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Statistical Properties of Dynamical Systems
From Statistical Mechanics to Hyperbolic Geometry

Statistical Properties of Dynamical Systems:

From Statistical Mechanics to Hyperbolic Geometry.

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ABSTRACT

Broadly, this thesis treats the statistical properties of dynamical systems in two different contexts. That is, we characterise asymptotic behaviour, independence, and randomness in two distinct settings.

First we consider two models for diffusion of gasses: the random Lorentz gas and the random wind-tree model. Understanding how typical particles diffuse outwards is one of the central aims of the field. In both these contexts our main results state that (in a particular scaling limit), when considered over long times, the typical particle trajectory converges in distribution to a Brownian motion. We use novel coupling methods to approximate these trajectories by Markovian walks and thus prove these invariance principles.

For the second half, we consider a general discrete hyperbolic subgroup. Therefore these groups may be 'thin'. Then we consider the orbit of a point in hyperbolic half-space by this group. The main results concern characterising the limiting local statistics of these orbits in a number of different contexts. We extend methods from homogeneous dynamics to the thin group setting and use Patterson-Sullivan theory and equidistribution of expanding horospheres to characterise the limiting behaviour of these group orbits. This work has applications to sphere packings, Diophantine approximation, continued fractions, and more.

ACKNOWLEDGEMENTS

First, I would like to thank my advisor Jens Marklof. Whose patient and kind guidance have helped me to accomplish things which would have been impossible otherwise - and to have fun along the way. Looking back on where I started I am tremendously grateful for Jens' persistence in instilling in me some of the habits necessary to become a mathematician.

The same is true of my advisor and collaborator Bálint Tóth. I am very grateful to Bálint for working with me through such a challenging problem and helping me understand the challenges faced in mathematics and in collaboration. The enthusiasm, rigour, and cheerful manner in which both Jens and Bálint work has been an inspiration.

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I owe many thanks to my extended family in Bristol, Gidleigh, Valdosta and all over the world, who have offered homes away from home and support when it was needed. Lastly I would like to thank my father, mother and brother for always being there for me, believing in me, and representing the examples I needed to pursue this degree.

Furthermore I am very grateful to my examiners Thomas Jordan and Mark Pollicott for their thorough reading of my thesis, and the discussion that followed.

– To all those who have supported and put up with me, thank you. –

DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Chapter 1

General Introduction and Plan

The study of dynamical systems is an attempt to understand the behaviour of a physical or mathematical system as it evolves over time. Given this general definition, dynamical systems is a topic which permeates many areas of mathematics. As such, there are many examples of dynamical systems ranging from models for physical systems to fundamental mathematical objects. In the real world, dynamical systems have been used to study: solar systems, the weather, crystal growth, financial markets, traffic *etc.* While from a mathematical point of view some examples and applications of dynamical systems include: dynamics on the space of infinite sequences (symbolic coding), modelling the motion of gas particles, solutions to Diophantine equations, billiard tables, *etc.* In general, dynamicists are in search of general properties of the system, such as asymptotic behaviour and invariance.

It would be ambitious to try and pin down the first example of a modern dynamical systems approach appearing in mathematics, but many authors identify Poincaré as an early pioneer, who (in 1890 [Poi90]) discussed the problem later known as the Poincaré recurrence theorem (however later proved by Carathéodory [Car19]). This attribution owes to the fact that Poincaré was concerned with the asymptotic properties of a wide-class of systems. Following this, Von Neumann continued the study of dynamical systems from a functional analysis point of view. However, since many of the most powerful theorems in ergodic theory are measure theoretic, the modern treatment of dynamical systems was truly started by Kolmogorov (c. 1958) who introduced probabilistic methods into the subject. The probabilistic point of view has motivated a plethora of recent work by many great mathematicians. While the problems addressed are wide-ranging, the philosophy is often similar: namely dynamicists seek to use measure theory to characterise different behaviour (i.e symmetry, invariance, or asymptotic properties) of a dynamical system. See [KH95] for an excellent introduction to dynamical systems and its development.

