a lattice, this exponent $\delta = m$. One can say more using spectral theory, as follows.

The norm $\|\cdot\|$ on $\mathbb{R}^n$ is invariant under a maximal compact subgroup $K \cong \mathrm{SO}(m)$, and the locally symmetric space $X := \Gamma\backslash G/K$ is endowed with a Riemannian metric corresponding to the Killing form on the Lie algebra of G; let Δ denote the Laplace operator (quadratic Casimir) on X; see §2.3 for details and normalization. Then the spectrum of Δ below 1 consists of finitely many "exceptional" eigenvalues

$$0 \leq \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k < 1.$$

Our normalization of the Laplace operator is such that the tempered spectrum starts at 1. Further, let s_i be the positive root $s_i = m\sqrt{\lambda_i}$. Using either homogeneous dynamics or the meromorphic continuation of mirabolic Eisenstein series, it is more or less standard to prove a result of the following form.

Theorem 1. *There exist constants $c_0 > 0$, $c_1, \ldots, c_k$ and $\eta_m > 0$ such that*

$$N_\Gamma(T) = c_0 T^m + c_1 T^{(m-s_1)} + \cdots + c_k T^{(m-s_k)} + O(T^{m-\eta_m}). \tag{1.2}$$

Our goal in this paper is to demonstrate a soft abstract spectral-theoretic technique in higher rank by giving a novel proof of Theorem 1. (For a rank-one instance of this technique, see [KL22].) While this method does not recover the same exponent as the explicit spectral method of meromorphic continuation of higher-rank Eisenstein series, it avoids the technicalities thereof, while producing error exponents that are stronger than those obtained from more traditional homogeneous dynamics approaches. The latter have proven much more flexible in situations where an explicit spectral decomposition is not readily available.

Using homogeneous dynamics to attack such orbital counting problems is well studied in the literature, see, e.g., Margulis [Mar04], Duke-Rudnick-Sarnak [DRS93], and Eskin-McMullen [EM93]. A typical strategy is as follows. Writing $H = \mathrm{Stab}_G(o)$ for the stabilizer of o in G, let

$$\chi_T(g) = \mathbb{1}_{\|og\|<T}$$

be the indicator function of the count in question. This is a function on $H\backslash G/K$, that is, it is left-invariant by the stabilizer subgroup, and right-invariant under the maximal compact K, since the norm $\|\cdot\|$ is (bi-)K-invariant. We then create the automorphic function:

$$F_T(g) = \sum_{\gamma \in \Gamma_H\backslash\Gamma} \chi_T(\gamma g),$$

where $\Gamma_H := \Gamma \cap H$ is the stabilizer of o in Γ; the assumed discreteness of the orbit $\mathcal{O}$ implies that Γ_H is a lattice in H. Then $F_T(e)$ is exactly the count $N_\Gamma(T)$. To approximate $N_\Gamma(T)$, we smooth $F_T(e)$ as follows. Noting that F_T takes values in $\Gamma\backslash G/K$, we fix a bump function Ψ on $\Gamma\backslash G/K$; then $F_T(e)$ is approximated by

$$\widetilde{N}(T) = \int_{\Gamma\backslash G/K} \Psi(g) F_T(g) dg.$$

Unfolding, and writing an Iwasawa-type decomposition $G = HAK$, leads to:

$$\widetilde{N}(T) = \int_A \mathbb{1}_{\|oa\|<T} \left[\int_{\Gamma_H\backslash H} \Psi(ha) dh \right] da$$

The bracketed term is the expansion by $a \in A$ of the finite H-volume quotient $\Gamma_H\backslash H$, and is typically analyzed by a second smoothing process, namely, thickening and applying a wavefront-type lemma. There is significant loss in this analysis from this second smoothing, and this is what we are able to avoid here.

