# Spherical density for real semisimple groups

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**Abstract**

Let $G$ be a connected semisimple real Lie group with finite center and let $(\Gamma_N)$ be a tower of cocompact lattices. We prove that the weak injective radius property implies the spherical density hypothesis with the same parameter. Together with the converse theorem of Golubev and Kamber, this gives an equivalence in the cocompact Archimedean setting. The proof uses a uniform spherical-mass estimate near spectral walls and a positive pre-trace kernel, but no tempered-spectrum estimate or a priori spectral gap.

## 1 Introduction

Let $G$ be a connected semisimple real Lie group with finite center and let $K$ be a maximal compact subgroup. Let $X_N = \Gamma_N \backslash G/K$ be finite covers of a fixed compact locally symmetric space; thus $\Gamma_N < \Gamma_1$ has finite index, with no normality assumption. Put $V_N = [\Gamma_1 : \Gamma_N]$, and let $m(\pi, \Gamma_N)$ be the multiplicity of $\pi$ in $L^2(\Gamma_N \backslash G)$. If $\pi$ is an irreducible unitary representation of $G$, let $p(\pi)$ be the infimum of the $p \geq 2$ for which the $K$-finite matrix coefficients of $\pi$ belong to $L^{p+\epsilon}(G)$ for every $\epsilon > 0$.

Let $\widehat{G}_{\mathrm{sph}}$ be the set of spherical irreducible unitary representations. Fix a positive elliptic Casimir operator $\Omega$. If $\chi_\pi(\Omega)$ is the scalar by which it acts on $\pi$, set $\Lambda(\pi) = 1 + |\chi_\pi(\Omega)|$. For $L \geq 0$ and $p \geq 2$, put

$$M_{\mathrm{sph}}(L, \Gamma_N, p) = \sum_{\substack{\pi \in \widehat{G}_{\mathrm{sph}} \\ \Lambda(\pi) \leq L,\ p(\pi) \geq p}} m(\pi, \Gamma_N). \tag{1.1}$$

Using the Cartan decomposition, we write $g = k \exp(H) k'$ with $H$ in the closed positive Cartan chamber and set $\ell(g) = 2\rho(H)$, where $\rho$ is the half-sum of the positive restricted roots, counted with multiplicity. Golubev and Kamber introduced a weak injective radius property, recalled in Definition 2.2, and formulated their spherical density hypothesis as an upper bound for $M_{\mathrm{sph}}(L, \Gamma_N, p)$ with the Sarnak–Xue exponent. They conjectured that the former implies the latter [9, Conjecture 1.7], and proved this over non-Archimedean fields and in real rank one. In higher real rank the missing input was a spherical lower bound uniform near Weyl walls; see [9, Conjecture 3.12].

Our main theorem proves the Archimedean spherical implication.

**Theorem 1.1** (Weak injective radius implies spherical density)**.** *Let $G$ be a connected semisimple real Lie group with finite center, and let $(\Gamma_N)$ be a tower of cocompact lattices satisfying the weak injective radius property with parameter $0 < \alpha \leq 1$. There is a constant $C = C(G) > 0$ such that, for every $L \geq 0$, $p \geq 2$, and $\epsilon > 0$,*

$$M_{\mathrm{sph}}(L, \Gamma_N, p) \ll_{G,\Gamma_1,\alpha,\epsilon} (1+L)^C V_N^{1-\alpha(1-2/p)+\epsilon}. \tag{1.2}$$

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Golubev and Kamber proved the converse implication [9, Theorem 1.6]. Their argument preserves the parameter $\alpha$, and hence we obtain the following.

**Corollary 1.2** (Archimedean spherical equivalence)**.** *For towers of cocompact lattices in $G$, the weak injective radius property with parameter $\alpha$ is equivalent to the spherical density hypothesis with parameter $\alpha$.*

## Relation to previous work

Sarnak and Xue introduced density bounds as a quantitative substitute for temperedness [18]. Golubev and Kamber placed these estimates in a geometric framework and isolated spherical density and weak injective radius as two sides of the same conjectural picture [9].

There has also been substantial arithmetic progress. Blomer proved strong density estimates for $\mathrm{GL}_n$ [5]; Assing and Blomer treated principal congruence subgroups [2]; and Assing–Blomer–Nelson obtained arbitrary level and transferred the result to certain compact inner forms [3]. Further results include horizontal families of arithmetic lattices [7], paramodular quotients [1], and irreducible lattices in products of rank-one groups [13]. These arguments use arithmetic trace formulae, transfer, or the structure of the automorphic spectrum. The theorem here is an implication from a geometric counting property and applies to an arbitrary cocompact tower.

The positive-kernel construction has a more direct lineage. It refines Oh's pre-trace argument for the exceptional spectrum of congruence covers [15, Section 2.3]. In the author's earlier density theorem for thin groups [14], the same argument was sharpened by stopping the kernel before the first nonidentity return. The new feature here is the localization of the kernel in a critical Cartan direction, together with uniform control as the spectral parameter approaches a wall.

The representation-theoretic inputs are different as well. The polyhedral description of $p(\pi)$ is the same geometry which appears in [16]. The expansion at singular spectral parameters is due to Narayanan, Pasquale, and Pusti [17]. Their construction is local in the spectral parameter. The required global uniformity is obtained below from their explicit regularization and a finite induction over affine root hyperplanes.

## Proof mechanism

The invariant $p(\pi)$ measures the decay of the matrix coefficients of $\pi$; larger values correspond to slower decay. Let $\varphi_\lambda$ be the normalized spherical function and put

$$\vartheta(\lambda) = 1 - \frac{2}{p(\pi_\lambda)} \qquad \text{and} \qquad Q_\lambda = 2 + \|\lambda\|.$$

By Lemma 2.1, $\vartheta(\lambda)$ is the maximum over the finitely many normalized extreme rays of the positive Cartan chamber. The rays and the polyhedral formula are explained in Section 2.1; this is the same description used in [16]. Choose a direction where the maximum is attained and move it a logarithmic distance into the interior of the chamber. The move costs only a power of $Q_\lambda R$, and the Harish–Chandra expansion then gives a uniform amount of spherical mass. At a singular parameter we use the all-parameter expansion of Narayanan, Pasquale, and Pusti [17]; a finite induction over the singular hyperplanes prevents cancellation of the leading term. We obtain a cutoff $\chi_{\lambda,R}$ supported where $\ell(g) = R + O(1)$ and

$$\int_G \chi_{\lambda,R}(g) \mathrm{e}^{\ell(g)/p(\pi_\lambda)} |\varphi_\lambda(g)|^2 \, \mathrm{d}g \gg Q_\lambda^{-C} R^{-C} \mathrm{e}^{R(1-1/p(\pi_\lambda))}.$$

Define

$$\beta_{\lambda,R}(g) = \chi_{\lambda,R}(g)\mathrm{e}^{\ell(g)/p(\pi_\lambda)}\overline{\varphi_\lambda(g^{-1})}, \qquad \Phi_{\lambda,R} = \beta_{\lambda,R}^* * \beta_{\lambda,R}.$$

Then $\Phi_{\lambda,R}$ is positive on the unitary dual,

$$\widehat{\Phi}_{\lambda,R}(\lambda) \gg Q_\lambda^{-C} R^{-C} \mathrm{e}^{2R(1-1/p(\pi_\lambda))},$$

while the convolution estimate of [9] gives

$$|\Phi_{\lambda,R}(g)| \ll Q_\lambda^C R^C \mathrm{e}^{R-\ell(g)/2} \mathbf{1}_{\ell(g)\le 2R+O(1)}.$$

This is the pre-trace construction of [15, Section 2.3], with the kernel localized in the critical Cartan direction. A polynomially sized net of these kernels amplifies every parameter being counted. The weak injective radius property bounds the geometric trace by $V_N^{1+\epsilon}\mathrm{e}^R$, up to polynomial factors, whereas each counted representation contributes at least $\mathrm{e}^{2R(1-1/p)}$. Taking $R = \alpha \log V_N + O(1)$ gives (1.2).