One example of a dynamical system is a billiard table, where one considers the motion of a point particle in a compact region flying according to Newtonian dynamics, and colliding elastically with the walls of the table (see Figure 1.1). While the rules governing the dynamics of the particle are easy to compute, the behaviour exhibited by these simple systems can vary dramatically, and is not fully understood to this day. From the point of view of physicists, similar systems have been used to study the dynamics of clouds of particle systems, dating back to the work of Boltzmann in the 1870s.

Another fascinating topic in modern mathematics is the study of tilings of the plane or of the disk (see Figure 1.1). It is perhaps surprising that tilings (a stationary object) can be thought of in terms of the evolution of a dynamical system. However, dynamical systems can be used to characterise properties of given tilings, the space of all tilings, and the motion of particles in periodic environments. This point of view has led to a number of surprising breakthroughs in a variety of settings, ranging from geometric objects introduced by the Greeks (e.g Apollonian circle packings) to questions about the approximation

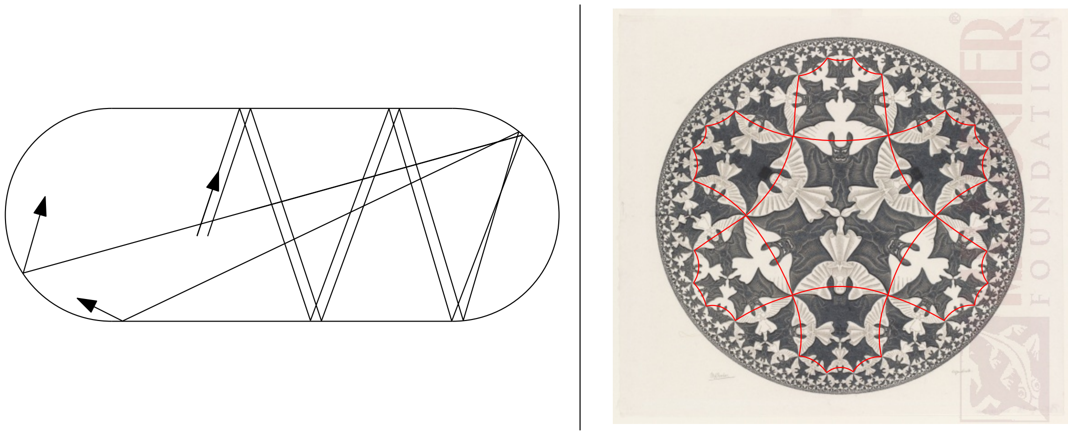

Figure 1.1: Here we show two examples of dynamical systems. On the left is the motion of two particles in a 'Bunimovich stadium' - an example of a billiard table. On the right is the print 'Circle Limit IV' (or Heaven and Hell) by M.C Escher. The image shows a hexagonal tiling of the disk using hyperbolic geometry. Indeed Escher benefited greatly from discussions with mathematician Donald Coxeter in constructing these hyperbolic tilings.[1]

of irrationals by rational numbers (Diophantine approximation) raised by mathematicians in the 19^{th} Century.

Now, if we want to consider the statistical properties of dynamical systems, there are at least two ways to do this. One option is to generate an orbit deterministically, for example the centres of the hexagonal tiles in Figure 1.1. Then to ask how a typical observer would see this orbit. For example, place an observer at a random position in the image on the right hand side of Figure 1.1 and ask what distance is the observer from the centre of the nearest tile. Thus the randomness is in the observation. Alternatively, one could introduce some randomness in the initial set-up of the dynamical system and measure properties of the orbit. For example, in Figure 1.1 one could consider randomly chosen initial conditions for the particle in the Bunimovich stadium and how quickly such trajectories diverge. The goal for dynamicists is then to understand what can be said about typical (with respect to either of these sources of randomness - or both) statistics. In the remainder of this introduction (prior to starting Part 1) we will present a brief and informal introduction to the two topics which we study in each of the subsequent parts.