Our method is based on a technique using abstract spectral theory. In our application, the bracketed function, extended to G, that is, the map

$$g \mapsto \int_{\Gamma_H\backslash H} \Psi(hg) dh$$

is a function on the double-coset $H\backslash G/K$, which is one-dimensional (just a function of a one-parameter $a \in A$). Then if Ψ were an eigenfunction of the quadratic Casimir operator (the others, it turns out, need not enter the analysis!), it would satisfy a quadratic ODE which can be solved explicitly. This observation, together with L^2-techniques developed by the authors in [Kon09, KL22, Lut22], can be turned into a proof. The main technical innovation in this paper is an analysis of the Lie algebra structure, explicit Casimir operator, and Haar measure resulting from the choice of parametrization of the group suitable for our application. The key calculation is that, in our $H \times A \times K_H\backslash K$ coordinates (see (2.1) and Theorem 2), the Haar measure on G decomposes as $dg = dh\, da\, dk$, in which dh is Haar measure on H. This fact is crucial for the analysis carried out in §3.1.

Remark. In smooth form (see Theorem 4 below), our error term exhibits square-root cancellation, that is, has size $T^{m/2}$, which is optimal in the sense that it reaches the tempered spectrum.

Remark. Our proof of Theorem 1 produces the error exponent:

$$\eta_m = \frac{2m}{(m+2)(m-1)+4}. \tag{1.3}$$

For example $\eta_2 = 1/2$, $\eta_3 = 3/7$, $\eta_4 = 4/11$, $\eta_5 = 5/16$, and $\eta_6 = 3/11$. In the special case that $\Gamma = \mathrm{SL}_m(\mathbb{Z})$ and $o = e_m = (0, \ldots, 0, 1)$, this problem amounts to counting primitive lattice points in the T-ball. That is, let $r_m^*(n)$ denote the number of ways to express an integer, n, as the sum of m squares which share no common factor,

$$r_m^*(n) := \#\{\mathbf{a} \in \mathbb{Z}^m \ : \ \|\mathbf{a}\|^2 = n \text{ with } (a_1, \dots, a_m) = 1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. A classic problem is to obtain an asymptotic formula for the number

$$N_m(T) := \sum_{n=1}^{T^2} r_m^*(n).$$

The main term is known to be $c_m T^m$, where $c_m = 1/\zeta(m)$ [Chr56], however finding optimal estimates for the error term is a challenging problem. When $m = 2$ the best known error term is due to Wu [Wu02] (assuming the Riemann hypothesis) who shows that $\eta_2 < \frac{387}{304} \approx 1.273\dots$. For $m = 3$ the best known result is that of Goldfeld-Hoffstein [GH85] that $\eta_3 < 39/32 \approx 1.219\dots$, which follows from the fact that N_3 can be related to the first moment of the quadratic Dirichlet L-function $L(\frac{1}{2}, \chi_{8d})$ (similar estimates were achieved by Young [You09] in the smooth case). For $m \geq 4$, one can use Möbius inversion to compare the primitive lattice point count to the Gauss circle problem. Then one can easily show that the asymptotic estimate above holds for any value of $\eta_m < 1$. All of these results are much stronger that what one can achieve in the generality of Theorem 1, and are possible due to the explicit nature of the lattice $\mathrm{SL}_m(\mathbb{Z})$.

1.1 Plan of paper

In Section 2, we present some preliminaries in Lie algebras, groups, and decompositions thereof, along with the main Structure Theorem (see Theorem 2) for the Haar measure and Casimir operator in these coordinates. Finally, in Section 3, we prove Theorem 1.

2 Preliminaries

Without loss of generality (conjugating Γ), we can choose our base point to be $o = e_m := (0, \dots, 0, 1) \in \mathbb{R}^m$. Let

$$G := \mathrm{SL}_m(\mathbb{R}) := \left\{ g = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ & \vdots & \vdots & \\ x_{(m-1)1} & x_{(m-1)2} & \dots & x_{(m-1)m} \\ a_1 & a_2 & \dots & a_m \end{pmatrix} \ : \ \det g = 1 \right\},$$

with coordinates as specified. Further let $H := \mathrm{Stab}_G(o) = \{g \in G : e_m g = e_m\}$; then $H \cong \mathrm{ASL}_{m-1}(\mathbb{R})$, or more explicitly,

$$H \;=\; \{g \in G \;:\; a_1 = \cdots = a_{m-1} = 0\;,\; a_m = 1\}\,.$$

Now fix Γ and let $\Gamma_H := \Gamma \cap H$. By the assumed discreteness of the orbit $\mathcal{O}$, the stabilizer Γ_H is a lattice in H. Then our count can be expressed as

$$N_\Gamma(T) := \#\{\gamma \in \Gamma_H\backslash\Gamma \;:\; a_1^2 + \cdots + a_m^2 \le T^2\}.$$