## 2 Preliminaries

We recall the notation from spherical harmonic analysis that is used in the proof. Compact factors of $G$ may be included in $K$, so we assume that $G$ has no compact factors. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$. The Cartan decomposition of the Lie algebra is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Let $\Sigma$ be the restricted root system, with root multiplicities $m_\alpha$, and choose positive roots $\Sigma^+$ with simple roots $\Delta$. The open positive Cartan chamber is

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for every } \alpha \in \Delta\}.$$

We write $\overline{\mathfrak{a}^+}$ for its closure and put

$$\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha.$$

### Notation

The normalizer $N_K(\mathfrak{a})$ consists of the elements of $K$ which preserve $\mathfrak{a}$, and the centralizer $Z_K(\mathfrak{a})$ consists of those which act trivially on it. Their quotient

$$W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$$

is the *restricted Weyl group*. Fix a $W$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$, let $\alpha^\sharp \in \mathfrak{a}$ be defined by $\alpha(H) = \langle \alpha^\sharp, H\rangle$. The *coroot* of $\alpha$ is

$$\alpha^\vee = \frac{2\alpha^\sharp}{\langle \alpha^\sharp, \alpha^\sharp\rangle}; \qquad \alpha(\alpha^\vee) = 2.$$

Thus $\lambda(\alpha^\vee)$ means simply that the functional $\lambda$ is evaluated at the vector $\alpha^\vee$. The inner product identifies $\mathfrak{a}$ with $\mathfrak{a}^*$ and gives norms on these spaces and on their complexifications. Write $\mathfrak{a}_\mathbb{C}^* = \mathfrak{a}^* \otimes_\mathbb{R} \mathbb{C}$. On $\mathfrak{a}_\mathbb{C}^*/W$ we use the quotient distance

$$d_W(\lambda,\nu) = \min_{w\in W}\|\lambda - w\nu\|.$$

A functional $\eta \in \mathfrak{a}^*$ is *dominant* if

$$\eta(\alpha^\vee) \geq 0 \qquad (\alpha \in \Delta),$$

and every $W$-orbit in $\mathfrak{a}^*$ has a dominant representative. For $X, Y \in \overline{\mathfrak{a}^+}$, we write $X \preceq Y$ if

$$Y - X = \sum_{\alpha \in \Delta} t_\alpha \alpha^\vee \quad \text{with } t_\alpha \geq 0.$$

This is the dominance order on the closed chamber.

We write $A \ll_S B$ if $|A| \leq C_S B$, and $A \asymp_S B$ if both $A \ll_S B$ and $B \ll_S A$. The indicator of a set $E$ is denoted by $\mathbf{1}_E$. The space $C_c^\infty(G)$ consists of the smooth, compactly supported functions on $G$, and $C_c^\infty(K\backslash G/K)$ is its subspace of functions satisfying $f(k_1 g k_2) = f(g)$ for $k_1, k_2 \in K$. We call these functions $K$-bi-invariant. We normalize Haar measure on $K$ to have mass one. Our convolution and involution conventions are

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\, \mathrm{d}h, \qquad f^*(g) = \overline{f(g^{-1})}.$$

The Cartan decomposition says that every $g \in G$ can be written as $g = k \exp(H) k'$ with $k, k' \in K$ and a unique $H \in \overline{\mathfrak{a}^+}$. This $H$ is the *Cartan projection* $\mu(g)$. We use the scalar length

$$\ell(g) = 2\rho(\mu(g)).$$

The vector-valued triangle inequality for this order, $\mu(gh) \preceq \mu(g) + \mu(h)$, is proved, for example, in [12, Remark 3.36(ii)]. Since $\rho$ is dominant, it gives

$$\ell(g^{-1}) = \ell(g), \qquad \ell(gh) \leq \ell(g) + \ell(h). \tag{2.1}$$

In particular, $\ell$ is a proper, continuous, $K$-bi-invariant length.

## 2.1 Spherical parameters and the $p$-invariant

For $\pi \in \widehat{G}_{\mathrm{sph}}$, choose a unit vector $v \in \pi^K$ and write

$$\varphi_\pi(g) = \langle \pi(g)v, v\rangle, \qquad \varphi_\pi(e) = 1.$$

Normalized spherical functions are parameterized by $\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W$, where the quotient identifies $\lambda$ with $w\lambda$ for $w \in W$. We write $\varphi_\lambda$ and $\pi_\lambda$ for the corresponding function and representation. We normalize the parameter so that the exponents in the Harish–Chandra expansion are $w\lambda - \rho$; in the convention which writes $iw\lambda - \rho$, our $\lambda$ is replaced by $i\lambda$. In particular, $\varphi_\lambda = \varphi_{w\lambda}$. When useful, we choose a representative for which $\operatorname{Re}\lambda$ is dominant. By the Helgason–Johnson theorem, a bounded spherical function has a representative with

$$\operatorname{Re}\lambda \in \operatorname{conv}(W\rho).$$

Here $\operatorname{conv}(W\rho)$ is the convex hull of the finite Weyl orbit of $\rho$; see [10, Theorem IV.8.1]. We use the spectral size

$$Q_\lambda = 2 + \|\lambda\|.$$

For the positive elliptic Casimir operator $\Omega$ fixed in the introduction, let $\chi_\lambda(\Omega)$ be its scalar action on $\pi_\lambda$. Thus

$$\Lambda(\pi_\lambda) = 1 + |\chi_\lambda(\Omega)|.$$

Since the real part of a unitary spherical parameter lies in a fixed compact set, the Harish–Chandra isomorphism [10, Theorem II.5.18] gives

$$Q_\lambda^2 \asymp_G 1 + \Lambda(\pi_\lambda) \qquad (\pi_\lambda \in \widehat{G}_{\mathrm{sph}}). \tag{2.2}$$

The closed chamber $\overline{\mathfrak{a}^+}$ is a cone cut out by the linear inequalities $\alpha(H) \geq 0$, $\alpha \in \Delta$. A ray $\mathbb{R}_{\geq 0}H$ in this cone is *extreme* if $H = H' + H''$ with $H', H'' \in \overline{\mathfrak{a}^+}$ forces $H'$ and $H''$ to lie on the same ray. Equivalently, the extreme rays are the one-dimensional faces of the cone. Choose one vector $H_i$ on each extreme ray and normalize it by $2\rho(H_i) = 1$. If there are $r = \dim \mathfrak{a}$ such rays, define

$$\vartheta(\lambda) = \max_{w \in W} \max_{1 \leq i \leq r} \frac{\mathrm{Re}(w\lambda)(H_i)}{\rho(H_i)}. \tag{2.3}$$

**Lemma 2.1** (Polyhedral $p$-formula)**.** *For a spherical unitary representation $\pi_\lambda$,*

$$\vartheta(\lambda) = 1 - \frac{2}{p(\pi_\lambda)}.$$

*Here $p(\pi_\lambda) = \infty$ when $\vartheta(\lambda) = 1$.*

*Proof.* The leading-exponent criterion for matrix coefficients [9, Theorem 9.2] identifies the right-hand side with the maximum of $\max_w \mathrm{Re}(w\lambda)(H)/\rho(H)$ over nonzero $H$ in the positive chamber. This is the polyhedral norm used in [16, Proposition 3.3]. Because the quotient is unchanged when $H$ is multiplied by a positive scalar, it is enough to consider the compact simplex $2\rho(H) = 1$. On that simplex the function is convex and piecewise linear. It is therefore maximized at a vertex, and these vertices are precisely the normalized extreme rays $H_i$. □

For $f \in C_c^\infty(K\backslash G/K)$, its *spherical transform* is

$$\widehat{f}(\lambda) = \int_G f(g)\varphi_\lambda(g^{-1})\,\mathrm{d}g.$$

The integrated operator $\pi_\lambda(f) = \int_G f(g)\pi_\lambda(g)\,\mathrm{d}g$ acts on the line $\pi_\lambda^K$ by the scalar $\widehat{f}(\lambda)$. For an arbitrary unitary representation $\pi$, the same notation means $\pi(f) = \int_G f(g)\pi(g)\,\mathrm{d}g$. Thus $\widehat{f^* * f}(\lambda) = |\widehat{f}(\lambda)|^2$ on the spherical unitary dual. Moreover, if $f$ is $K$-bi-invariant, then $\mathrm{Tr}\,\pi_\lambda(f) = \widehat{f}(\lambda)$ for a spherical representation, while $\pi(f) = 0$ for a nonspherical representation.

