Exceptional eigenvalue density for thin groups

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Abstract

We prove a limit multiplicity conjecture of Hee Oh for principal congruence covers of geometrically finite hyperbolic manifolds, with an explicit power-saving rate. The rate is governed by the return of Patterson–Sullivan shadows through a fixed compact core, giving a geometric interpretation to the exceptional eigenvalue density.

1 Introduction

Let $G := \mathrm{SO}(n,1)^o$, with maximal compact subgroup $K < G$. Let $\rho = \frac{n-1}{2}$, and let $\Gamma < G(\mathbb{Z})$ be a torsion-free, geometrically finite, Zariski dense subgroup. Write $X = \Gamma \backslash \mathbb{H}^n$ and let δ_Γ denote the critical exponent for Γ; we assume $\delta_\Gamma > \rho$. Elstrodt, Patterson, and Sullivan [Els73a, Els73b, Pat76, Sul87] showed that δ_Γ is equal to the Hausdorff dimension of the limit set and moreover, satisfies

$$\lambda_0(\Gamma) = \delta_\Gamma(n-1-\delta_\Gamma)$$

where $\lambda_0(\Gamma)$ is the bottom of the $L^2(X)$-spectrum of the Laplacian (equal to 0 for lattices and greater than or equal to 0 for non-lattice subgroups). If X is noncompact, the essential spectrum of the Laplacian starts at ρ^2; if X is compact, its spectrum is discrete. A major topic of research is the presence of possible *exceptional eigenvalues* below ρ^2 (see for example Selberg's eigenvalue conjecture [IK04]).

A natural question[1] is whether the exceptional eigenvalues have any geometric interpretation. To which the answer is yes. In particular, we show that the density is controlled by the return of Patterson-Sullivan shadows through a compact core. For convex-cocompact groups the Patterson-Sullivan measure is comparable to the Hausdorff measure. Hence, this allows us to give one possible geometric interpretation of the exceptional spectrum.

For $s \in (\rho, n-1]$, put $\lambda_s = s(n-1-s)$. If $\Gamma' < \Gamma$ has finite index, let $\mathcal{N}(\lambda_s, \Gamma')$ denote the number of Laplace eigenvalues of $L^2(\Gamma' \backslash \mathbb{H}^n)$ in $[0, \lambda_s]$, counted with multiplicity. For $q \geq 1$, set

$$\Gamma(q) := \{\gamma \in \Gamma \ : \ \gamma \equiv e \mod q\}, \qquad D_q := [\Gamma : \Gamma(q)].$$

In [Oh15, Conjecture 1.11], Hee Oh conjectured that the normalized exceptional eigenvalue count tends to zero along the principal congruence tower. We prove this conjecture with polynomial decay.

[1] We thank Dennis Sullivan for posing this question to the author

Theorem 1 (Oh's Limit Multiplicity Conjecture)**.** *For every $s \in (\rho, \delta_\Gamma]$ and every $q \geq 3$,*

$$\frac{\mathcal{N}(\lambda_s, \Gamma(q))}{D_q} \ll_s q^{-(s-\rho)}. \tag{1.1}$$

Note that, for all but finitely many primes, strong approximation gives $D_q \asymp q^g$ where $g = \dim(G)$ [MVW84]. Hence, Theorem 1 gives

$$\mathcal{N}(\lambda_s, \Gamma(q)) \ll_s D_q^{1-(s-\rho)/g}. \tag{1.2}$$

1.1 Geometry and spectral density

We now state the geometric result behind Theorem 1. Let

$$p_i : X_i \to X$$

be finite covers of degree D_i. Fix $s \in (\rho, \delta_\Gamma]$. We say that (X_i) has the *uniform collar property at* s if there exists a compact subset $\Omega = \Omega_s \subset X$ and a constant $c_s > 0$ such that every L^2-normalized eigenfunction ϕ on X_i with spectral parameter $u \in [s, \delta_\Gamma]$ satisfies

$$\int_{p_i^{-1}(\Omega)} |\phi|^2 \mathrm{d}x \geq c_s. \tag{1.3}$$

This prevents exceptional eigenfunctions from escaping into the ends. By the collar lemma of Gamburd and Magee [Gam02, Mag15], principal congruence covers have the uniform collar property at every $s \in (\rho, \delta_\Gamma]$.

