POISSONIAN PAIR CORRELATION FOR THE THREE-PARTICLE SUTHERLAND MODEL

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Abstract. We prove a Berry–Tabor theorem for the repulsive three-particle trigonometric Sutherland Hamiltonian. For fixed total momentum, its relative spectrum is an affine translate of the positive-definite A_2-form. For a Diophantine coupling, the desymmetrized spectrum has Poissonian pair correlation. The proof reduces the problem to a finite-congruence and Weyl-sector version of Marklof's theorem on inhomogeneous quadratic forms.

1. Introduction

In 1977 Berry and Tabor [BT77] conjectured that a generic integrable quantum system has Poissonian local spectral statistics once its exact symmetries have been removed. We prove this at the level of pair correlation for a natural interacting model: the three-particle trigonometric Sutherland Hamiltonian with interaction coupling $g > 1$,

$$H_g = -\frac{1}{2}\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2} + \frac{g(g-1)}{4}\sum_{1\le i<j\le 3}\frac{1}{\sin^2((x_i-x_j)/2)}. \tag{1}$$

This describes three particles on a circle with repulsive inverse-square chord interaction. The model is quantum integrable and exactly solvable: its quasimomenta form an affine translate of the dominant weight lattice, and its eigenfunctions can be expressed in terms of Jack polynomials [Sut71, Sut72, OP83, BF97, Hal24]. An explicit treatment of the three-particle system, including formulas for the corresponding Jack-polynomial eigenfunctions, is given by Mangazeev [Man07].

Fixing the total momentum $K \in \mathbb{Z}$ and subtracting the center-of-mass energy $K^2/6$, the relative spectrum is

$$E_{p,q} := \frac{1}{3}\big((p+g)^2 + (p+g)(q+g) + (q+g)^2\big), \qquad p, q \in \mathbb{Z}_{\ge 0}, \quad p + 2q \equiv K \pmod 3. \tag{2}$$

This is the three-particle specialization of the standard Sutherland quasimomentum formula; see [Hal24, Section 4.2]. Thus the interaction parameter is a shift of the positive-definite A_2-form. The only remaining issues are the congruence condition, the Weyl symmetries (taking $p, q \ge 0$ restricts to one chamber), and the reflection symmetry when $K \equiv 0 \pmod 3$.

We remove this last symmetry at the outset. Put

$$\mathcal{P}_K := \begin{cases} \{(p,q) \in \mathbb{Z}^2_{\ge 0} : p + 2q \equiv K \pmod 3\}, & K \not\equiv 0 \pmod 3, \\ \{(p,q) \in \mathbb{Z}^2_{\ge 0} : p + 2q \equiv 0 \pmod 3,\ p \ge q\}, & K \equiv 0 \pmod 3, \end{cases}$$

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and let $\mathcal{E}_{K,g} := \{E_{p,q} : (p,q) \in \mathcal{P}_K\}$. Its mean density is

$$\rho_K := \begin{cases} \dfrac{\pi}{3\sqrt{3}}, & K \not\equiv 0 \pmod 3, \\ \dfrac{\pi}{6\sqrt{3}}, & K \equiv 0 \pmod 3. \end{cases}$$

Write

$$N_{K,g}(X) := \#\{E \in \mathcal{E}_{K,g} : E \le X\},$$

and, for $f \in C_c^\infty(\mathbb{R})$, define the pair correlation

$$R_{K,g}(X,f) := \frac{1}{N_{K,g}(X)} \sum^{*}_{\substack{E,E' \in \mathcal{E}_{K,g} \\ E,E' \le X}} f\big(\rho_K(E-E')\big),$$

where the $*$ denotes that $E \neq E'$

For every irrational $g > 1$ and every $K \in \mathbb{Z}$, the following Weyl law applies:

$$N_{K,g}(X) = \rho_K X + O_{g,K}(X^{1/2}). \tag{3}$$

Call $g \in \mathbb{R}$ *Diophantine of finite type* if there are $c > 0$ and $\kappa < \infty$ such that

$$\left| g - \frac{a}{b} \right| \ge c b^{-\kappa} \qquad (a \in \mathbb{Z},\ b \in \mathbb{Z}_{\ge 1}). \tag{4}$$

This includes every algebraic irrational and almost every real number. Some Diophantine restriction is necessary in general: without it pair correlation can fail along subsequences [Mar03b, Theorem 1.13]; see also the higher-dimensional correction [Mar03a].

Theorem 1. *Let $g > 1$ be Diophantine of finite type, and let $K \in \mathbb{Z}$. Then*

$$\lim_{X \to \infty} R_{K,g}(X,f) = \int_{\mathbb{R}} f(x)\,dx$$

for every $f \in C_c^\infty(\mathbb{R})$. In particular, $\mathcal{E}_{K,g}$ has Poissonian pair correlation.

Remark 2. For the N-particle model, after fixing the total momentum, the relative energies come from a shifted positive-definite $(N-1)$-ary quadratic form. For $N \ge 4$ this creates additional collisions.