## 2.2 The two density properties

Normalize the Haar measure by $\mathrm{vol}(\Gamma_1\backslash G) = 1$, so that $\mathrm{vol}(\Gamma_N\backslash G) = V_N$.

**Definition 2.2** (Weak injective radius)**.** The tower $(\Gamma_N)$ has the weak injective radius property with parameter $0 < \alpha \leq 1$ if there exists $C_0 > 0$ such that, uniformly for $0 \leq d \leq 2\alpha \log V_N - C_0$,

$$\frac{1}{V_N}\int_{\Gamma_N\backslash G} \#\{\gamma \in \Gamma_N : \ell(x^{-1}\gamma x) \leq d\}\,\mathrm{d}x \ll_\epsilon V_N^\epsilon \mathrm{e}^{d/2}. \tag{2.4}$$

For each $\epsilon > 0$, the implied constant is independent of $N$ and $d$.

**Remark 2.3** (Continuous and discrete formulations)**.** For a cocompact base lattice, Definition 2.2 is equivalent to the weak injective radius property of [9, Section 2.2], with the same parameter $\alpha$. Indeed, choose a compact fundamental set $\Omega \subset G$ for $\Gamma_1\backslash G$ and decompose $\Gamma_N\backslash G$ into its $V_N$ translates. The two averages differ only by conjugation by an element $y \in \Omega$, and

$$|\ell(y^{-1}gy) - \ell(g)| \leq 2\ell(y) \leq C_\Omega.$$

Thus each count is bounded by the other with the radius changed by at most $C_\Omega$. This preserves both the estimate and the parameter $\alpha$.

**Remark 2.4.** The parameter $\alpha$ can be estimated directly in arithmetic examples. Golubev and Kamber do this for principal congruence towers and, using general results on congruence subgroups, obtain an explicit positive parameter for a broader class of congruence towers; see [9, Corollary 2.2 and Theorem 2.3]. Thus Theorem 1.1 gives unconditional spherical density bounds for the cocompact congruence towers covered by their results.

**Definition 2.5** (Spherical density)**.** With $M_{\mathrm{sph}}$ as in (1.1), the tower has the spherical density hypothesis with parameter $\alpha$ if there is $C = C(G) > 0$ such that

$$M_{\mathrm{sph}}(L, \Gamma_N, p) \ll_\epsilon (1+L)^C V_N^{1-\alpha(1-2/p)+\epsilon}$$

for all $L \geq 0$, $p \geq 2$, and $\epsilon > 0$.

The exponent $C$ is independent of $p$ and $\epsilon$. This uniformity in the spectral aspect is used in Section 5.

### 2.3 The geometric trace bound

For $g \in G$, the right-regular action on $L^2(\Gamma_N \backslash G)$ is $R_{\Gamma_N}(g)u(x) = u(xg)$. For $F \in C_c^\infty(G)$, its integrated operator is

$$(R_{\Gamma_N}(F)u)(x) = \int_G F(g)u(xg)\,\mathrm{d}g.$$

Its automorphic kernel is $K_F(x,y) = \sum_{\gamma \in \Gamma_N} F(x^{-1}\gamma y)$. Because $\Gamma_N \backslash G$ is compact, the right-regular representation has a discrete spectral decomposition. For smooth compactly supported $F$,

$$\operatorname{Tr} R_{\Gamma_N}(F) = \sum_\pi m(\pi, \Gamma_N) \operatorname{Tr} \pi(F), \tag{2.5}$$

where the sum is over irreducible unitary representations occurring in $L^2(\Gamma_N \backslash G)$. If $F$ is $K$-bi-invariant, only spherical representations contribute.

**Lemma 2.6** (Geometric trace bound)**.** *There is $C_1 = C_1(G, C_0)$ with the following property. Suppose $R \leq \alpha \log V_N - C_1$ and $F = F^* \in C_c^\infty(G)$ satisfies*

$$|F(g)| \leq A_R \mathrm{e}^{R-\ell(g)/2} \mathbf{1}_{\ell(g) \leq 2R+C_1}. \tag{2.6}$$

*Then*

$$|\operatorname{Tr} R_{\Gamma_N}(F)| \ll_{G,\epsilon} A_R (1+R) V_N^{1+\epsilon} \mathrm{e}^R.$$

*Proof.* This is the shell argument used in [15, Section 2.3]; compare also [9, proof of Proposition 3.9 and Corollary 5.4]. Integrating the automorphic kernel on the diagonal gives

$$|\operatorname{Tr} R_{\Gamma_N}(F)| \leq \int_{\Gamma_N \backslash G} \sum_{\gamma \in \Gamma_N} |F(x^{-1}\gamma x)|\,\mathrm{d}x.$$

Take $C_1 \geq C_0 + 2$ and split the support into the shells $m \leq \ell < m+1$, $m \in \mathbb{Z}_{\geq 0}$. Then every radius $m+1$ below is in the range of (2.4). On the $m$th shell, (2.6) and (2.4) give

$$\begin{aligned}
|\operatorname{Tr} R_{\Gamma_N}(F)| &\leq A_R \mathrm{e}^R \sum_{\substack{m \in \mathbb{Z}_{\geq 0} \\ m \leq 2R+C_1}} \mathrm{e}^{-m/2} \int_{\Gamma_N \backslash G} \#\{\gamma \in \Gamma_N : \ell(x^{-1}\gamma x) < m+1\}\,\mathrm{d}x \\
&\ll_{G,\epsilon} A_R \mathrm{e}^R V_N^{1+\epsilon} \sum_{\substack{m \in \mathbb{Z}_{\geq 0} \\ m \leq 2R+C_1}} \mathrm{e}^{-m/2} \mathrm{e}^{(m+1)/2} \\
&\ll_{G,\epsilon} A_R (1+R) V_N^{1+\epsilon} \mathrm{e}^R.
\end{aligned}$$

□

We also use the local Weyl bound

$$M_{\mathrm{sph}}(L, \Gamma_N, 2) \ll_{G,\Gamma_1} V_N(1+L)^d \tag{2.7}$$

for some $d = d(G)$. This follows from the heat-kernel proof of Weyl's law; see [11]. The diagonal heat kernel on each cover is locally the lift of the heat-kernel parametrix on the fixed base. Its diagonal upper bound is therefore independent of the cover, and integration over $\Gamma_N \backslash G/K$ introduces the factor $V_N$.