Put $\Omega_i = p_i^{-1}(\Omega)$. For $x = \Gamma_i g K \in \Omega_i$, set $r_g(\gamma) = d(go, \gamma go)$ and define the normalized return energy to be

$$\mathcal{E}_i(L) := \frac{1}{D_i} \int_{\Omega_i} \sum_{\substack{\gamma \in \Gamma_i \\ r_g(\gamma) \leq L}} e^{-\rho r_g(\gamma)} \mathrm{d}x, \tag{1.4}$$

where the sum is independent of the representative g.

Theorem 2 (Shadow-return transfer)**.** *Suppose (X_i) has the uniform collar property at s. Then, for every $L \geq 2$,*

$$\frac{\mathcal{N}(\lambda_s, \Gamma_i)}{D_i} \ll_s e^{-(s-\rho)L} \mathcal{E}_i(L). \tag{1.5}$$

The implied constant is independent of i and L.

Remark. We call this a shadow return lemma for the following reason. Given $z, w \in \mathbb{H}^n$ and $R > 0$, the *shadow* of $B_w(R)$ viewed from z is defined to be

$$\mathrm{Sh}_z(B_w(R)) := \{\xi \in \partial\mathbb{H}^n \ : \ [z, \xi) \cap B_w(R) \neq \varnothing\},$$

where $[z, \xi)$ denotes the geodesic ray connecting z to ξ. Thus, the shadow consists of all boundary points whose geodesic ray from z intersects $B_w(R)$. Let (ν_x) denote the Patterson-Sullivan density (see below) and fix R_0 sufficiently large. Then the shadow lemma [Sul87] gives, uniformly over lifted compact cores,

$$e^{-\rho r_g(\gamma)} \asymp \nu_{go}(\mathrm{Sh}_{go}(B_{\gamma go}(R_0)))^{\rho/\delta_\Gamma}. \tag{1.6}$$

Hence, $\mathcal{E}_i(L)$ is an averaged return energy for the Patterson-Sullivan shadows.

1.2 Previous work

[Oh15, Conjecture 1.11] is known to be true if Γ is a cocompact lattice by DeGeorge and Wallach [DW78], or if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ by Sarnak [Sar82]. It does not seem to be known for a general arithmetic lattice, although Savin proved it for those s whose corresponding eigenfunctions are cusp forms [Sav89]. We also refer to [FLM15], where an analogous problem was answered positively for principal congruence subgroups of GL_n and SL_n.

The quantitative density theorems of Sarnak–Xue and Oh give the following comparison.

Theorem 3 (Sarnak–Xue [SX91]; Oh [Oh15])**.**

1. *If Γ is a cocompact arithmetic subgroup of $\mathrm{SO}(n,1)^o$, with $n = 2$ or 3, then, for every fixed $s \in (\rho, n-1]$ and every $\epsilon > 0$,*
$$\mathcal{N}(\lambda_s, \Gamma(q)) \ll_{s,\epsilon} D_q^{((n-1)-s)/\rho+\epsilon}. \tag{1.7}$$

2. *Under the hypotheses of this paper, there exists $\eta > 0$ such that, for every fixed $s \in (\rho, \delta_\Gamma]$ and every prime q,*
$$\mathcal{N}(\lambda_s, \Gamma(q)) \ll_s D_q^{(\delta_\Gamma - s)/\eta}. \tag{1.8}$$
Here η may be any positive number smaller than the uniform spherical spectral gap of the family.

The estimate (1.8) is stronger than Theorem 1 when s is sufficiently close to δ_Γ, but it gives limit multiplicity only when $\delta_\Gamma - s < \eta$. Theorem 1 covers the full exceptional interval without a uniform spectral gap.

The classical limit-multiplicity theorem of DeGeorge and Wallach normalizes by total volume [DW78], while later work treats Benjamini–Schramm convergence [ABB+17]. In the infinite-volume setting, the collar property prevents escape of mass and the degree of one fixed compact core replaces total volume. The return condition is closely related to the weak injective-radius property of Golubev–Kamber [GK23]. The distinction here is that the spectral density is controlled directly by a boundary-geometric quantity, namely the return energy of Patterson–Sullivan shadows.

Notation

The identity element of G is denoted by e, and $o = eK \in \mathbb{H}^n$. Haar measure on G is normalized so that the induced measure on G/K is hyperbolic volume. We write $A \ll_\vartheta B$ if $|A| \le C_\vartheta B$ for a constant depending only on the indicated parameters and the fixed base manifold. The notation $B_w(R)$ denotes the hyperbolic ball of radius R centered at w.