1.1 Non-Equilibrium Statistical Mechanics

The first half of the thesis concerns non-equilibrium statistical mechanics. Generally speaking, statistical mechanics is the study of large ensembles of particles, starting with rules governing how the particles interact. Indeed in 1900 David Hilbert presented a list of 23 problems [Hil02] which were unsolved at the time and which he felt were central to mathematics. Hilbert's 6^{th} problem concerns the axiomisation of the laws of physics. Specifically, it states (in part) the need to develop "mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua". In other words, Hilbert was concerned with how microscopic laws, governing interaction between particles can result in the laws which we observe around us (the organisation of clouds, the flow of liquids and heat, nucleation of stars, flows of traffic, *etc.*) - *i.e* macroscopic behaviour.

[1]Regarding Circle Limit IV: All M.C. Escher works © 2019 The M.C. Escher Company - the Netherlands. All rights reserved. Used by permission. www.mcescher.com - The author is very grateful to the M.C Escher Company for this permission.

While Hilbert gave a very elegant phrasing of a very general and open-ended problem, he was not the first person to study it. In the 19^{th} Century, Ludwig Boltzmann studied an intermediate problem of this nature. Specifically, Boltzmann asked: given the rules for inter-particle interactions and an initial ensemble, what can be said about the evolution of the ensemble? In other words how does the particle *density* evolve? We call this the mesoscopic regime, and Boltzmann was interested in how we move from the microscopic to the mesoscopic; as oppose to Hilbert who asked how to go from microscopic straight to macroscopic. Moving from mesoscopic to macroscopic presents a problem in itself, but generally speaking this is better understood than the transition from microscopic to mesoscopic.

Given the rules for inter-particle interaction, there are two fundamental questions commonly asked. Firstly, given a cloud (or density) of these particles, is there a partial differential equation describing the evolution of such a cloud? Boltzmann gave a heuristic argument that, in a particular scaling limit (the low density limit), for a particular particle system, the particle density evolves according to, what became known as the Boltzmann equation. Proving this relation rigourously has been the topic of a great deal of research in the 20^{th} and 21^{st} centuries. The second question asks: consider the trajectory of a single particle inside of such an ensemble for a very long time, if we scale this appropriately does the trajectory 'look like' a Brownian motion? In other words, on large time scales and suitably 'zoomed-out' do these trajectories look random? In a sense, both of these questions are asking whether one can approximate the individual particle trajectories by random, independent, Markovian trajectories. In the first instance we ask if the bulk can be well-modelled by a gas of independent molecules. The second asks if the typical trajectory converges to a Brownian motion (a purely probabilistic object) when viewed on long time scales.

In this thesis we are primarily concerned with *non-interacting* particle systems, in particular two models:

- **The random Lorentz gas:** Given an array of fixed, infinite-mass, spherical obstacles of a given radius, randomly arranged in $\mathbb{R}^3$, consider the trajectory of a point particle which begins at the origin and travels in straight lines until it collides with an obstacle, whereupon it reflects elastically off of the sphere. Then the particle continues flying in straight lines until the next collision (see the left side of Figure 1.2).

- **The random wind-tree:** Here we also consider a random array of obstacles. However in this case the obstacles are hard cubes. Then, as for the random Lorentz gas, we consider the trajectory of a point particle with a given initial velocity travelling through this array in straight lines reflecting off of the cubes (see the right side of Figure 1.2).