# 3 A uniform spherical lower bound

This section proves the analytic estimate used by the positive kernel. We need a lower bound for $\varphi_\lambda$ which is uniform in both the Cartan variable $H$ and the spectral parameter $\lambda$. There are two possible losses of uniformity. First, the direction which controls $p(\pi_\lambda)$ may lie on the boundary of the positive Cartan chamber. Second, the usual Harish–Chandra formula has singular-looking coefficients when $\lambda$ approaches certain hyperplanes in the spectral parameter space. We deal with the first problem by moving the Cartan direction a distance $O(\log(Q_\lambda R))$ into the chamber. We deal with the second by using the expansion of Narayanan, Pasquale, and Pusti at singular parameters and then comparing nearby parameters.

## 3.1 The critical direction

Choose $H_\circ \in \mathfrak{a}^+$ with $2\rho(H_\circ) = 1$. This is a fixed direction in the interior of the chamber. By Weyl invariance, we choose the representative of $\lambda$ for which $\operatorname{Re}\lambda$ is dominant. We call an index $i$ *critical* if $H_i$ attains the maximum in (2.3). For such an index,

$$\operatorname{Re}\lambda(H_i) = \frac{1}{2}\vartheta(\lambda).$$

Let $a_0 = a_0(G)$ be a sufficiently large constant, to be fixed in the proof of Lemma 3.3, and put

$$\tau_{\lambda,R} = a_0 \log(2 + Q_\lambda R), \qquad H_{i,\lambda,R}(u) = (u - \tau_{\lambda,R})H_i + \tau_{\lambda,R}H_\circ. \tag{3.1}$$

The variable $u$ is the radial variable for our length: since $2\rho(H_i) = 2\rho(H_\circ) = 1$, we have

$$2\rho(H_{i,\lambda,R}(u)) = u.$$

The point $uH_i$ may lie on a wall. Formula (3.1) moves it by $\tau_{\lambda,R}(H_\circ - H_i)$ while keeping the same value of $u$. If $u \geq 2\tau_{\lambda,R}$, every positive root satisfies

$$\alpha(H_{i,\lambda,R}(u)) \gg_G \log(2 + Q_\lambda R), \tag{3.2}$$

and

$$\operatorname{Re}\lambda(H_{i,\lambda,R}(u)) \geq \frac{1}{2}\vartheta(\lambda)u - C_G \log(2 + Q_\lambda R). \tag{3.3}$$

Indeed,

$$\alpha(H_{i,\lambda,R}(u)) = (u - \tau_{\lambda,R})\alpha(H_i) + \tau_{\lambda,R}\alpha(H_\circ) \geq \tau_{\lambda,R}\alpha(H_\circ),$$

which proves (3.2). Since $i$ is critical,

$$\begin{aligned}\operatorname{Re}\lambda(H_{i,\lambda,R}(u)) &= (u - \tau_{\lambda,R})\frac{\vartheta(\lambda)}{2} + \tau_{\lambda,R}\operatorname{Re}\lambda(H_\circ) \\ &\geq \frac{1}{2}\vartheta(\lambda)u - C_G\tau_{\lambda,R}.\end{aligned}$$

Here we used $\operatorname{Re}\lambda \in \operatorname{conv}(W\rho)$, so $|\operatorname{Re}\lambda(H_\circ)| \ll_G 1$. This proves (3.3).

Let

$$\ker\rho = \{Z \in \mathfrak{a} : \rho(Z) = 0\}.$$

Fix bounded open rectangular boxes, centered at the origin,

$$B_0 \Subset B_1 \Subset B_2 \subset \ker\rho,$$

where $B_j \Subset B_{j+1}$ means that the closure of $B_j$ is contained in the interior of $B_{j+1}$. The variable $Z \in \ker\rho$ moves within a level set of the length. Choose $A_0 = A_0(G)$ sufficiently large compared with $a_0$ and suppose that

$$R \geq A_0 \log(2 + Q_\lambda R), \tag{3.4}$$

By (3.2), the following sets lie in the open chamber after $A_0$ is enlarged if necessary. Choose a smooth function supported in the open chamber which is one on

$$\{H_{i,\lambda,R}(u) + Z : R-2 \leq u \leq R-1,\ Z \in B_1\}$$

and is supported in

$$\{H_{i,\lambda,R}(u) + Z : R-3 \leq u \leq R,\ Z \in B_2\}.$$

Extend this function to a smooth $W$-invariant function on $\mathfrak{a}$ by Weyl reflection and by zero outside the reflected supports. Its composition with the Cartan projection is then a smooth, nonnegative, $K$-bi-invariant cutoff $\chi_{\lambda,i,R}$ on $G$. Its support satisfies

$$\operatorname{supp}\chi_{\lambda,i,R} \subset \{g : R-3 \leq \ell(g) \leq R\}.$$

In Cartan coordinates, Haar measure has the form

$$\int_G f(g)\,\mathrm{d}g = c_G \int_K \int_{\mathfrak{a}^+} \int_K f(k\exp(H)k')J(H)\,\mathrm{d}k\,\mathrm{d}H\,\mathrm{d}k', \qquad J(H) = \prod_{\alpha\in\Sigma^+} (\sinh\alpha(H))^{m_\alpha}.$$

The positive constant $c_G$ depends on the normalization of Haar measure. Every positive root is large throughout the support of the cutoff. Hence

$$J(H_{i,\lambda,R}(u) + Z) \asymp_G \mathrm{e}^u \tag{3.5}$$

there, because $2\rho(H_{i,\lambda,R}(u) + Z) = u$ and $\sinh\alpha(H) \asymp \mathrm{e}^{\alpha(H)}$ for $\alpha \in \Sigma^+$.

### 3.2 A finite exponential estimate

We first record the only finite-dimensional estimate used below. The important point is that the number of exponents and the polynomial degrees are fixed.

**Lemma 3.1** (Separated exponential polynomials)**.** *Let $B \subset \mathbb{R}^n$ be a box with nonempty interior, and fix $M, D, C_0 \geq 1$. Suppose that the $P_j$ are complex polynomials, $\zeta_j \in \mathbb{C}^n$, and*

$$E(x) = \sum_{j=1}^{m} P_j(x)\mathrm{e}^{\zeta_j\cdot x}, \qquad m \leq M, \quad \deg P_j \leq D,$$

*where $\|\operatorname{Re}\zeta_j\| \leq C_0$, $\|\zeta_j\| \leq Q$ with $Q \geq 2$, and $\|\zeta_j - \zeta_k\| \geq \delta$ for $j \neq k$. If $\mathcal{C}(E)$ is the largest absolute value of a coefficient of one of the $P_j$, then*

$$\|E\|_{L^2(B)} \geq c\,Q^{-C}\delta^C\mathcal{C}(E),$$

*where $c, C$ depend only on $B, M, D, C_0$.*

This is the fixed-complexity form of the standard quantitative independence estimate for quasipolynomials. We use it as a black box. A reference for the multivariable Bernstein estimate is [6]; the coefficient estimate, including polynomial factors, follows from the bound for inverse confluent Vandermonde matrices in [4, Theorem 1].

### 3.3 The all-parameter expansion

We first describe the hyperplanes which occur in the spectral parameter space. Use the inner product to put

$$\lambda_\alpha = \frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} = \frac{1}{2}\lambda(\alpha^\vee), \qquad \alpha \in \Sigma.$$

For $n \in \mathbb{Z}$, the set

$$\mathcal{H}_{\alpha,n} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_\alpha = n\} \tag{3.6}$$

is an *affine root hyperplane*. It is called affine because it is a translate of the linear root hyperplane $\lambda_\alpha = 0$. These hyperplanes lie in the spectral parameter space; they should not be confused with the walls $\alpha(H) = 0$ of the Cartan chamber.