There is a substantial physics literature on many-particle analogues of the Berry–Tabor conjecture. Poisson statistics have been observed numerically in integrable Heisenberg, t–J, and Hubbard models [PZB+93]. Noninteracting many-particle spectra were studied in [MnFM+06]. For interacting bosons, the level statistics of the Lieb–Liniger model were investigated analytically and numerically in [SIB21], and an N-particle Berry–Tabor trace formula was derived in [UKR23]. That said, these works do not give a rigorous Poissonian pair-correlation limit for a deterministic interacting Hamiltonian. To our knowledge, Theorem 1 is the first rigorous proof of Poissonian pair correlation for a deterministic quantum Hamiltonian with multiple interacting particles.

Rigorous work on Berry–Tabor statistics has largely concerned free particles on flat tori and boxes. Sarnak proved pair correlation for almost every positive binary quadratic form [Sar97]; Eskin, Margulis and Mozes obtained fixed-form results under explicit Diophantine hypotheses [EMM05]. Marklof proved Poissonian pair correlation for shifted lattice norms using theta functions and Ratner theory [Mar02, Mar03b]. Effective refinements were developed by Strömbergsson–Vishe and Lindenstrauss–Mohammadi–Wang [SV20, LMW23]. In dimension three, Kim, Marklof and Welsh recently proved the pair-correlation conjecture for a Diophantine family of boxes [KMW26].

The closest input to our argument is Marklof's inhomogeneous theorem, particularly its symmetry-reduced cases [Mar03b, Appendix A]. In what follows, we pass from the square lattice to a rational A_2-form, a finite-index coset, and a sharp Weyl sector. The first two are handled by the finite Weil representation, and the sector by smooth approximation and nonescape of mass. The weighted theta mean-value statement used below is proved in [GM19, Proposition 8.5].

2. Exact spectrum and center-of-mass reduction

We work in the bosonic Hilbert space for three particles on the circle. Fix labels

$$\lambda = (\lambda_1, \lambda_2, \lambda_3), \qquad \lambda_1 \geq \lambda_2 \geq \lambda_3, \tag{5}$$

then the quasimomenta are given by

$$\kappa_i = \lambda_i + \frac{g}{2}(4-2i), \qquad (\kappa_1, \kappa_2, \kappa_3) = (\lambda_1 + g, \lambda_2, \lambda_3 - g).$$

The energy and total momentum are

$$E_\lambda = \frac{1}{2}\sum_{i=1}^{3} \kappa_i^2, \qquad K_\lambda = \sum_{i=1}^{3} \kappa_i = \lambda_1 + \lambda_2 + \lambda_3.$$

One often indexes Jack polynomials by nonnegative partitions [Man07, Section 2].

For fixed K and λ, we write

$$p = \lambda_1 - \lambda_2, \qquad q = \lambda_2 - \lambda_3,$$

so $p, q \geq 0$. The total-momentum condition is

$$K = 3k + p + 2q, \qquad p + 2q \equiv K \pmod 3, \tag{6}$$

which gives

$$\lambda = (k + p + q, k + q, k).$$

The following lemma is the standard quasimomentum formula for the Sutherland model [Man07, Section 2].

Lemma 3. *For fixed K, the energy after subtracting $K^2/6$ is*

$$E^{\mathrm{rel}}_{p,q} = E_\lambda - \frac{K^2}{6} = \frac{1}{3}Q_g(p,q), \tag{7}$$

where

$$\begin{aligned} Q_g(p,q) &= (p+g)^2 + (p+g)(q+g) + (q+g)^2 \\ &= p^2 + pq + q^2 + 3g(p+q) + 3g^2. \end{aligned}$$

The allowed labels are exactly

$$\mathcal{L}_K = \{(p,q) \in \mathbb{Z}^2_{\geq 0} : p + 2q \equiv K \pmod 3\}. \tag{8}$$

Proof. For any $(a_1, a_2, a_3) \in \mathbb{R}^3$ one has

$$\sum_{i=1}^{3} a_i^2 - \frac{1}{3}\Big(\sum_{i=1}^{3} a_i\Big)^2 = \frac{1}{3}\sum_{1 \leq i < j \leq 3} (a_i - a_j)^2.$$

Apply this identity with $a_i = \kappa_i$ and use $E_\lambda = \frac{1}{2}\sum_i \kappa_i^2$ and $\sum_i \kappa_i = K$. We obtain

$$E_\lambda - \frac{K^2}{6} = \frac{1}{6}\big((\kappa_1 - \kappa_2)^2 + (\kappa_1 - \kappa_3)^2 + (\kappa_2 - \kappa_3)^2\big).$$

The definitions of p and q give

$$\kappa_1 - \kappa_2 = p + g, \qquad \kappa_2 - \kappa_3 = q + g, \qquad \kappa_1 - \kappa_3 = p + q + 2g.$$

Consequently

$$\begin{aligned} 6\Big(E_\lambda - \frac{K^2}{6}\Big) &= (p+g)^2 + (q+g)^2 + (p+q+2g)^2 \\ &= 2\{p^2 + pq + q^2 + 3g(p+q) + 3g^2\} = 2Q_g(p,q), \end{aligned}$$

which proves (7).