The Lorentz gas was originally proposed [Lor05] as a model to study the motion of electrons through metals. The model may seem simplistic at first, but it is very fundamental and of great importance. In particular, the dream for mathematicians is to model particle systems by independent probabilistic objects with no memory. If that were the case, then Hilbert's 6^{th} problem would essentially be solved, as the macroscopic laws governing a fluid of independent particles with no memory are well-understood. The random Lorentz gas is compose of independent point particles which have memory – in that as the particle explores its environment, it might return to a certain position and recollide with a previously encountered scatterer. Therefore the goal is to control these memory effects and say that the Lorentz gas behaves similarly enough to the purely random gas. As such, the Lorentz gas presents one of the most tractable examples of a complex particle system which exhibits physically relevant phenomena (e.g diffusion). Similarly, the wind-tree model was introduced by Paul and Tatiana Ehrenfest [EE59] as a model for diffusion. The challenge with the wind-tree process is that since the obstacles are square, there is less randomness introduced at every collision, hence understanding the role played by memory effects presents additional challenges.

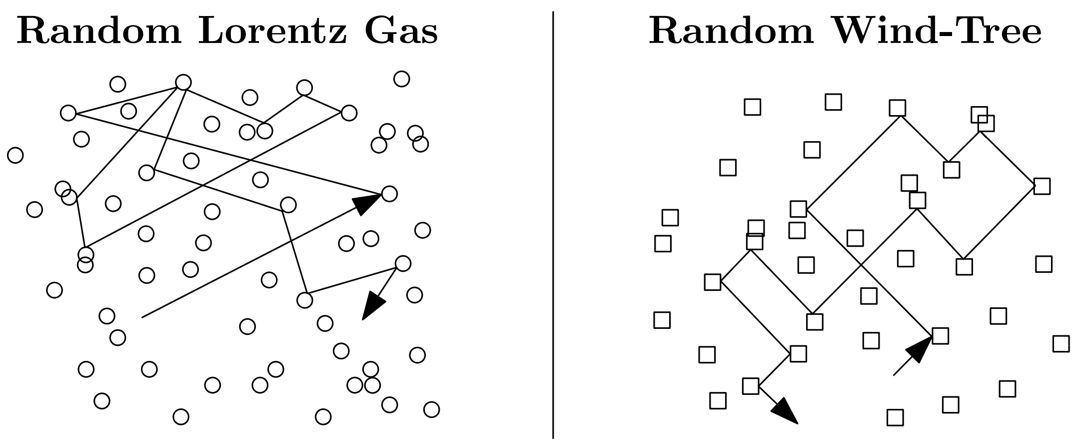

Figure 1.2: On the left we show a typical trajectory of the random Lorentz gas while on the right we show a typical trajectory of the random wind-tree model. Note that in either case the obstacles do not move throughout the trajectory. The key difference in the dynamics is that in the wind-tree model the velocities are restricted to a finite set (in 2 dimensions there are only 4 possible velocities), while in the Lorentz gas model the velocities can be anything in the unit sphere.

In both contexts, the first question we asked: "are clouds of Lorentz gas/wind-tree particles governed by a (linear) Boltzmann equation in the low-density limit?" has been positively answered (see [Gal70, Spo78, BBS83]) in some generality. However the second question "convergence to a Brownian motion in the diffusive limit" is one of the main open questions in the field. In Part I of the thesis, we will address this open problem and prove an intermediate result towards this second question for the random Lorentz gas and the wind-tree model.

1.2 Orbits of Thin Groups

Since the late 19^{th} Century and the work of Poincaré, Klein, and other pioneers of the field, mathematicians have studied the orbits of discrete hyperbolic groups (discrete groups acting on the hyperbolic half-plane). This research has had numerous consequences - in particular when the groups concerned are lattices. However until recently there has less research concerning the counter-part to lattices: infinite co-volume discrete (or thin) subgroups (see Chapter 5). This disparity owes more to the lack of tools for handling thin groups rather than to any disparity in the applications or relevance. Recently some of the tools classically used to study lattices have been generalised to the thin group setting. As a consequence, the topic has become the focus of a great deal of modern mathematical research.