If $\lambda$ lies on none of these hyperplanes, the Harish–Chandra expansion has the form

$$\varphi_\lambda(\exp H) = \sum_{w \in W} c(w\lambda) \mathrm{e}^{(w\lambda - \rho)(H)} \sum_{\mu \in \Lambda^+} \Gamma_\mu(w\lambda) \mathrm{e}^{-\mu(H)}. \tag{3.7}$$

Here $c$ is the Harish–Chandra $c$-function,

$$\Lambda^+ = \left\{ \sum_{\alpha \in \Delta} n_\alpha \alpha : n_\alpha \in \mathbb{Z}_{\geq 0} \right\}$$

is the positive root lattice, and $\mathrm{ht}(\mu) = \sum_{\alpha \in \Delta} n_\alpha$ is the height of $\mu$. The coefficients $c(w\lambda)$ and $\Gamma_\mu(w\lambda)$ may have poles on hyperplanes of the form (3.6), although their Weyl sum, which is the spherical function, is regular. When exponents meet on such a hyperplane, regularization replaces some exponentials by a polynomial in $H$ times an exponential. The elementary limit

$$\frac{\mathrm{e}^{zt} - \mathrm{e}^{-zt}}{z} \longrightarrow 2t \qquad (z \longrightarrow 0)$$

is the rank-one model for this phenomenon.

Fix a compact, convex, $W$-invariant neighborhood $U_0 \subset \mathfrak{a}^*$ of $\mathrm{conv}(W\rho)$ and put

$$\mathcal{U} = U_0 + i\mathfrak{a}^*.$$

The singular hyperplanes of the coefficients in (3.7) have integer translates as in (3.6). Since $U_0$ is compact, only finitely many of them meet $\mathcal{U}$. Enlarge this finite collection to be $W$-invariant and denote it by $\mathcal{H}$. A *stratum* of $\mathcal{H}$ is a connected component of the set of parameters which lie on one fixed collection of hyperplanes and on no others. A *proper substratum* is obtained by imposing at least one additional hyperplane equation. Distances are measured using the norm fixed in the notation subsection.

**Proposition 3.2** (Uniform all-parameter expansion)**.** *There are $C, d \geq 1$ with the following property. Let $S$ be a stratum, $\lambda \in S$, suppose that $\mathrm{Re}\,\lambda$ is dominant, and suppose, for some $0 < \delta \leq 1$, that $\lambda$ has distance at least $\delta$ from every proper substratum of $S$. For $H \in \mathfrak{a}^+$ with $\min_{\alpha \in \Delta} \alpha(H) \geq 1$,*

$$\varphi_\lambda(\exp H) = \mathcal{L}_\lambda(H) + \mathcal{R}_\lambda(H),$$

*where the following hold.*

1. *The leading part $\mathcal{L}_\lambda$ consists of the terms with $\mu = 0$. After equal exponentials have been grouped, it is a sum of at most $d$ terms $P(H)\mathrm{e}^{(w\lambda-\rho)(H)}$, with $\deg P \le d$.*

2. *The coefficients of these polynomials, and their parameter derivatives of order at most $d$, are bounded by $(Q_\lambda/\delta)^C$.*

3. *Let $P_{\lambda,e}(H)$ be the polynomial multiplying $\mathrm{e}^{(\lambda-\rho)(H)}$ after all equal exponentials have been grouped. At least one coefficient of $P_{\lambda,e}$ has absolute value at least $(Q_\lambda/\delta)^{-C}$.*

4. *For every $A > 0$, if $\min_{\alpha\in\Delta} \alpha(H) \ge a_A \log(2 + Q_\lambda/\delta)$, then*

$$|\mathcal{R}_\lambda(H)| \ll_{G,A} (1 + \|H\|)^C (Q_\lambda/\delta)^{-A} \mathrm{e}^{-\rho(H)+\max_w \operatorname{Re}(w\lambda)(H)}. \tag{3.8}$$

*All constants are uniform over the finitely many strata.*

*Proof.* The regularization at a singular parameter is [17, Proposition 2.5], the resulting all-parameter expansion is [17, Theorem 2.11], and the nonvanishing of the polynomial $P_{\lambda,e}$ in part (3) is [17, Corollary 2.10]. We record why the constants in those formulas grow at most polynomially in $Q_\lambda$ and $\delta^{-1}$.

The possible poles are simple and lie in $\mathcal{H}$ by [17, Theorem 1.2 and Lemma 1.9]. On a fixed stratum, the regularizing polynomials $p$ and $\pi$ of [17, Proposition 2.5] are products of the affine linear forms defining the singular hyperplanes which contain that stratum. Multiplication by $p$ removes the poles along those hyperplanes before the differential operator $\partial(\pi)$ is applied. Every other singular hyperplane is at distance at least $\delta$. Consequently, the terms with bounded $\operatorname{ht}(\mu)$, together with the finitely many parameter derivatives used in the regularization, are bounded by a fixed power of $Q_\lambda/\delta$. The gamma-product formula for $c$ and Stirling's formula give the same polynomial bound as the imaginary part of $\lambda$ grows.

For larger $m = \operatorname{ht}(\mu)$, the recurrence defining the Harish–Chandra coefficient, recorded in [17, equation (12) and Lemma 1.4], has denominator

$$|\langle \mu, \mu - 2\lambda\rangle| \ge \langle \mu, \mu - 2\operatorname{Re}\lambda\rangle \gg_G (1+m)^2,$$

once $m$ is larger than a fixed constant. There are $O_G(1 + m)$ terms in the recurrence, and their numerators are $O_G(Q_\lambda + m)$. If $\Gamma_{\mu,S}$ denotes the coefficient after regularization on $S$, induction on $m$, followed by induction on the number of parameter derivatives, gives

$$|\partial_\lambda^\kappa \Gamma_{\mu,S}(\lambda)| \le \left(C(Q_\lambda/\delta)^C\right)^{1+m}, \qquad |\kappa| \le d.$$

Since $\mu(H) \gg_G m \min_{\alpha\in\Delta} \alpha(H)$, each increase in $m$ also supplies exponential decay in $H$. The logarithmic lower bound in part (4) makes this decay larger than the coefficient growth. The remaining sum over $m$ is then a geometric series, which gives (3.8).

Finally, in the notation of [17, Lemma 2.6], $b_0(\lambda) = \pi_0(\lambda)c(\lambda)$ is neither zero nor infinite on the stratum. Corollary 2.10 of that paper shows that the highest-degree coefficient of $P_{\lambda,e}$ is a nonzero constant multiple of $b_0(\lambda)$. The product formula for $c$ in [17, equation (19)], Stirling's formula, and the separation from the remaining hyperplanes give the lower bound in part (3). □

## 3.4 Uniformity at spectral walls

The previous proposition is uniform as long as the parameter is not too close to a smaller stratum. We now remove that condition. Away from a smaller stratum we apply the expansion directly. Near a smaller stratum we move the parameter onto it and use continuity. Since there are only finitely many hyperplanes, this process stops after finitely many steps.

**Lemma 3.3** (Finite wall induction). *Let $\lambda \in \mathcal{U}$, choose its Weyl representative so that* $\operatorname{Re}\lambda$ *is dominant, and suppose* (3.4) *holds. Then, for every $i$,*

$$\int_{R-2}^{R-1} \int_{B_0} |\varphi_\lambda(\exp(H_{i,\lambda,R}(u) + Z))|^2 \, \mathrm{d}Z \, \mathrm{d}u \gg_G Q_\lambda^{-C} R^{-C} \mathrm{e}^{-R+2\operatorname{Re}\lambda(H_{i,\lambda,R}(R))}. \tag{3.9}$$

*Proof.* Put $X = 2 + Q_\lambda R$ and

$$E_{\lambda,i,R} = \exp\left(-\frac{R}{2} + \operatorname{Re}\lambda(H_{i,\lambda,R}(R))\right).$$

We prove by induction on the dimension of the stratum containing $\lambda$ that the square root of the left-hand side of (3.9) is at least $X^{-A_S} E_{\lambda,i,R}$ for some exponent $A_S$ depending only on the stratum $S$. Since $\mathcal{H}$ is finite, the exponents may be replaced at the end by one constant depending only on $G$.