If λ has momentum K, then (6) implies $p + 2q \equiv K \pmod 3$. Conversely, given $p, q \geq 0$ satisfying this congruence, the integer

$$k = \frac{K - p - 2q}{3}$$

defines a weakly decreasing integral triple $\lambda = (k+p+q, k+q, k)$ of momentum K. These two constructions are inverse to one another, proving (8). □

Next, the change of variables

$$u = p + q, \qquad v = p - q \tag{9}$$

puts the form in the especially useful shape

$$4Q_g(p,q) = 3(u+2g)^2 + v^2.$$

The image of $\mathcal{L}_K$ is the finite-index lattice coset

$$u \equiv v \pmod 2, \qquad v \equiv K \pmod 3, \qquad u \geq |v|. \tag{10}$$

The first two conditions are a parity sublattice and a residue class; the last condition is the closed A_2 Weyl chamber.

3. Arithmetic: multiplicities and mean density

Let $Q_0(p,q) = p^2 + pq + q^2$. The next lemma states that irrationality of g and working in the positive Weyl chamber, the only remaining symmetry to be removed is the reflection when $K \equiv 0$.

Lemma 4. *If g is irrational and $Q_g(p,q) = Q_g(r,s)$ for nonnegative integral labels, then*

$$(r,s) = (p,q) \quad \text{or} \quad (r,s) = (q,p).$$

If both labels lie in $\mathcal{L}_K$ and the two labels are distinct, the second possibility can occur only when $K \equiv 0 \pmod 3$.

Proof. The equality is

$$Q_0(p,q) - Q_0(r,s) + 3g(p+q-r-s) = 0.$$

Both coefficients of 1 and g are integers, so irrationality gives $p+q = r+s$ and $Q_0(p,q) = Q_0(r,s)$. Since $Q_0(p,q) = (p+q)^2 - pq$, we also have $pq = rs$ which forces $(r,s) = (p,q)$ or (q,p).

Suppose that the distinct labels (p,q) and (q,p) both belong to $\mathcal{L}_K$. Since $p + 2q \equiv K \pmod 3$, the reflected congruence is $q + 2p \equiv -K \pmod 3$. Thus the reflected point lies in $\mathcal{L}_K$ only if $K \equiv -K \pmod 3$, which is equivalent to $K \equiv 0 \pmod 3$. □

We remove this remaining symmetry by restricting to $\mathcal{P}_K$. The following Weyl law is an easy consequence of lattice point equidistribution.

Lemma 5. *For every irrational $g > 1$ and $K \in \mathbb{Z}$,*

$$N_{K,g}(X) = \rho_K X + O_{g,K}(X^{1/2}).$$

Proof. Let

$$B = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \qquad Q_0(x,y) = (x,y)B(x,y)^t.$$

Since $\det B = 3/4$, the change $w = B^{1/2}(x,y)^t$ gives

$$\mathrm{vol}\{(x,y) \in \mathbb{R}^2 : Q_0(x,y) \le Y\} = \frac{\pi Y}{\sqrt{\det B}} = \frac{2\pi Y}{\sqrt{3}}.$$

The B-inner product of $(1,0)$ and $(0,1)$ is $1/2$, and both vectors have B-norm one. Thus the angle between the two boundary rays of the positive quadrant is $\arccos(1/2) = \pi/3$ in the B-metric. The quadrant therefore occupies one sixth of the ellipse, so

$$\mathrm{vol}\{(x,y) \in \mathbb{R}^2_{\ge 0} : Q_0(x,y) \le Y\} = \frac{\pi Y}{3\sqrt{3}}. \tag{11}$$

In fact

$$Q_g(p,q) = Q_0(p+g, q+g),$$

so the relevant ellipse is translated by the fixed vector (g,g). Hence

$$\Omega_{g,Y} := \{z \in \mathbb{R}^2_{\ge 0} : Q_g(z) \le Y\}$$

has boundary consisting of a bounded number of C^1 arcs of total length $O_g(Y^{1/2})$. The planar Lipschitz principle gives, for every fixed full-rank sublattice $L \subset \mathbb{Z}^2$ and coset $c + L$,

$$\#((c+L) \cap \Omega_{g,Y}) = \frac{\mathrm{vol}(\Omega_{g,Y})}{[\mathbb{Z}^2 : L]} + O_{g,L}(Y^{1/2}).$$

Translating the ellipse by the fixed vector (g,g) changes its intersection with the quadrant only in a strip of bounded width along a boundary of length $O_g(Y^{1/2})$. It follows from (11) that

$$\mathrm{vol}(\Omega_{g,Y}) = \frac{\pi Y}{3\sqrt{3}} + O_g(Y^{1/2}).$$

The congruence $p + 2q \equiv K \pmod 3$ defines a coset of the sublattice

$$L_0 = \{(p,q) \in \mathbb{Z}^2 : p + 2q \equiv 0 \pmod 3\}, \qquad [\mathbb{Z}^2 : L_0] = 3.$$

Taking $Y = 3X$ because $E^{\mathrm{rel}} = Q_g/3$, we therefore obtain

$$\#\{(p,q) \in \mathcal{L}_K : Q_g(p,q) \le 3X\} = \frac{1}{3}\frac{\pi(3X)}{3\sqrt{3}} + O_{g,K}(X^{1/2}) = \frac{\pi X}{3\sqrt{3}} + O_{g,K}(X^{1/2}).$$