From the arithmetic side, the development of strong and super-strong approximation (See Chapter 5, Section 5.3.1) has allowed mathematicians to extend sieving theory to thin groups and prove local global principles. Without entering into the definitions and details, this development has opened thin groups up to arithmetic techniques. From the geometric point of view, Patterson-Sullivan theory gave rise to the development of measure theory and ergodic theory in the thin (hyperbolic) subgroups setting (Chapter 5, Section 5.5). This second step means that the techniques from homogeneous dynamics can now be applied to the infinite volume setting. Therefore, at around the same time as thin groups have become increasingly relevant to modern mathematics because of their number theoretic applications, the ergodic tools for studying their group orbits have been developed. These advances in the theory of thin groups have recently been successfully applied to Apollonian circle packings, Pythagorean triples, continued fractions, group theory, *etc.*

Since thin groups have taken a more central role in mathematics, it makes sense to study the statistics of these groups. In particular we ask, what can be said about the distribution of distances between points in the orbit of a thin group? In general the study of local statistics of point sets has numerous applications to quantum systems, random matrices, the Riemann hypothesis *etc.*. Moreover understanding the local statistics of group orbits is akin to understanding the fine-scale behaviour of the group. The goal for Part II is to characterise the fine-scale statistics of general discrete (possibly thin) hyperbolic subgroups.

An Explicit Example: For the sake of concreteness we will present one example of such a thin group which is relevant to our research, and to a long-studied problem: the Apollonian circle packing (or Apollonian gasket) - named after Apollonius of Perga (c. 200 B.C) who (among other things) was interested in the tangencies of 'kissing circles'. Given three mutually tangent circles, it is always possible to draw two more circles which are tangent to all three circles (in Figure 1.3, on the left hand side we show three mutually tangent circles, and in dotted line the two circles tangent to all three).

In the 1640s Descartes studied this relationship, indeed Descartes even wrote to Princess Elizabeth of Bohemia on the subject. Descartes found that given the radii of three mutually tangent circles there is a formula for the radius of the the fourth circle which is mutually tangent to all three - thus relating the problem to a problem in Diophantine equations. That is, if we consider circles with empty interior to have positive radius, and circles with empty exterior to have negative radius (therefore in Figure 1.3 in the middle, only the outer-most circle has negative radius):

Theorem 1.2.1 (Descartes–Princess Elizabeth)**.** *Given three mutually tangent circles with radii* $r_1, r_2, r_3 > 0$ *and assume the fourth mutually tangent circle has radius* $r_4 > 0$*. Then*

$$2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2. \tag{1.2.1}$$

Subsequently, in the 1930s Nobel prize winning chemist, Frederick Soddy considered the problem (and even eulogised it in a poem in Nature [Sod36]). He was the first to consider packings where one continued to inscribe circles into the diagram (see Figure 1.3). That is, start from a large circle and two smaller circles all mutually tangent (i.e starting with the left hand image in Figure 1.3), then add the two mutually tangent circles to the diagram. Then select three mutually tangent circles from the packing and add to the diagram the circle tangent to all three until all of the 'holes' in the picture have been filled. This generates an Apollonian packing (as in the middle image of Figure 1.3).

These packings have been extensively studied and we will return to them later. For this introduction we simply note that the Apollonian gasket can be viewed as the orbit of an initial configuration by a (thin) group (called the Apollonian group). Specifically, consider 4 mutually tangent circles (see the right hand side of Figure 1.3), this will be our initial configuration. Given three circles in this configuration we call the circle which passes through the tangency points of this triple the *dual* circle (associated to the triple). Therefore there are 4 dual circles to the initial configuration. Given a dual circle we can consider the inversion through that circle - that is, a natural mapping from the outside to the interior of the circle (and vice versa) (see Figure 1.3). Therefore, given an initial configuration there are 4 inversion maps. These four maps generate a (thin) group. Moreover, the action of this group on the initial configuration produces the entire Apollonian gasket.