During the induction we also allow the box $B_0$ to be translated by the small displacements produced when the parameter is moved to a substratum. An induction chain has length at most $\#\mathcal{H}$, so these translations remain in a fixed compact family of boxes contained in $B_1$; bounded values of $X$ are absorbed into the final constant. The direct estimate below is uniform over this family because Lemma 3.1 is uniform over a compact family of translated boxes. We denote the accumulated translation by $Z_0$; at the start of the induction $Z_0 = 0$.

We first give the direct argument. Write $s = u - R$ and $H_R = H_{i,\lambda,R}(R)$. Then

$$H_{i,\lambda,R}(u) + Z = H_R + sH_i + Z, \qquad -2 \le s \le -1.$$

Suppose that $\lambda$ is at distance at least $\delta$ from every proper substratum of $S$. By Proposition 3.2, the leading part can be written as

$$\mathcal{L}_\lambda(H_R + sH_i + Z) = \mathrm{e}^{(\lambda-\rho)(H_R)} \sum_{j=1}^{m} \widetilde{P}_j(s, Z) \mathrm{e}^{\zeta_j(s,Z)},$$

where $m$ and the degrees of the polynomials $\widetilde{P}_j$ are bounded in terms of $G$. More explicitly, a term $P(H)\mathrm{e}^{(w\lambda-\rho)(H)}$ contributes

$$\widetilde{P}_j(s, Z) = P(H_R + sH_i + Z)\mathrm{e}^{(w\lambda-\lambda)(H_R)}, \qquad \zeta_j(s, Z) = (w\lambda - \rho)(sH_i + Z).$$

Because $\operatorname{Re}\lambda$ is dominant and $H_R$ is in the positive chamber, $\operatorname{Re}(w\lambda - \lambda)(H_R) \le 0$. Translation of the polynomials by $H_R$ changes their coefficient norms by at most a fixed power of $R$. Part (3) of Proposition 3.2 therefore implies that at least one coefficient in the displayed exponential polynomial has absolute value at least $(X/\delta)^{-C}$.

The decomposition $\mathfrak{a} = \mathbb{R}H_i \oplus \ker\rho$ shows that restriction to the variables $(s, Z)$ is an isomorphism on linear functionals. Consequently, after equal exponentials are grouped, distinct $\zeta_j$ are separated by $c_G\delta$. Their real parts are bounded and their absolute values are $O_G(Q_\lambda)$. Applying Lemma 3.1 gives

$$\|\mathcal{L}_\lambda\|_{L^2([-2,-1]\times(B_0+Z_0))} \gg_G X^{-C}\delta^C E_{\lambda,i,R}. \tag{3.10}$$

Here and below the functions are evaluated at $H_R + sH_i + Z$. Choose the exponent $A$ in (3.8) larger than the fixed powers in (3.10). Since $\min_{\alpha\in\Delta} \alpha(H_R + sH_i + Z) \gg_G a_0 \log X$, choosing $a_0$ sufficiently large makes the hypothesis of (3.8) valid and gives

$$\|\mathcal{R}_\lambda\|_{L^2([-2,-1]\times(B_0+Z_0))} \le \frac{1}{2}\|\mathcal{L}_\lambda\|_{L^2([-2,-1]\times(B_0+Z_0))}.$$

This proves the desired estimate whenever $\lambda$ is $\delta$-separated from the proper substrata. For a stratum with no proper substrata, take $\delta = 1$; this starts the induction.

Now assume the result for every proper substratum of $S$, with a common exponent $A_{<S}$. The spherical function is entire in its parameter. If $v$ is a unit direction in $\mathfrak{a}_{\mathbb{C}}^*$, differentiation of the spherical integral formula introduces the factor $v(A(k \exp H))$, where $A(k \exp H)$ is the Iwasawa projection. Since $\|A(k \exp H)\| \ll_G \|H\|$, the integral formula and the standard spherical-function estimate [8, Proposition 4.6.1] give, uniformly for $\zeta \in \mathcal{U}$,

$$|\partial_v \varphi_\zeta(\exp H)| \ll_G (1 + \|H\|)^C \mathrm{e}^{-\rho(H) + \max_w \mathrm{Re}(w\zeta)(H)}; \tag{3.11}$$

see [10, Chapter IV, Section 4]. Choose $D > A_{<S} + C + 2$ and put $\delta = X^{-D}$. If $\lambda$ is $\delta$-separated from every proper substratum, the direct argument proves the claim, with a new exponent which may depend on $D$.

Otherwise choose $\lambda_0'$ on a proper substratum with $\|\lambda - \lambda_0'\| \ll_G \delta$. Choose $w \in W$ so that $\mathrm{Re}(w\lambda_0')$ is dominant and put $\lambda_0 = w\lambda_0'$. The distance between two dominant representatives is their quotient distance. Since $\mathrm{Re}\,\lambda$ is already dominant,

$$\|\mathrm{Re}\,\lambda_0 - \mathrm{Re}\,\lambda\| = d_W(\mathrm{Re}\,\lambda_0', \mathrm{Re}\,\lambda) \le \|\mathrm{Re}\,\lambda_0' - \mathrm{Re}\,\lambda\| \ll_G \delta.$$

Moreover, $\varphi_{\lambda_0} = \varphi_{\lambda_0'}$ and $Q_{\lambda_0} \asymp_G Q_\lambda$, so (3.4) also holds at $\lambda_0$ after a fixed change of constants. Along the segment joining $\lambda$ to $\lambda_0'$, the exponent on the right of (3.11) differs from $-\rho(H) + \mathrm{Re}\,\lambda(H)$ by $O_G(R\delta)$. The mean-value theorem therefore gives

$$\|\varphi_\lambda - \varphi_{\lambda_0}\|_{L^2} \ll_G X^{-D+C} E_{\lambda,i,R}, \tag{3.12}$$

where the norm is taken over $-2 \le s \le -1$ and $Z \in B_0 + Z_0$.

The difference

$$H_{i,\lambda,R}(u) - H_{i,\lambda_0,R}(u) = (\tau_{\lambda,R} - \tau_{\lambda_0,R})(H_\circ - H_i)$$

is independent of $u$, belongs to $\ker \rho$, and is $O_G(\delta)$. It is therefore absorbed by the allowed translation $Z_0$. The induction hypothesis at $\lambda_0$ gives

$$\|\varphi_{\lambda_0}\|_{L^2} \gg_G X^{-A_{<S}} E_{\lambda,i,R};$$

the exponent at $\lambda_0$ is comparable with the one displayed because their logarithms differ by $O_G(R\delta)$. Our choice of $D$ makes (3.12) at most half this lower bound. This completes the induction. Squaring the resulting estimate proves (3.9). Finally, since only finitely many strata occur, $a_0$ may be chosen once so that the remainder estimate used above holds in every direct step. We then enlarge $A_0$ so that (3.4) implies $R - 3 \ge 2\tau_{\lambda,R}$. $\square$

**Remark 3.4.** The polynomial $P_{\lambda,e}$ in part (3) of Proposition 3.2 is essential. Its quantitative nonvanishing supplies the coefficient to which Lemma 3.1 is applied; the size of an exponential alone would not give a lower bound for their sum.

### 3.5 The mass estimate

For a unitary parameter $\lambda$, write $p_\lambda = p(\pi_\lambda)$.