2.1 Group decomposition for $m = 3$

For the reader's benefit, we first express everything completely explicitly in the case $m = 3$. Let $\mathfrak{g} := \mathfrak{sl}_3(\mathbb{R})$ be the Lie algebra associated to G. It is convenient to decompose $\mathfrak{g}$ according to the following basis:

$$X_{H,1} := \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix}, \quad X_{H,2} := \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, \quad X_{H,3} := \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix},$$

$$X_{H,4} := \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}, \quad X_{H,5} := \begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix},$$

$$X_A := \begin{pmatrix} -1/2 & & \\ & -1/2 & \\ & & 1 \end{pmatrix}$$

$$X_{K,1} := \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}, \quad X_{K,2} := \begin{pmatrix} & & \\ & & 1 \\ & -1 & \end{pmatrix}.$$

The basis elements $X_{H,i}$ generate $\mathfrak{h} = \mathrm{Lie}(H)$, and we denote their matrix exponentials by $n_1(x_1) = \exp(x_1 X_{H,1}), n_2(x_2), n_3(x_3), a_H(t), k_H(\theta)$ respectively. We denote the exponential of X_A by $\widetilde{a}(t) = \exp(X_A t)$. Rather than work with the t variable, we prefer to work with $r = e^t > 0$; thus we set

$$a(r) := \widetilde{a}(\log r) = \begin{pmatrix} r^{-1/2} & & \\ & r^{-1/2} & \\ & & r \end{pmatrix}.$$

Finally $X_{K,1}$ and $X_{K,2}$ correspond to two rotations, and we denote their exponentials by $k_1(\theta_1) = \exp(\theta_1 X_{K,1})$ and similarly for $k_2(\theta_2)$. Thus, given a $g \in G$, we can write

$$g = n_H(\mathbf{x}) a_H(t) k_H(\theta) a(r) k_1(\theta_1) k_2(\theta_2),$$

where $n_H(\mathbf{x}) = n_1(x_1)n_2(x_2)n_3(x_3)$. Note that $a(r)$ commutes with $k_H(\theta)$, and if we move k_H to the right, then $k_H k_1 k_2$ generate $\mathrm{SO}(3)$.

2.2 Group decomposition for general m

In general, we parametrize $G = \mathrm{SL}_m(\mathbb{R})$ via the map

$$H \times A \times K_H \backslash K \to G,$$

where A is a *one*-parameter diagonal group, and $K_H = K \cap H \cong \mathrm{SO}(m-1)$ (note that this subgroup commutes with A). We furthermore decompose H in standard Iwasawa coordinates,

$$H = N_H \times A_H \times K_H,$$

leading to the G-coordinate system:

$$N_H \times A_H \times K_H \times A \times K_H \backslash K \to G.$$

More explicitly, we decompose G into: $H \cong \mathrm{ASL}_{m-1}(\mathbb{R}) \times$ a one-parameter diagonal subgroup $\times$ a product of $(m-1)$ one-parameter rotations. That is, we again let

$$a(r) = \mathrm{diag}(r^{-1/(m-1)}, \dots, r^{-1/(m-1)}, r).$$

Let $X_{K,i}$ be the element of the Lie algebra with $(X_{K,i})_{mi} = -(X_{K,i})_{im} = 1$ for $i = 1, \dots, m-1$, and let $k_i(\theta_i) = \exp(\theta_i X_{K,i})$. Then in a neighborhood of the identity in G, we can write $g \in G$ as

$$g = h a(r) k_1(\theta_1) \cdots k_{m-1}(\theta_{m-1}), \tag{2.1}$$

for some $h \in H$, $r > 0$, and $\theta_i \in [0, 2\pi)$. We denote by $k(\boldsymbol{\theta}) := k_1(\theta_1) \cdots k_{m-1}(\theta_{m-1})$, corresponding to a choice of Euler coordinates on the sphere $K_H \backslash K \cong \mathbb{S}^{m-1}$.