To conclude, given an initial configuration there is a group which generates the Apollonian gasket. Hence studying the statistical properties of the gasket is akin to studying the statistical properties of an orbit of the group. Moreover, using Theorem 1.2.1, studying the statistical properties of solutions to equation (1.2.1) is also akin to studying this group orbit. This is just one example of an interesting thin group-orbit of which there are many. We return to this topic in Part II. For a more detailed

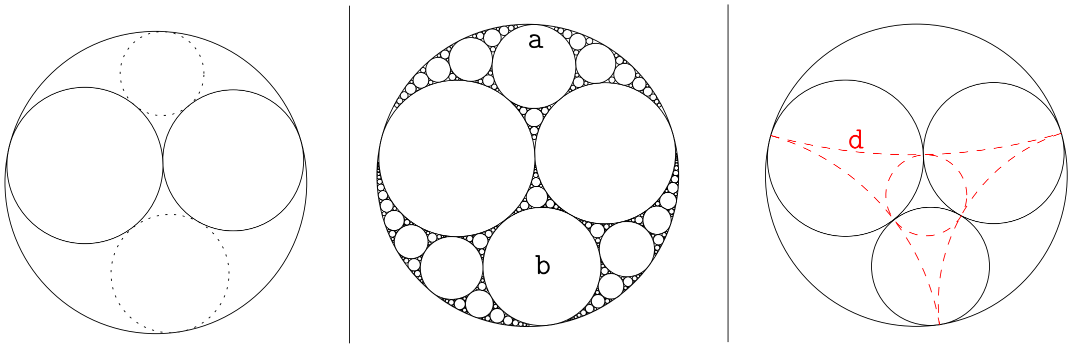

Figure 1.3: On the left hand side, we show three mutually tangent circles, along with the two circles tangent to all three. In the middle we show a diagram of an Apollonian circle packing. Beginning with the image on the left hand side we construct the packing by repeatedly filling in the holes with circles mutually tangent with 3 of the existing circles. On the right hand side we show an initial configuration of 4 mutually tangent circles. In dashed lines we show (segments of) the dual circles to all 4 triples of circles in the initial configuration. The circle labelled a is the image of the circle labelled b under inversion by the dual circle labelled d.

exposition of Apollonian packings and their history see [Pol15].

1.3 Plan and Organisation

This thesis is based on 2 rather independent research projects. Each comprising 2 papers. To reduce the confusion we treat each of these research projects independently in two parts. Each chapter will be more-or-less self-contained. Therefore there is some repetition.

Part I: presents my work with Bálint Tóth on non-interacting particle systems [LT18, LT19].

- *Chapter 2* is a formal introduction to the Lorentz gas and wind-tree processes, as well as some historic background concerning similar research. Then, in Section 2.5 I present some preliminary theorems and definitions.

- *Chapter 3* (joint with Bálint Tóth) concerns the random Lorentz gas, in this chapter we show that an invariance principle holds for the random Lorentz gas in an intermediate scaling regime. That is, under appropriate scaling we show that a typical random Lorentz gas particle converges to a Brownian motion. This does not fully solve the problem stated in this introduction as we need to work in a regime intermediate between the kinetic regime and the diffusive one. Thus this represents partial progress towards resolution of this central problem.

- *Chapter 4* (joint with Bálint Tóth) concerns the random wind-tree process. We show that the random wind-tree process also satisfies an invariance principle in the same intermediate scaling regime as Chapter 3. In particular it is interesting to note that while the wind-tree model has less of a defocusing mechanism built into the dynamics, the same type of limiting behaviour is observed in both the wind-tree and Lorentz models. We emphasise that the methodology in this chapter is very similar to that of Chapter 3. Indeed the central ideas of the proof are present in Chapter 3 however there are a number of complications in the application due to the change in dynamics.

Part II: contains work from 2 papers written by myself [Lut18, Lut19] concerning the local statistics of the orbits of thin groups.