When $K \equiv 0 \pmod 3$, both the coset and Q_g are invariant under $(p,q) \mapsto (q,p)$. The Weyl chamber splits into two open half-chambers with the same cardinality. The wall $p = q$ contains $O_g(X^{1/2})$ labels below energy X. Thus restriction to $p \ge q$ halves the main term without changing the order of the error. This gives the stated value of ρ_K. $\square$

4. Theta sums and Marklof's theorem

The Jacobi quotient and the affine horocycle. Write $\mathrm{e}(t) = e^{2\pi i t}$. For $k \geq 1$ let

$$G^k = \mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^{2k}.$$

We write an element of $\mathbb{R}^{2k}$ as $\xi = {}^t(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$, and let $M \in \mathrm{SL}(2, \mathbb{R})$ act on each of the k column vectors ${}^t(x_j, y_j)$. Thus the group law is

$$(M; \xi)(M'; \xi') = (MM'; \xi + M\xi').$$

Put

$$n(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad a(v) = \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \quad (v > 0), \qquad k(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

If $f \in \mathcal{S}(\mathbb{R}^k)$ and $g = (n(u)a(v)k(\phi); {}^t(\boldsymbol{x}, \boldsymbol{y}))$, define

$$\Theta_f(g) = v^{k/4} \sum_{m \in \mathbb{Z}^k} f_\phi((m - \boldsymbol{y})v^{1/2}) \mathrm{e}\Big(\frac{u}{2}\|m - \boldsymbol{y}\|^2 + m \cdot \boldsymbol{x}\Big). \tag{12}$$

Here $f_\phi = \widetilde{R}(i, \phi)f$, where $\widetilde{R}$ is the metaplectic representation; only the facts $f_0 = f$ and $\|f_\phi\|_2 = \|f\|_2$ will be used. Define Γ_θ^k to be the common stabilizer

$$\Big\{\gamma \in \mathrm{SL}(2, \mathbb{Z}) \ltimes (\tfrac{1}{2}\mathbb{Z})^{2k} : (\Theta_f \overline{\Theta_{f'}})(\gamma g) = (\Theta_f \overline{\Theta_{f'}})(g) \text{ for all } f, f' \in \mathcal{S}(\mathbb{R}^k),\ g \in G^k\Big\}.$$

The theta transformation formulas obtained from Poisson summation show that Γ_θ^k has finite index in $\mathrm{SL}(2, \mathbb{Z}) \ltimes (\frac{1}{2}\mathbb{Z})^{2k}$ [Mar03b, Lemmas 4.11–4.12]. Consequently each product is a function on the finite-volume quotient $\Gamma_\theta^k \backslash G^k$.

The rational dependence relevant here has a particularly concrete description. We call a subgroup of G^2 *arithmetic* if it is commensurable with $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^4$; replacing $\mathbb{Z}^4$ by a rational dilate gives the same class. Inside G^2 set

$$H = \{(M; {}^t(x, 0, y, 0)) : M \in \mathrm{SL}(2, \mathbb{R}),\ x, y \in \mathbb{R}\}.$$

The map $(M; {}^t(x, 0, y, 0)) \mapsto (M; {}^t(x, y))$ identifies H with G^1. For an arithmetic subgroup $\Gamma < G^2$, put $\Gamma_H = \Gamma \cap H$ and

$$Y = \Gamma_H \backslash H.$$

The group Γ_H is a lattice in H, so Y has finite Haar measure. In the coordinates (u, v, ϕ, x, y) above this measure is a constant multiple of

$$\frac{du\, dv\, d\phi\, dx\, dy}{v^2};$$

we write μ_Y for its normalization to total mass one. For $\alpha \in \mathbb{R}$ let

$$g_\alpha = (I; {}^t(0, 0, -\alpha, 0)) \in H.$$

The point represented in Iwasawa coordinates by

$$(u + iv, 0; {}^t(0, 0, -\alpha, 0))$$

is precisely $\Gamma_H g_\alpha n(u)a(v)$. Thus the limit $v \to 0$ is an expanding affine horocycle in Y.

Lemma 6. *Let M be a discrete set and let $\lambda_m > 0$ for $m \in M$. For $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}_{>0})$ define*

$$S_{j,X}(u) = X^{-1/2} \sum_{m \in M} \psi_j\left(\frac{\lambda_m}{X}\right) \mathrm{e}\Big(\frac{u}{2}\lambda_m\Big).$$

For $h \in C_c(\mathbb{R})$ set

$$\widehat{h}(s) = \int_{\mathbb{R}} h(u)\mathrm{e}(us/2)\, du.$$

Then

$$\int_{\mathbb{R}} S_{1,X}(u)\overline{S_{2,X}(u)}h(u)\, du = \frac{1}{X} \sum_{m,n \in M} \psi_1\left(\frac{\lambda_m}{X}\right) \psi_2\left(\frac{\lambda_n}{X}\right) \widehat{h}(\lambda_m - \lambda_n). \tag{13}$$

If λ_m is a positive quadratic polynomial on a lattice coset, the left side of (13), with $v = X^{-1}$, is a finite linear combination of integrals of products of the form $\Theta_f\overline{\Theta_{f'}}(g_\alpha n(u)a(v))$.