2 Preliminaries

2.1 Patterson–Sullivan densities

For $\xi \in \partial\mathbb{H}^n$, let
$$\beta_\xi(z,w) = \lim_{y\to\xi}\big(d(z,y) - d(w,y)\big)$$

denote the Busemann cocycle. A Patterson–Sullivan density of dimension δ_Γ is a family $(\nu_z)_{z\in\mathbb{H}^n}$ of finite measures supported on the limit set $\Lambda(\Gamma)$ satisfying

$$\gamma_*\nu_z = \nu_{\gamma z}, \qquad \frac{\mathrm{d}\nu_z}{\mathrm{d}\nu_w}(\xi) = e^{-\delta_\Gamma \beta_\xi(z,w)}. \tag{2.1}$$

We fix one such density. For a geometrically finite, Zariski dense group it is unique up to scale. The shadow lemma gives

$$\nu_z(\mathrm{Sh}_z(B_{\gamma z}(R_0))) \asymp e^{-\delta_\Gamma d(z,\gamma z)} \tag{2.2}$$

for R_0 sufficiently large, uniformly when the image of z ranges over a fixed compact subset of X [Pat76, Sul87]. Raising (2.2) to the power ρ/δ_Γ gives (1.6).

2.2 Exceptional spectrum and positive kernels

We take the Laplacian to be nonnegative. Its spectrum below ρ^2 is discrete, with finite multiplicities, and every eigenvalue in this range has a unique parameter $u \in (\rho, 2\rho]$ such that

$$\lambda_u = u(2\rho - u) = u(n-1-u).$$

Since $u \mapsto \lambda_u$ is decreasing on $[\rho, 2\rho]$, $\mathcal{N}(\lambda_s, \Gamma')$ counts precisely the exceptional parameters $u \geq s$, with multiplicity.

Let ψ_u be the positive-definite spherical function corresponding to the parameter u, normalized by $\psi_u(e) = 1$. If $f \in L^1(K\backslash G/K)$, its spherical transform is

$$\widehat{f}(\lambda_u) = \int_G f(g)\psi_u(g)\,\mathrm{d}g. \tag{2.3}$$

We write $\check{f}(g) = \overline{f(g^{-1})}$. Then $f * \check{f}$ is of positive type and has spherical transform $|\widehat{f}|^2 \geq 0$. For every compact interval $J \subset (\rho, 2\rho]$, one has, uniformly for $u \in J$,

$$\psi_u(a_t) \asymp_J e^{(u-2\rho)t}, \tag{2.4}$$

where $d(o, a_t o) = t$.

For each q, let $s_{1,q} < \delta_\Gamma$ be the largest spherical complementary-series parameter below δ_Γ occurring in $L^2(\Gamma(q)\backslash\mathbb{H}^n)$, with $s_{1,q} = \rho$ if there is no such parameter. The family has a *uniform spherical spectral gap* if

$$\liminf_q (\delta_\Gamma - s_{1,q}) > 0.$$

2.3 Congruence covers and compact cores

Fix the defining integral representation $G \subset \mathrm{GL}_{n+1}$ and put $G(\mathbb{Z}) = G \cap \mathrm{GL}_{n+1}(\mathbb{Z})$. A subgroup of G is arithmetic if it is commensurable with the integral points of a $\mathbb{Q}$-form of G. The group $\Gamma(q)$ is the kernel of reduction modulo q, and is therefore normal in Γ.

For a finite-index subgroup $\Gamma_i < \Gamma$, we write $X_i = \Gamma_i \backslash \mathbb{H}^n$. If $\Omega \subset X$ is the compact set in the collar property and $\Omega_i = p_i^{-1}(\Omega)$, then

$$\mathrm{vol}(\Omega_i) = D_i \,\mathrm{vol}(\Omega).$$

We refer to Ω as the compact core. It is the compact set furnished by the collar property and need not be the convex core. If $x = \Gamma_i gK$, replacing g by another representative conjugates the indexing group in (1.4); consequently, the return energy is well-defined on Ω_i.