- *Chapter 5* is a background/introductory chapter which introduces hyperbolic geometry, and dynamics therein; thin groups; and the measure theory and some of the important theorems in that context.

- *Chapter 6* presents my work [Lut18] characterising the local statistics of directions in group orbits. That is, given the orbit of a point under the action of a thin group, and an observer placed in space, how do the directions of the points of the orbit distribute when viewed from the position of the observer.

- *Chapter 7* presents my work [Lut19] characterising the local statistics of generalised Farey sequences. That is, what can be said about the local statistics of the orbit of the point at infinity by a thin group. This gives rise to some surprising applications to continued fractions and Diophantine approximation.

1.3.1 Authorship

To avoid cluttering the exposition below, we explain here how the various sections were written and which sections are taken from previously disseminated papers.

- *Chapter 2:* This chapter expands on the historical background given in [LT18] and [LT19], some of the explanations are given verbatim and some are given by myself here. In addition there are some classical definitions and theorems presented here.

- *Chapter 3:* This chapter is an expanded and modified version of [LT18], the paper was originally written by Bálint Tóth and myself.

- *Chapter 4:* This chapter is an expanded and modified version of [LT19], the paper was originally written by Bálint Tóth and myself.

- *Chapter 5:* This background/introductory chapter was written for this thesis - with the exception of Chapter 5, Section 5.4 - 5.6 which are taken almost verbatim from [Lut18]

- *Chapter 6:* presents my work in [Lut18] and is taken almost verbatim, with some modification where appropriate.

- *Chapter 7:* presents my work in [Lut19] and is taken almost verbatim, with some modification where appropriate.

The original research for this thesis can be found in Chapters 3, 4, 6, and 7, while Chapters 2 and 5 are background.

Part I

Kinetic Theory and Diffusion in Lorentz Gas Models

Chapter 2

Kinetic Theory of Gases and Motivation

Statistical mechanics is the study of how small-scale laws governing particle interactions can determine global behaviour of a larger body built up from these particles. In order to understand how a gas of particles behaves mathematically, one requires several pieces of information. Firstly one requires a rule for how particles fly through space when unimpeded (this could be free flight in straight lines or, for charged particles in a magnetic field – in circles). Then one requires rules about how the particles interact with each other (e.g non-interacting particles, colliding spheres, Coulomb interactions...) and the environment. Lastly one requires an initial state. Rather than prescribe one particular initial state it is more common to give the initial state as a probability distribution on the phase space. Alternatively, one can view such a distribution as a particle cloud. For more information on the general picture and approach used we suggest the excellent and detailed monograph [Spo91].

As we discussed in Chapter 1, the central aim of mathematical statistical mechanics is to begin with particle dynamics and derive solutions to equations describing the continuous fluid. The first step in doing this is to ask what happens if the particles are independent and fly with no memory. Under this assumption it is simple to derive the linear Boltzmann equation to describe the particle density; and, with the diffusive scaling, to derive the heat equation. The challenge is then to use these independent dynamics to approximate the true dynamics of the model with which we are concerned (or to explain how these heuristics do not approximate the true dynamics).

2.1 Boltzmann and Heat Equations

2.1.1 Boltzmann's Heuristic

In 1872 Boltzmann [Bol72] used a heuristic argument to derive the non-linear Boltzmann equation for interacting particle systems. Later, Lorentz [Lor05] used the same argument to derive the linear Boltzmann equations for the non-interacting particle systems that he was studying. As our research will focus on Lorentz gas it is more informative to see Lorentz's application of Boltzmann's heuristic argument.