Proof. Because the ψ_j have compact support and the quadratic polynomial is proper, both sums are finite. We may therefore multiply them and integrate term by term. The summand indexed by (m,n) contains

$$\int_{\mathbb{R}} h(u)\mathrm{e}\Big(\frac{u}{2}(\lambda_m - \lambda_n)\Big)\, du = \widehat{h}(\lambda_m - \lambda_n),$$

which proves (13). If $\lambda_m = (m - \boldsymbol{y})^t A(m - \boldsymbol{y})$, take a Schwartz function whose value at $w = (m - \boldsymbol{y})X^{-1/2}$ is $\psi_j(w^t A w)$. Formula (12) then gives the asserted theta representation. A congruence coset and rational A require finitely many component theta functions; these are constructed in Lemma 9 below. □

Height, domination, and nonescape. Let $\Gamma_\infty = \{n(m) : m \in \mathbb{Z}\}$. If $\tau = u + iv$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, put

$$v_\gamma = \mathrm{Im}(\gamma\tau) = \frac{v}{|c\tau + d|^2}, \qquad {}^t(\boldsymbol{x}_\gamma, \boldsymbol{y}_\gamma) = \gamma\, {}^t(\boldsymbol{x}, \boldsymbol{y}).$$

The modular height is the well-defined function

$$\mathcal{H}(\Gamma g) = \sup_{\gamma \in \mathrm{SL}(2,\mathbb{Z})} v_\gamma.$$

For a nonnegative, even, rapidly decreasing function $f_0 \in C(\mathbb{R})$ and $R > 1$, define Marklof's cusp majorant

$$F_R(g) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{m \in \mathbb{Z}} f_0((y_{1,\gamma} + m)v_\gamma^{1/2})v_\gamma \mathbf{1}_{[R,\infty)}(v_\gamma). \tag{14}$$

The sum is invariant under $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$. A continuous function F on an arithmetic Jacobi quotient is said to be *dominated by* F_R if there is an $L > 1$ such that, for all sufficiently large R,

$$|F(g)|\mathbf{1}_{\{\mathcal{H}(g) \geq R\}} \leq L + F_R(g) \qquad (g \in G^k). \tag{15}$$

This is Marklof's precise definition of moderate growth in the cusp [Mar03b, Section 7.2]. The following lemma combines [Mar03b, Theorem 5.7], [Mar03b, Lemma 6.6 and Proposition 6.5], and [Mar03b, Corollaries 7.4 and 7.6].

Lemma 7. *Let $\Gamma_H < H$ be commensurable with $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, let μ_Y be Haar probability measure on $Y = \Gamma_H \backslash H$, and let $\alpha \notin \mathbb{Q}$. For every bounded continuous $F : Y \to \mathbb{C}$ and every $h \in C_c(\mathbb{R})$,*

$$\lim_{v \to 0} \int_{\mathbb{R}} F(\Gamma_H g_\alpha n(u)a(v))h(u)\, du = \left(\int_Y F\, d\mu_Y\right)\left(\int_{\mathbb{R}} h(u)\, du\right). \tag{16}$$

Suppose in addition that α is Diophantine of type κ in the sense of (4). If $h_0 \in C_c(\mathbb{R})$ is nonnegative, $0 < \epsilon < 1$, and $0 < \epsilon' < 1/(\kappa - 1)$, then the function (14) satisfies

$$\limsup_{v \to 0} \int_{|u| > v^{1-\epsilon}} F_R(g_\alpha n(u)a(v))h_0(u)\, du \ll_{\alpha, f_0, h_0, \epsilon, \epsilon'} R^{-\epsilon'/2}. \tag{17}$$

Consequently, if F is continuous and dominated by F_R, then

$$\lim_{v\to 0}\int_{|u|>v^{1-\epsilon}} F(g_\alpha n(u)a(v))h(u)\,du = \left(\int_Y F\,d\mu_Y\right)\left(\int_{\mathbb{R}} h(u)\,du\right). \tag{18}$$

The same statements hold on a finite cover of Y and on a finite union of rational translates of such covers.

Lemma 8. *For $f_1, f_2 \in \mathcal{S}(\mathbb{R}^2)$, the function $F = \Theta_{f_1}\overline{\Theta_{f_2}}$ satisfies (15) for a majorant of the form (14). This remains true for every component obtained by a rational congruence condition and for every smooth angular cutoff.*

Proof. Define

$$f_0(t) = \sup_{s\in\mathbb{R}}\sup_{\phi\in\mathbb{R}} |(f_1)_\phi(t,s)(f_2)_\phi(t,s)|.$$

The metaplectic rotations of a Schwartz function are uniformly Schwartz for ϕ in the compact circle. Hence, for every $A>0$, $f_0(t) \ll_A (1+|t|)^{-A}$. Replace $f_0(t)$ by $f_0(t)+f_0(-t)$ to make it even and nonnegative.