We further decompose H into a product of an upper triangular matrix $n_H(\mathbf{x})$, where $\mathbf{x}$ has dimension $\frac{m(m-1)}{2}$ $\times$ a diagonal matrix $a_H(\mathbf{t})$ (of dimension $m-2$) $\times$ an element of $K_H \cong \mathrm{SO}(m-1)$ that we denote $k_H(\boldsymbol{\varphi})$ (of dimension $\frac{(m-1)(m-2)}{2}$). Thus we write

$$g = n_H(\mathbf{x}) a_H(\mathbf{t}) k_H(\boldsymbol{\varphi}) a(r) k(\boldsymbol{\theta}). \tag{2.2}$$

Crucially, note that k_H commutes with $a(r)$. This allows us to multiply together the

a_H and $a(r)$ matrices and change coordinates to the more standard Iwasawa coordinates (see [Gol06]) where the Haar measure and Casimir operator are known, resulting in the following.

Theorem 2 (Structure Theorem of the Haar measure and Casimir operator)**.** *The Haar measure on G in the coordinates of (2.2) is given by $dg = dh\, da\, dk$, or more explicitly:*

$$\mathrm{d}g = (\rho_1(\mathbf{x}, \mathbf{t}, \boldsymbol{\varphi})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{t}\mathrm{d}\boldsymbol{\varphi}) \left(r^{m-1}\mathrm{d}r\right) (\rho_2(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}), \tag{2.3}$$

where ρ_1 is the Haar measure density on $\mathrm{ASL}_{m-1}(\mathbb{R})$ and ρ_2 is bounded. Meanwhile the quadratic Casimir operator, acting on left-H-invariant and right-K-invariant functions $f(r) = f(ha(r)k)$, is given in these coordinates by

$$\Delta f(r) = \frac{4}{m^2}(r^2\partial_{rr} + r\partial_r)f(r). \tag{2.4}$$

Proof. The Haar measure is well-known in standard Iwasawa coordinates, see, e.g. [Gol06, Theorem 1.6.1]. Thus to prove (2.3), all that is needed is a change of coordinates and an inductive argument.

As for the Casimir operator, let $X_{H,1}, \ldots, X_{H,(m-1)m}$ be a basis for $\mathfrak{h}$, let X_A be the Lie element $\mathrm{diag}(\frac{1}{m-1}, \ldots, \frac{1}{m-1}, 1)$, and let $X_{K,1}, \ldots, X_{K,m-1}$ be the basis elements corresponding to k_i.

The quadratic Casimir operator (as an element of the universal enveloping algebra of $\mathfrak{g}$) is then given by:

$$\Delta = \sum_{i=1}^{(m-1)m} X_{H,i}^* X_{H,i} + X_A^* X_A + \sum_{i=1}^{m-1} X_{K,i}^* X_{K,i},$$

where X^* is the dual element (that is $B(X_i, X_j^*) = \delta_{i,j}$ where B is the Killing form). Each basis element, X corresponds to a differential operator given by

$$D_X f(g) = \frac{d}{dt} f(g\exp(tX))\bigg|_{t=0}.$$

Using the fact that $a(r)$ commutes with the almost all of H, one can show that the only contribution to the Casimir (when acting on H-invariant function) is given by $X_A^* X_A = c_m X_A^2$, for some constant c_m. Note that for the differential operator X_A, we can use the method in [BKS10, Proof of Lemma 2.7] (also used in [KL22, Proof of Theorem 8]) to compute X_A. The normalization can be derived by acting on the I function [Gol06, Definition 2.4.1] and matching eigenvalues. □