3 The shadow-return estimate

We prove Theorem 2. Let

$$B_T = \{g \in G : d(o, go) \le T\}.$$

Lemma 4 (Positive kernel)**.** *For every $T \ge 1$ there is a bi-K-invariant function F_T of positive type, supported in B_{2T}, such that*

$$\widehat{F}_T(\lambda_u) \gg_s e^{4(s-\rho)T} \qquad (u \in [s, \delta_\Gamma]), \tag{3.1}$$

$$0 \le F_T(g) \ll_s e^{2(s-\rho)T} e^{-\rho d(o,go)}. \tag{3.2}$$

Proof. Let $f_T = \mathbb{1}_{B_T}\psi_s$ and $F_T = f_T * \check{f}_T$. Then $\widehat{F}_T = |\widehat{f}_T|^2 \ge 0$. Uniformly for $u \in [s, \delta_\Gamma]$, (2.4) and the Jacobian ($\asymp e^{2\rho t}$) give

$$\widehat{f}_T(\lambda_u) = \int_{B_T} \psi_s(g)\psi_u(g)\,\mathrm{d}g \gg_s e^{(s+u-2\rho)T} \ge e^{2(s-\rho)T}.$$

This proves (3.1). The support statement is immediate. The standard hyperbolic shell-intersection estimate gives (3.2); see [SX91] and [Oh15, (2.3)]. $\square$

Proof of Theorem 2. Put $T = L/2$ and periodize F_T on $\Gamma_i \backslash G$:

$$K_{i,T}(g,h) = \sum_{\gamma \in \Gamma_i} F_T(g^{-1}\gamma h).$$

Since F_T is bi-K-invariant, this is a kernel on X_i. The pre-trace formula, positivity, (3.1), and the collar property (1.3) give

$$\int_{\Omega_i} K_{i,T}(g,g)\,\mathrm{d}x \gg_s e^{4(s-\rho)T}\mathcal{N}(\lambda_s, \Gamma_i). \tag{3.3}$$

The omitted discrete and continuous spectral terms are nonnegative. On the geometric side, (3.2) and the support of F_T give

$$\begin{aligned}\frac{1}{D_i}\int_{\Omega_i} K_{i,T}(g,g)\,\mathrm{d}x &\ll_s e^{2(s-\rho)T}\frac{1}{D_i}\int_{\Omega_i} \sum_{\substack{\gamma\in\Gamma_i \\ r_g(\gamma)\le 2T}} e^{-\rho r_g(\gamma)}\,\mathrm{d}x \\ &= e^{2(s-\rho)T}\mathcal{E}_i(2T).\end{aligned}$$

Comparing this with (3.3) and using $L = 2T$ proves (1.5). $\square$

4 Principal congruence covers

It remains to show that a principal congruence cover has no nontrivial compact-core return below logarithmic scale.

Lemma 5 (Compact-core congruence injectivity)**.** *There is a $B = B(\Gamma, \Omega)$ such that, for every $q \geq 3$,*

$$\inf_{\substack{x=\Gamma(q)gK\in\Omega_q \\ e\neq\gamma\in\Gamma(q)}} d(go, \gamma go) \geq \log q - B. \tag{4.1}$$

Proof. Choose a compact set $C \subset G$ whose image in X contains Ω. Every point of Ω_q has a representative αgK, with $\alpha \in \Gamma$ and $g \in C$. If $\eta \in \Gamma(q)$, then

$$d(\alpha go, \eta\alpha go) = d(go, \alpha^{-1}\eta\alpha go),$$

and $\alpha^{-1}\eta\alpha \in \Gamma(q)$ by normality. It therefore suffices to prove a uniform estimate for $g \in C$.

If $e \neq \gamma \in \Gamma(q)$, then some entry of $\gamma - I$ is a nonzero multiple of q. For any fixed matrix norm, $\|\gamma\| \gg q$. The Cartan decomposition in the defining representation gives

$$\|\gamma\| \ll_G e^{d(o,\gamma o)},$$

and therefore $d(o, \gamma o) \geq \log q - O_G(1)$. For $g \in C$,

$$d(go, \gamma go) \geq d(o, \gamma o) - 2d(o, go) \geq \log q - B.$$

The preceding reduction gives the same estimate on every sheet of Ω_q. □

Proof of Theorem 1. Fix $s \in (\rho, \delta_\Gamma]$ and take the compact set $\Omega = \Omega_s$ furnished by the collar lemma. For all sufficiently large q, so that the quantity below is at least 2, put

$$L = \log q - B - 1.$$

By Lemma 5, the identity is the only term in (1.4), and hence

$$\mathcal{E}_q(L) = \mathrm{vol}(\Omega).$$

Theorem 2 gives

$$\frac{\mathcal{N}(\lambda_s, \Gamma(q))}{D_q} \ll_s e^{-(s-\rho)(\log q - B - 1)} \ll_s q^{-(s-\rho)}.$$

The remaining finitely many levels are absorbed into the implied constant. □

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