In the standard cusp $v>R$, expand both theta sums using (12). Cauchy–Schwarz in the second lattice coordinate and the definition of f_0 bound their product by

$$v\sum_{m\in\mathbb{Z}} f_0((m-y_1)v^{1/2}) + O_A(v^{-A}).$$

Move an arbitrary point of height at least R to the standard cusp and sum this estimate over $\Gamma_\infty\backslash \mathrm{SL}(2,\mathbb{Z})$. The first term becomes (14), and the rapidly decreasing error is absorbed in the constant L in (15). This is Marklof's construction in [Mar03b, Sections 7.2 and 8.4.3].

A rational component is a finite linear combination of theta functions with rational characteristics, so the same estimate holds after increasing its constant. Multiplication by a smooth angular cutoff replaces f_j by a Schwartz function and does not affect the argument. □

Rational theta data and the smoothed limit.

Definition 4.1 (Collision relation)**.** Let $P_\alpha(m) = P_0(m) + \alpha\ell(m) + c$ on a lattice coset, where P_0 is rational-valued, ℓ is a rational linear form, and $\alpha \notin \mathbb{Q}$. A pair (m,n) is a *bulk collision* if

$$\ell(m) = \ell(n), \qquad P_0(m) = P_0(n).$$

The diagonal $m=n$ is always a bulk collision. A *collision graph* is the graph of a map σ for which $(m,\sigma(m))$ is a bulk collision for all labels in its domain.

The terminology has a direct analytic meaning. In the Haar integral of a theta product, integration over the compact translation variable forces $\ell(m)=\ell(n)$, and integration over the modular horocycle forces $P_0(m)=P_0(n)$.

Let A be a positive-definite symmetric 2×2 matrix with rational entries, let $L\subset\mathbb{Z}^2$ be a finite-index sublattice, and fix $c\in\mathbb{Z}^2$. After interchanging the coordinates if necessary, write the affine shift as $\boldsymbol{\alpha} = (\alpha,\beta)$, where $\alpha\notin\mathbb{Q}$ and $\beta\in\mathbb{Q}$. For $f\in\mathcal{S}(\mathbb{R}^2)$ and $\tau = u+iv\in\mathbb{H}$, put

$$\Theta_{A,L,c,\boldsymbol{\alpha};f}(u,v) = v^{1/2}\sum_{m\in c+L} f\Big(v^{1/2}A^{1/2}(m+\boldsymbol{\alpha})\Big)\,\mathrm{e}\Big(\frac{u}{2}(m+\boldsymbol{\alpha})^t A(m+\boldsymbol{\alpha})\Big). \tag{19}$$

Lemma 9. *The function in (19) is a finite linear combination of theta components on finite-volume arithmetic quotients of G^2. Along the associated affine horocycle, each component lies on one of finitely many rational translates of Y, with irrational translation coordinate $q\alpha + r$, where $q\in\mathbb{Q}^\times$ and $r\in\mathbb{Q}$. Hence, if α is Diophantine of finite type, Lemmas 7 and 8 apply componentwise.*

Proof. Choose an integer clearing the denominators of A and β. Expanding the congruence condition by

$$\mathbf{1}_{c+L}(m) = \frac{1}{[\mathbb{Z}^2 : L]} \sum_{\chi \in \widehat{\mathbb{Z}^2/L}} \overline{\chi(c)}\chi(m)$$

expresses (19), after a fixed rescaling of τ, as a finite linear combination of theta functions with rational characteristics. The standard finite Weil representation shows that their products with conjugates descend to finite-volume arithmetic quotients of G^2; see [LV80, BS98].

The rational coordinate β, the congruence characters, and the rational characteristics contribute only finitely many rational translates of the subgroup H defined above. On each translate, the remaining translation coordinate is obtained from α by a rational affine change and is therefore $q\alpha + r$ for some $q \in \mathbb{Q}^\times$ and $r \in \mathbb{Q}$. Rational affine changes preserve finite Diophantine type. The majorant argument in Lemma 8 is componentwise, while a finite sum changes only its constants. Thus the equidistribution and nonescape statements apply to every component product. □

By a *smooth angular weight* we mean a bounded function $w \in C^\infty(\mathbb{R}^2)$ for which there exist $R_0 > 0$ and $\omega \in C^\infty(S^1)$ such that

$$w(r\theta) = \omega(\theta) \qquad (\theta \in S^1,\ r \geq R_0).$$

Thus w is homogeneous of degree zero outside the ball $B(0, R_0)$.

Lemma 4.2. *Let $\{\lambda_m : m \in M\}$ be represented by the rational theta data of Lemma 9, and let w be a smooth angular weight. Suppose that, for w and w^2, the weighted Weyl law*

$$\lim_{X\to\infty} \frac{1}{X} \sum_{m\in M} w(m)\varphi(\lambda_m/X) = d(w) \int_0^\infty \varphi(t)\, dt \tag{20}$$

holds for every $\varphi \in C_c^\infty(\mathbb{R}_{>0})$. If the irrational affine coordinate is Diophantine and $m = n$ is the only bulk collision on $\operatorname{supp}(w) \times \operatorname{supp}(w)$*, then*

$$\begin{aligned} &\lim_{X\to\infty} \frac{1}{X} \sum_{\substack{m,n\in M \\ m\neq n}} w(m)w(n)\psi_1\left(\frac{\lambda_m}{X}\right)\psi_2\left(\frac{\lambda_n}{X}\right)\widehat{h}(\lambda_m - \lambda_n) \\ &= d(w)^2 \left(\int_{\mathbb{R}} \widehat{h}(s)\, ds\right)\left(\int_0^\infty \psi_1(t)\psi_2(t)\, dt\right) \end{aligned} \tag{21}$$

for $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}_{>0})$ and $h \in C_c(\mathbb{R})$.