2.3 Decomposition of $L^2(\Gamma\backslash G/K)$ into irreducibles and the spectral theorem

The Riemannian metric on the locally symmetric space $\Gamma\backslash G/K$ has an associated Laplace-Beltrami operator. With respect to the right-regular representation of G on $L^2(\Gamma\backslash G)$, the quadratic Casimir operator Δ agrees on the subspace $\mathscr{H} := L^2(\Gamma\backslash G/K)$ of right K-invariant functions, with the Laplacian. This operator, Δ, is positive and self-adjoint, and thus its spectrum lies in $\mathbb{R}_{\geq 0}$. We have the following abstract spectral theorem (see e.g., [Rud73, Ch. 13])

Theorem 3 (Abstract Spectral Theorem)**.** *There exists a spectral measure $\widehat{\mu}$ on $\mathbb{R}_{\geq 0}$ and a unitary spectral operator $\widehat{\ }: \mathscr{H} \to L^2([0,\infty), d\widehat{\mu})$ such that:*

i) Abstract Parseval's Identity: for $\phi_1, \phi_2 \in \mathscr{H}$

$$\langle \phi_1, \phi_2 \rangle_{\mathscr{H}} = \langle \widehat{\phi_1}, \widehat{\phi_2} \rangle_{L^2([0,\infty),d\widehat{\mu})}, \tag{2.5}$$

and

ii) The spectral operator is diagonal with respect to L: for $\phi \in \mathscr{H}$ and $\lambda \geq 0$,

$$\widehat{L\phi}(\lambda) = \lambda\widehat{\phi}(\lambda). \tag{2.6}$$

Moreover, if λ is in the point specturm of L with associated eigenspace $\mathscr{H}_\lambda$, then for any $\psi_1, \psi_2 \in \mathscr{H}$ one has

$$\widehat{\psi_1}(\lambda)\widehat{\overline{\psi}_2}(\lambda) = \langle \operatorname{Proj}_{\mathscr{H}_\lambda} \psi_1, \operatorname{Proj}_{\mathscr{H}_\lambda} \psi_2 \rangle, \tag{2.7}$$

where Proj refers to the projection to the subspace $\mathscr{H}_\lambda$. In the special case that $\mathscr{H}_\lambda$ is one-dimensional and spanned by the normalized eigenfunction ϕ_λ, we have that

$$\widehat{\psi_1}(\lambda)\widehat{\overline{\psi}_2}(\lambda) = \langle \psi_1, \phi_\lambda \rangle \langle \phi_\lambda, \psi_2 \rangle. \tag{2.8}$$

The group G acts by right regular representation on $\mathscr{H}$. The Hilbert space $\mathscr{H}$ decomposes into components as follows

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_k \oplus \mathscr{H}^{tempered}$$

where $\mathscr{H}_i$ is a finite dimensional eigenspace with Δ-eigenvalue λ_i and $\mathscr{H}^{tempered}$ denotes the tempered spectrum.

3 Proof of Theorem 1

We follow the standard procedure described in the introduction of smoothing the counting function, as follows. Let

$$\chi_T(g) := \mathbb{1}_{\|e_m g\|<T} = \begin{cases} 1 & \text{if } r \le T \\ 0 & \text{otherwise,} \end{cases}$$

where g is decomposed as above into $g = n_H a_H k_H a(r) k$. Let

$$F_T(g) = \sum_{\gamma \in \Gamma_H \backslash \Gamma} \chi_T(\gamma g),$$

whence $N_\Gamma(T) = F_T(e)$.

For $\varepsilon > 0$, choose a smooth, nonnegative, right-K-invariant bump function $\psi = \psi_\varepsilon$ supported in an ε-neighborhood of the identity coset of G/K, with $\int_{G/K} \psi = 1$, so that, for any $\gamma \in \Gamma$,

$$\int_{G/K} \chi_T(\gamma g)\psi(g)dg = \begin{cases} 1 & \text{if } \|e_m \gamma\| < T(1 - c\varepsilon) \\ 0 & \text{if } \|e_m \gamma\| > T(1 + c\varepsilon). \end{cases} \tag{3.1}$$