**Proposition 3.5** (Uniform critical mass)**.** *There are $B, C > 0$, depending only on $G$, with the following property. Let $\vartheta = \vartheta(\lambda) > 0$ and suppose*

$$R\vartheta \ge C \log(2 + Q_\lambda R). \tag{3.13}$$

*Then a critical index $i$ satisfies*

$$I_{\lambda,i,R} := \int_G \chi_{\lambda,i,R}(g) \mathrm{e}^{\ell(g)/p_\lambda} |\varphi_\lambda(g)|^2 \,\mathrm{d}g \gg_G Q_\lambda^{-C} R^{-C} \mathrm{e}^{R(1-1/p_\lambda)}. \tag{3.14}$$

*If $\nu$ is a spherical unitary parameter and*

$$d_W(\nu,\lambda) \le Q_\lambda^{-B} R^{-B}, \tag{3.15}$$

*then, for*

$$\beta_{\lambda,i,R}(g) = \chi_{\lambda,i,R}(g) \mathrm{e}^{\ell(g)/p_\lambda} \overline{\varphi_\lambda(g^{-1})},$$

*one has*

$$|\widehat{\beta}_{\lambda,i,R}(\nu)| \gg_G Q_\lambda^{-C} R^{-C} \mathrm{e}^{R(1-1/p_\nu)}. \tag{3.16}$$

*Finally, on the support of $\chi_{\lambda,i,R}$,*

$$\mathrm{e}^{\ell(g)/p_\lambda} |\varphi_\lambda(g)| \ll_G R^C Q_\lambda^C. \tag{3.17}$$

*Proof.* We first need an upper bound. The standard uniform estimate for the elementary spherical function is

$$|\varphi_\lambda(\exp H)| \ll_G (1+\|H\|)^C \mathrm{e}^{-\rho(H)+\max_w \mathrm{Re}(w\lambda)(H)} \tag{3.18}$$

see [8, Proposition 4.6.1]. By Lemma 2.1,

$$\max_w \mathrm{Re}(w\lambda)(H) \le \vartheta(\lambda)\rho(H) \qquad (H \in \overline{\mathfrak{a}^+}).$$

Since $1-\vartheta(\lambda) = 2/p_\lambda$ and $\ell(\exp H) = 2\rho(H)$, the exponential in (3.18) is at most $\mathrm{e}^{-\ell(\exp H)/p_\lambda}$. On the support of the cutoff, the remaining polynomial is $O_G(Q_\lambda^C R^C)$. This proves (3.17).

The constant in (3.13) is chosen large enough that (3.4) holds. For a critical index $i$, Lemma 3.3 and (3.3) give

$$\int_{R-2}^{R-1} \int_{B_0} |\varphi_\lambda(\exp(H_{i,\lambda,R}(u)+Z))|^2 \,\mathrm{d}Z \,\mathrm{d}u \gg_G Q_\lambda^{-C} R^{-C} \mathrm{e}^{-(1-\vartheta)R}.$$

On the region of integration, (3.5) gives $J(H) \asymp_G \mathrm{e}^R$ and $\mathrm{e}^{\ell(g)/p_\lambda} \asymp_G \mathrm{e}^{R/p_\lambda}$. Hence

$$I_{\lambda,i,R} \gg_G Q_\lambda^{-C} R^{-C} \mathrm{e}^{-(1-\vartheta)R+R+R/p_\lambda} = Q_\lambda^{-C} R^{-C} \mathrm{e}^{R(1-1/p_\lambda)}.$$

The definition of the spherical transform gives, at $\nu=\lambda$,

$$\widehat{\beta}_{\lambda,i,R}(\lambda) = I_{\lambda,i,R}.$$

Indeed, $\overline{\varphi_\lambda(g^{-1})}\varphi_\lambda(g^{-1}) = |\varphi_\lambda(g)|^2$ for a unitary spherical parameter. Choose the Weyl representative of $\nu$ for which $\|\nu-\lambda\| = d_W(\nu,\lambda)$, and let $\zeta$ lie on the segment from $\lambda$ to $\nu$. Since (2.3) is the maximum of finitely many linear functions,

$$|\vartheta(\zeta)-\vartheta(\lambda)| \ll_G \|\zeta-\lambda\|.$$

For $B \ge 1$, condition (3.15) therefore implies, on the support of $\beta_{\lambda,i,R}$,

$$-\rho(H) + \max_w \mathrm{Re}(w\zeta)(H) \le -\frac{\ell(\exp H)}{p_\lambda} + O_G(1).$$

Differentiating the spherical integral formula as in [10, Chapter IV, Section 4] introduces a factor of size $O_G(1+\|H\|)$; the absolute value of the resulting integral is therefore at most $O_G(1+\|H\|)\varphi_{\mathrm{Re}\,\zeta}(\exp H)$. Hence (3.18) and (3.17) give

$$|\beta_{\lambda,i,R}(g)|\|\nabla_\zeta\varphi_\zeta(g^{-1})\| \ll_G Q_\lambda^C R^C \mathrm{e}^{-\ell(g)/p_\lambda}$$

on this support. Cartan integration gives $\mathrm{vol}\{g : \ell(g) \le R\} \ll_G R^C \mathrm{e}^R$. Integration now yields

$$\|\nabla_\zeta \widehat{\beta}_{\lambda,i,R}(\zeta)\| \ll_G Q_\lambda^C R^C \mathrm{e}^{R(1-1/p_\lambda)} \tag{3.19}$$

along the segment; here $\nabla_\zeta$ is the gradient for the fixed inner product on the parameter space. Choose $B$ larger than the sum of the fixed powers in (3.14) and (3.19). The mean-value theorem and (3.15) then give

$$|\widehat{\beta}_{\lambda,i,R}(\nu) - \widehat{\beta}_{\lambda,i,R}(\lambda)| \le \frac{1}{2} I_{\lambda,i,R}.$$

Finally, the Lipschitz estimate for $\vartheta$ and Lemma 2.1 give

$$|p_\nu^{-1} - p_\lambda^{-1}| \ll_G d_W(\nu,\lambda), \qquad R|p_\nu^{-1} - p_\lambda^{-1}| = O_G(1).$$

Thus replacing $p_\lambda$ by $p_\nu$ changes the final exponential by only a bounded factor. □

# 4 Positive kernels

We now use the mass estimate to build a positive test function for the pre-trace formula. This is the construction of [15, Section 2.3]; compare also the truncated kernel in [14]. The preceding section allows us to place the test function in a critical Cartan direction without losing uniformity in the spectral parameter.

For the index $i$ supplied by Proposition 3.5, define

$$\Phi_{\lambda,i,R} = \beta^*_{\lambda,i,R} * \beta_{\lambda,i,R}.$$

**Lemma 4.1** (One positive kernel)**.** *Under the hypotheses of Proposition 3.5,*

$$\operatorname{supp}\Phi_{\lambda,i,R} \subset \{g : \ell(g) \le 2R + C\}, \tag{4.1}$$

$$|\Phi_{\lambda,i,R}(g)| \ll_G Q_\lambda^C R^C \mathrm{e}^{R-\ell(g)/2}\mathbf{1}_{\ell(g)\le 2R+C}, \tag{4.2}$$

$$\widehat{\Phi}_{\lambda,i,R}(\nu) \gg_G Q_\lambda^{-C} R^{-C} \mathrm{e}^{2R(1-1/p_\nu)} \tag{4.3}$$

*whenever* (3.15) *holds. Moreover,* $\pi(\Phi_{\lambda,i,R}) \ge 0$ *for every unitary representation* $\pi$.