Proof. For the basic affine lattice, this is the weighted theta mean-value calculation of [GM19, Proposition 8.5], followed by subtraction of the diagonal; see also [Mar03b, Section 8.6]. By Lemma 9, the rational form and congruence coset give finitely many theta products on finite covers and rational translates. The Ratner and nonescape arguments apply to each component with the same estimates by Lemmas 7 and 8. Summing the components proves the formula. □

Lemma 4.3. *Let $\mathcal{D} \subset \mathbb{R}^2$ be a closed cone such that $\mathcal{D} \cap S^1$ is a finite union of closed arcs. Suppose Lemma 4.2 holds for smooth angular weights near $\mathcal{D}$, and that every nonidentity collision graph meeting $\mathcal{D} \times \mathcal{D}$ does so only on its boundary. Put*

$$M_{\mathcal{D}} = M \cap \mathcal{D}, \qquad N_{\mathcal{D}}(X) = \#\{m \in M_{\mathcal{D}} : \lambda_m \leq X\}.$$

If $N_{\mathcal{D}}(X) = DX + o(X)$ with $D > 0$, then, for every $f \in C_c^\infty(\mathbb{R})$,

$$\lim_{X\to\infty} \frac{1}{N_{\mathcal{D}}(X)} \sum_{\substack{m,n\in M_{\mathcal{D}}\ m\neq n \\ \lambda_m,\lambda_n\le X}} f(\lambda_m - \lambda_n) = D \int_{\mathbb{R}} f(t)\,dt.$$

Proof. Approximate the angular and radial indicators from above and below by smooth weights. The weighted Weyl law makes the boundary-strip and annular errors $O(\eta)$; Cauchy–Schwarz and Lemma 4.2 make every corresponding pair term $O_f(\eta^{1/2})$. Letting $\eta \to 0$ proves the claim. This is the standard approximation argument of [Mar03b, Section 8.6 and Appendix A.6]. ☐

5. A sector version of the inhomogeneous quadratic-form theorem

We isolate the homogeneous-dynamics input in the exact form required by the Sutherland spectrum. This makes explicit the three points not literally present in the most familiar statement of Marklof's theorem: the rational coefficient 3, the congruence coset, and the Weyl sector.

For $r \in \mathbb{Z}/3\mathbb{Z}$, put

$$\Lambda_r = \{(u,v) \in \mathbb{Z}^2 : u \equiv v \pmod 2,\ v \equiv r \pmod 3\},$$
$$\mathcal{C} = \{(u,v) \in \mathbb{R}^2 : u \ge |v|\}, \qquad \mathcal{C}^+ = \{(u,v) \in \mathbb{R}^2 : u \ge v \ge 0\}.$$

For $z = (u,v)$ define

$$\xi_g(z) = \frac{1}{12}\big(3(u+2g)^2 + v^2\big).$$

If $M \subset \mathbb{Z}^2$ and $\xi : M \to \mathbb{R}_{>0}$, write $N_M(X) = \#\{z \in M : \xi(z) \le X\}$. We say that its values have *pair-correlation intensity* D if, for every $F \in C_c^\infty(\mathbb{R})$,

$$\lim_{X\to\infty} \frac{1}{N_M(X)} \sum_{\substack{z,z'\in M \\ z\neq z',\ \xi(z),\xi(z')\le X}} F(\xi(z) - \xi(z')) = D \int_{\mathbb{R}} F(t)\,dt. \tag{22}$$

Proposition 10. *Let g be irrational and Diophantine of finite type. For $r = 1,2$, the values $\xi_g(z)$ with $z \in \Lambda_r \cap \mathcal{C}$ have Poissonian pair correlation of intensity $\pi/(3\sqrt{3})$. For $r = 0$, the values with $z \in \Lambda_0 \cap \mathcal{C}^+$ have Poissonian pair correlation of intensity $\pi/(6\sqrt{3})$.*

Proof. Set

$$L = \{(u,v) \in \mathbb{Z}^2 : u \equiv v \pmod 2, v \equiv 0 \pmod 3\}, \qquad A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{\alpha} = (2g, 0).$$

For $c_r = (r,r)$, where r is represented by $0, 1$, or 2, one has $\Lambda_r = c_r + L$. Indeed, subtraction of c_r makes the second coordinate divisible by three and preserves the parity relation, and the converse is immediate. Moreover,

$$12\xi_g(z) = (z+\boldsymbol{\alpha})^t A (z+\boldsymbol{\alpha}). \tag{23}$$

Thus A, L, and c_r satisfy the hypotheses of Lemma 9. The factor 12 only rescales u in Equation (13); it neither changes automorphy nor the nonescape estimate.