Since $G/K \cong N_H \times A_H \times A$ has dimension:

$$\frac{m(m-1)}{2} + (m-2) + 1 = \frac{(m+2)(m-1)}{2},$$

such a ψ can be constructed with

$$\|\psi\|_{L^2(G/K)} \ll \varepsilon^{-\frac{(m+2)(m-1)}{4}}.$$

Let $\Psi = \Psi_\varepsilon \in L^2(\Gamma\backslash G/K)$ denote the Γ-average of ψ_ε

$$\Psi(g) := \sum_{\gamma \in \Gamma} \psi(\gamma g).$$

It follows that

$$\|\Psi\|_{L^2(\Gamma\backslash G/K)} \ll \varepsilon^{-\frac{(m+2)(m-1)}{4}}. \tag{3.2}$$

Then our smoothed count is given by

$$\widetilde{N}(T) := \langle F_T, \Psi \rangle_{L^2(\Gamma\backslash G/K)}.$$

For this smooth count, we have the following asymptotic.

Theorem 4. *For any $\Gamma < \mathrm{SL}_m(\mathbb{Z})$ of finite co-volume we have*

$$\widetilde{N}(T) = c_0(\varepsilon)T^m + c_1(\varepsilon)T^{m-s_1} + \cdots + c_k(\varepsilon)T^{m-s_k} + O(\varepsilon^{-(m+2)(m-1)/4}T^{m/2}), \tag{3.3}$$

where for any $i = 1, \ldots, k$ we have that $c_i(\varepsilon) = C_i(1 + O(\varepsilon))$, where C_i are independent of ε.

Proof of Theorem 1 from Theorem 4. This argument is standard. After unfolding both F_T and Ψ, we have that:

$$\widetilde{N}(T) = \sum_{\gamma \in \Gamma_H \backslash \Gamma} \int_{G/K} \chi_T(\gamma g) \psi_\varepsilon(g) \mathrm{d}g.$$

It now follows from (3.1) that

$$\widetilde{N}(T(1 - c\varepsilon)) \leq N_\Gamma(T) \leq \widetilde{N}(T(1 + c\varepsilon)).$$

From here, we optimize the parameter ε by choosing $\varepsilon = T^{\frac{-2m}{(m+2)(m-1)+4}}$; this leads to Theorem 1 with the error term claimed in (1.3). □

The remainder of the paper is devoted to the proof of Theorem 4.

3.1 Unfolding and the differential equation

By unfolding, using the decomposition $g = ha(r)k$, and the calculation of Haar measure in (2.3), our smooth count becomes

$$\begin{aligned}
\widetilde{N}(T) &= \int_{\Gamma \backslash G/K} \sum_{\gamma \in \Gamma_H \backslash \Gamma} \chi_T(\gamma g) \Psi(g) \mathrm{d}g \\
&= \int_{\Gamma_H \backslash G/K} \chi_T(g) \Psi(g) \mathrm{d}g \\
&= \int_0^\infty \chi_T(r) r^{m-1} \left(\int_{\Gamma_H \backslash H} \Psi(ha(r)) \mathrm{d}h \right) \mathrm{d}r,
\end{aligned}$$

since Ψ is right K-invariant. Let $f(r) := \int_{\Gamma_H \backslash H} \Psi(ha(r)) \mathrm{d}h$ denote the quantity inside the brackets.

Then using Theorem 2 we know that, for any value of λ, f satisfies the differential equation

$$\left(\frac{4}{m^2}(r^2 \partial_{rr} + r\partial_r) - \lambda \right) f(r) = g(r) \tag{3.4}$$

with

$$g(r) := \int_{\Gamma_H \backslash H} (\Delta - \lambda) \Psi(ha(r)) \mathrm{d}h.$$

For $\lambda \neq 0$, the homogeneous case ($g \equiv 0$) in (3.4) has two solutions, namely $f_\pm(r) =$

$A_\pm\, r^{m-1\pm s}$, for some constants $A_\pm$; here we have written $\lambda = \frac{4}{m^2}s^2$ with $s > 0$. Write

$$\begin{aligned}\alpha_\pm(T) &:= \int_0^\infty \chi_T(r) r^{m-1\pm s} \mathrm{d}r \\ &= \frac{1}{m\pm s} T^{m\pm s}.\end{aligned}$$