*Proof.* By the definition of convolution,

$$\Phi_{\lambda,i,R}(g) = \int_G \overline{\beta_{\lambda,i,R}(h^{-1})}\beta_{\lambda,i,R}(h^{-1}g)\,\mathrm{d}h.$$

If the integrand is nonzero, then $h^{-1}$ and $h^{-1}g$ lie in the support of $\beta_{\lambda,i,R}$. Consequently, (2.1) gives

$$\ell(g) \le \ell(h) + \ell(h^{-1}g) = \ell(h^{-1}) + \ell(h^{-1}g) \le 2R + C,$$

which proves (4.1). By (3.17),

$$|\beta_{\lambda,i,R}(g)| \ll Q_\lambda^C R^C \mathbf{1}_{\ell(g)\le R+C}.$$

The Cartan convolution estimate

$$\mathbf{1}_{\ell\le R+C} * \mathbf{1}_{\ell\le R+C}(g) \ll_G R^C \mathrm{e}^{R-\ell(g)/2}\mathbf{1}_{\ell(g)\le 2R+C}$$

is [9, Lemma 4.18], and gives (4.2). On the spherical dual,

$$\widehat{\Phi}_{\lambda,i,R}(\nu) = |\widehat{\beta}_{\lambda,i,R}(\nu)|^2,$$

so (4.3) follows from (3.16). For an arbitrary unitary representation, $\pi(\beta^*) = \pi(\beta)^*$ and hence

$$\pi(\Phi_{\lambda,i,R}) = \pi(\beta_{\lambda,i,R})^*\pi(\beta_{\lambda,i,R}).$$

This is a positive semidefinite operator, which is what $\pi(\Phi_{\lambda,i,R}) \ge 0$ means. ☐

We now cover a spectral family. For $L \ge 0$ and $\eta > 0$, let

$$\mathcal{A}_{L,\eta} = \{\lambda : \pi_\lambda \in \widehat{G}_{\mathrm{sph}},\ \Lambda(\pi_\lambda) \le L,\ \vartheta(\lambda) \ge \eta\}.$$

This is a compact subset of the finite-dimensional parameter space $\mathfrak{a}^*_{\mathbb{C}}/W$, and $Q_\lambda \ll_G (1+L)^{1/2}$ on it by (2.2).

**Proposition 4.2** (A positive family)**.** *Suppose*

$$R\eta \ge C\log((2+L)(2+R)). \tag{4.4}$$

*There is $F_{R,L,\eta} = F^*_{R,L,\eta} \in C_c^\infty(K\backslash G/K)$ such that $\pi(F_{R,L,\eta}) \ge 0$ for every unitary $\pi$ and*

$$|F_{R,L,\eta}(g)| \ll_G (1+L)^C R^C \mathrm{e}^{R-\ell(g)/2}\mathbf{1}_{\ell(g)\le 2R+C}, \tag{4.5}$$

$$\widehat{F}_{R,L,\eta}(\nu) \gg_G (1+L)^{-C} R^{-C} \mathrm{e}^{2R(1-1/p_\nu)} \qquad (\nu \in \mathcal{A}_{L,\eta}). \tag{4.6}$$

*Proof.* Set $r_0 = (1+L)^{-B}R^{-B}$. An $r_0$-net is a finite set $\{\lambda_j\} \subset \mathcal{A}_{L,\eta}$ such that every $\nu \in \mathcal{A}_{L,\eta}$ satisfies $d_W(\nu,\lambda_j) \le r_0$ for some $j$. Since the parameter space has fixed dimension and $\mathcal{A}_{L,\eta}$ has diameter $O_G((1+L)^{1/2})$, a volume comparison gives a net whose cardinality is at most a fixed power of $(1+L)R$. For each net point, (4.4) implies the hypothesis of Proposition 3.5; choose a critical index $i_j$ and set

$$F_{R,L,\eta} = \sum_j \Phi_{\lambda_j,i_j,R}.$$

Every summand is positive. Given $\nu \in \mathcal{A}_{L,\eta}$, choose a net point within distance $r_0$. With $B$ large enough, this is the neighborhood in (3.15), and (4.3) gives (4.6). Summing (4.2) gives (4.5). The number of net points and all spectral-height losses are absorbed by the fixed powers of $(1+L)R$. ☐

# 5 Proof of the density theorem

*Proof of Theorem 1.1.* Fix $\epsilon > 0$ and put

$$\theta = 1 - \frac{2}{p}.$$

This is the threshold for $\vartheta(\lambda)$ corresponding to the condition $p(\pi_\lambda) \ge p$. If $\theta \le \epsilon/4$, then $\alpha\theta \le \epsilon/4$, and the local Weyl estimate (2.7) implies (1.2), since $V_N \le V_N^{1-\alpha\theta+\epsilon}$. We may therefore suppose

$\theta > \epsilon/4$. The trivial representation has multiplicity one; we remove it from the count and absorb this contribution into the final implied constant.

Choose

$$R = \alpha \log V_N - C_1,$$

where $C_1$ is sufficiently large for Lemma 2.6. When $R \le 1$, the volume $V_N$ is bounded in terms of $\alpha$ and $C_1$, and the assertion follows from (2.7) after changing the implied constant. We may therefore assume $R > 1$. First suppose

$$R\theta \ge C \log((2+L)(2+R)). \tag{5.1}$$

Use Proposition 4.2 with $\eta = \theta$. By Lemma 2.1, every representation counted by $M_{\mathrm{sph}}(L, \Gamma_N, p)$ has $\vartheta(\lambda) \ge \theta$ and therefore belongs to $\mathcal{A}_{L,\theta}$. The spectral trace formula (2.5), positivity, and (4.6) give

$$\operatorname{Tr} R_{\Gamma_N}(F_{R,L,\theta}) \gg_G (1+L)^{-C} R^{-C} \mathrm{e}^{2R(1-1/p)} M_{\mathrm{sph}}(L, \Gamma_N, p).$$

Indeed, $\vartheta(\nu) \ge \theta$ implies $p_\nu \ge p$, and hence $1 - 1/p_\nu \ge 1 - 1/p$.

On the geometric side, Lemma 2.6 and (4.5) give

$$\operatorname{Tr} R_{\Gamma_N}(F_{R,L,\theta}) \ll_G (1+L)^C R^C V_N^{1+\epsilon} \mathrm{e}^R.$$

Divide the geometric upper bound by the contribution of one representation in the spectral lower bound. Absorbing the fixed powers of $R$ into $V_N^\epsilon$ gives

$$\begin{aligned} M_{\mathrm{sph}}(L, \Gamma_N, p) &\ll_G (1+L)^C V_N^{1+\epsilon} \mathrm{e}^{-R(1-2/p)} \\ &\ll_{G,\Gamma_1,\alpha,\epsilon} (1+L)^C V_N^{1-\alpha(1-2/p)+2\epsilon}. \end{aligned}$$

Replacing $\epsilon$ by $\epsilon/2$ proves (1.2) in the long range.

It remains to consider the complementary, short range, in which (5.1) fails. Then

$$R\theta \ll_G \log(2+L) + \log(2+R). \tag{5.2}$$

Since $R = \alpha \log V_N + O_{G,\Gamma_1}(1)$, for all sufficiently large $V_N$ we have

$$C_G \log(2+R) + O_{G,\Gamma_1}(1) \le \frac{\epsilon}{2} \log V_N.$$

It follows from (5.2) that

$$V_N^{\alpha\theta - \epsilon/2} \ll_{G,\Gamma_1,\alpha,\epsilon} (1+L)^C$$

with a fixed $C = C(G)$. The remaining bounded values of $V_N$ are absorbed by the implied constant. Therefore (2.7) gives

$$M_{\mathrm{sph}}(L, \Gamma_N, p) \ll_{G,\Gamma_1} V_N (1+L)^d \ll_{G,\Gamma_1,\alpha,\epsilon} (1+L)^C V_N^{1-\alpha\theta+\epsilon}.$$

This completes the proof. □

*Proof of Corollary 1.2.* The forward implication is Theorem 1.1. The reverse implication is [9, Theorem 1.6 and Section 7.2]; the proof preserves the parameter $\alpha$. The two formulations of weak injective radius agree by Remark 2.3. □

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