Let χ be a smooth angular function supported in the interior of $\mathcal{C}$ or $\mathcal{C}^+$, and let $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}_{>0})$. Apply Lemma 6 to the sums

$$X^{-1/2} \sum_{z\in\Lambda_r} \chi(z+\boldsymbol{\alpha})\psi_j(\xi_g(z)/X)\mathrm{e}(u\xi_g(z)/2).$$

By (23) and Lemma 9, their product is a finite sum of automorphic theta products on finite covers of the quotient Y defined above. The irrational translation coordinate is $2g$. To check the Diophantine condition explicitly, if $a \in \mathbb{Z}$ and $b \geq 1$, then

$$\left|2g - \frac{a}{b}\right| = 2\left|g - \frac{a}{2b}\right| \geq 2c(2b)^{-\kappa}.$$

Hence $2g$ is Diophantine of the same finite type as g. By Lemmas 7 and 8, the theta products are therefore uniformly integrable and equidistribute on the explicit finite-volume H-quotients.

By Lemma 4, the sector restriction removes every nontrivial collision away from the fixed wall $p = q$, and that wall contains only $O_g(X^{1/2})$ labels by Lemma 5, so its contribution is absorbed by the error term.

The bijection $(p, q) \mapsto (u, v) = (p + q, p - q)$ and Lemma 5 give $D = \pi/(3\sqrt{3})$ for $r = 1, 2$ and $D = \pi/(6\sqrt{3})$ for $r = 0$.

For a smooth angular cutoff χ, the Lipschitz principle used in Lemma 5 proves (20); denote the resulting weighted density by $D_\chi = d(\chi)$. All hypotheses of Lemma 4.2 have now been verified. Its off-diagonal conclusion says

$$\lim_{X\to\infty} \frac{1}{D_\chi X} \sum_{z\neq z'} \chi(z+\boldsymbol{\alpha})\chi(z'+\boldsymbol{\alpha})\psi_1(\xi_g(z)/X)\psi_2(\xi_g(z')/X)\widehat{h}(\xi_g(z)-\xi_g(z')) = D_\chi \int_{\mathbb{R}} \widehat{h} \int_0^\infty \psi_1\psi_2.$$

Finally, Lemma 4.3 replaces the smooth weights by the sharp sector and replaces the radial weights by the conditions $\xi_g(z), \xi_g(z') \leq X$. Since $D_\chi \to D$ for the upper and lower angular approximations and $N_M(X)/(DX) \to 1$, the resulting formula is exactly (22). The theta weight is naturally a function of $z + \boldsymbol{\alpha}$, whereas the sharp cone condition is imposed on z. The symmetric difference between these two cone conditions lies in a strip of width $O_g(1)$ along the boundary rays. Below energy X the strip has area $O_g(X^{1/2})$, hence relative area $O_g(X^{-1/2})$; the discrepancy estimate in Lemma 4.3 shows that it contributes zero to the normalized pair limit. This proves the proposition. □

Remark 11. The Diophantine assumption is used only in (17). Equation (16) for bounded continuous observables requires only irrationality. Pair correlation is represented by the unbounded theta product, and a Liouville sequence of exceptionally short affine-lattice vectors can carry mass into the cusp. This is why irrationality alone is not enough for the present proof.

6. Proof of the main theorem

Proof of Theorem 1. By Lemma 3, the map $(p, q) \mapsto E_{p,q} = Q_g(p, q)/3$ gives every relative eigenvalue in the fixed momentum sector. The change of variables (9) is invertible on its image, with inverse

$$p = \frac{u+v}{2}, \qquad q = \frac{u-v}{2}.$$

The parity and cone conditions in (10) are exactly the conditions that these two numbers are nonnegative integers. The momentum congruence becomes $v \equiv K \pmod 3$. Hence the label set is in bijection with $\Lambda_{K \bmod 3} \cap \mathcal{C}$, and the identity $4Q_g = 3(u + 2g)^2 + v^2$ shows that its energy is $\xi_g(u, v)$.

When $K \equiv 0 \pmod 3$, the reflection $(p, q) \mapsto (q, p)$ becomes $(u, v) \mapsto (u, -v)$. Choosing the representative $p \geq q$ is therefore equivalent to imposing $v \geq 0$, namely to restricting to $\mathcal{C}^+$. By Lemma 4, no two distinct retained labels have the same energy. Thus the sets in Proposition 10 are precisely the desymmetrized spectra $\mathcal{E}_{K,g}$.

The Weyl law (3) is Lemma 5. Let $f \in C_c^\infty(\mathbb{R})$ and put $F(x) = f(\rho_K x)$. By the definition (22) and Proposition 10,

$$\lim_{X\to\infty} R_{K,g}(X,f) = \rho_K \int_{\mathbb{R}} F(x)\,dx = \rho_K \int_{\mathbb{R}} f(\rho_K x)\,dx = \int_{\mathbb{R}} f(y)\,dy,$$

where the last equality uses $y = \rho_K x$. This proves both assertions. $\square$

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