Using the same proof in [Kon09, Proof of Lemma 3.3], we can thus write

$$\widetilde{N}(T) = A_+\alpha_+(T) + A_-\alpha_-(T) + O(\|(\Delta-\lambda)\Psi\|). \tag{3.5}$$

However, note that we can trivially bound $\widetilde{N}(T) \ll T^m$, hence $A_+ = 0$. Thus we can in fact write

$$\widetilde{N}(T) = A\alpha(T) + O(\|(\Delta-\lambda)\Psi\|), \tag{3.6}$$

where $\alpha(T) = \alpha_-(T)$ and $A = A_-$. Hence we can solve for A and write

$$\widetilde{N}(T) = K_T(\lambda)\widetilde{N}(1) + O(\|(\Delta-\lambda)\Psi\|). \tag{3.7}$$

with $K_T(\lambda) = \frac{\alpha(T)}{\alpha(1)}$. The following theorem states that, since (3.7) holds for any Ψ and any λ, we can in fact show that the error vanishes. Since the proof is identical to the proof of [Kon09, Proposition 3.5] we omit it. Note that we can create an operator $K_T(\Delta)$ via a power series expansion of K_T.

Theorem 5 (Main Identity)**.** *For T large enough we have*

$$F_T(g) = K_T(\Delta)F_1(g) \tag{3.8}$$

almost everywhere. Moreover

$$K_T(\lambda) = \begin{cases} cT^{m-s} & \text{if } s < m/2 \\ cT^{m/2} & \text{if } s = m/2 + it. \end{cases} \tag{3.9}$$

3.2 Proof of Theorem 4

With the main identity at hand, we can proceed with the proof of Theorem 4. By Parseval's identity (2.5)

$$\begin{aligned}\widetilde{N}(T) &= \langle F_T, \Psi\rangle_\Gamma \\ &= \langle \widehat{F}_T, \widehat{\Psi}\rangle_{\mathrm{Spec}(\Gamma)} \\ &= \widehat{F_T}(\lambda_0)\widehat{\Psi}(\lambda_0) + \widehat{F_T}(\lambda_1)\widehat{\Psi}(\lambda_1) + \cdots + \widehat{F_T}(\lambda_k)\widehat{\Psi}(\lambda_k) \\ &\qquad + \int_{\mathcal{S}^{temp}} \widehat{F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda).\end{aligned}$$

Now for each point in the "exceptional" spectrum, $\widehat{\Psi}(\lambda_i)$ is the projection onto the i^{th} eigenspace, which is finite dimensional. Thus

$$\widehat{\Psi}(\lambda_i) = \langle \Psi, \phi_i \rangle,$$

and by the mean value theorem, we have that

$$\widehat{\Psi}(\lambda_i) = C_i + O(\varepsilon).$$

Moreover using (3.9), we have that

$$\begin{aligned}\widehat{F_T}(\lambda_i) &= T^{m-s_i}\langle F_1, \phi_i \rangle, \\ &= c_i T^{m-s_i},\end{aligned}$$

for some constants c_i.

As for the error term, we can use (2.6) and (3.9) to achieve the following bound

$$\begin{aligned}\int_{\mathcal{S}^{temp}} \widehat{F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda) &= \int_{\mathcal{S}^{temp}} \widehat{K_T(\Delta)F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda) \\ &= \int_{\mathcal{S}^{temp}} K_T(\lambda)\widehat{F_1}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda) \\ &\ll T^{m/2}\int_{\mathcal{S}^{temp}} \widehat{F_1}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda).\end{aligned}$$

From here we again apply Parseval's identity and Cauchy-Schwarz yielding

$$\int_{\mathcal{S}^{temp}} \widehat{F_1}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda) \ll T^{m/2}\|F_1\|\|\Psi\|.$$

Since $\Gamma\backslash G$ has finite volume and F_1 is bounded, the L^2 norm of F_1 is bounded. The proof follows on using (3.2).

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