Consider a single particle, moving classically with Hamiltonian and equations of motion

$$
\begin{aligned}
H(x,v) &:= \frac{1}{2}v^2 + U(x) \\
\dot{x}(t) = \partial_v H = v, \qquad & \dot{v} = -\partial_x H = -\nabla U(x).
\end{aligned}
\tag{2.1.1}
$$

For example, if $U \equiv 0$ then

$$
x(t) = x_0 + v_0 t. \tag{2.1.2}
$$

Now let $f_t(x,v)$ describe the phase-space density of a cloud of independent particles. Therefore

$$
\#\{\text{particles with } (x,v) \in A \text{ at time } t\ \} = \int_A f_t(x,v)dxdv. \tag{2.1.3}
$$

Applying the chain rule to (2.1.1) then gives

$$
(\partial_t + v \cdot \nabla_x) f_t = \nabla U(x) \cdot \nabla_v f_t(x,v), \tag{2.1.4}
$$

the *Liouville equation*. Note that if $\dot{v} = 0$ then the Liouville equation becomes the free transport equation

$$
(\partial_t + v \cdot \nabla_x) f_t = 0, \tag{2.1.5}
$$

with solution $f_t(x,v) = f_0(x - vt, v)$.

Now, define a random process on S_1^{d-1} with jump rate

$$
\sigma(v,u) : S_1^{d-1} \times S_1^{d-1} \to \mathbb{R}_+. \tag{2.1.6}
$$

That is, we consider a process on S_1^{d-1} which jumps from velocity v to u with rate (in the probabilistic sense) $\sigma(v,u)$. If we let $f_t(v)$ be the density of this jump process with velocity v then we have the following density evolution

$$
\partial_t f_t(v) = \int \sigma(v,u) \left[f_t(u) - f_t(v) \right] du. \tag{2.1.7}
$$

Putting the jump process and the free-evolution processes together: if we consider an array of independent particles which move according to free transport in between velocity jumps given by the jump rate σ, then the density of a cloud of these particles is given by the *linear Boltzmann equation*

$$
(\partial_t + v \cdot \nabla_x) f_t(x,v) = \int \sigma(v,u) \left[f_t(x,u) - f_t(x,v) \right] du. \tag{2.1.8}
$$

In this thesis we are primarily concerned with the linear Boltzmann equation. However, again starting from Newtonian dynamics, Boltzmann gave a heuristic argument ([Bol72]) that, when the particles described interact, then under suitable independence assumptions, the density should satisfy the *non-linear Boltzmann equation:*

$$\partial_t f_t + v \cdot \nabla_x f_t = \alpha Q(f_t, f_t) \tag{2.1.9}$$

where α is the inverse of the mean free flight length, and Q is the quadratic bi-linear operator:

$$Q(f_t, f_t) := \int_{S^{d-1}\times\mathbb{R}^d} [f' f_1' - f f_1] \, ((v - v_1)\cdot\omega) dv_1 d\omega, \tag{2.1.10}$$

where ω is the deflection angle and

$$f' = f_t(x, v'), \qquad f_1' = f_t(x, v_1'), \qquad f_1 = f_t(x, v_1),$$
$$f = f_t(x, v),$$

with

$$v' = v + \omega \cdot (v_1 - v)\omega, \qquad v_1' = v_1 - \omega \cdot (v_1 - v)\omega.$$

The non-linear Boltzmann equation can be understood in the same way as the linear Boltzmann equation. The left hand side of (2.1.9) is a free transport term. The right hand side can be separated into a gain and loss term which account for the particles adopting velocity v and those losing velocity v. The difference is that now the particles are inter-dependent.

2.1.2 Heat Equation

Let v_i be i.i.d $\mathbb{R}$-valued random variables with $\mathbf{E}(v_i) = 0$ and $\mathbf{E}\left(v_i^2\right) = \sigma^2$ (if the random variables are d-dimensional, then the variance is described by a co-variance matrix). Let

$$S_n := \sum_{i=1}^{n} v_i, \qquad \mathbf{E}\left(S_n^2\right) = n\sigma^2. \tag{2.1.11}$$

For $T > 0$ a fixed macroscopic time let

$$n := \lfloor T\epsilon^{-1} \rfloor,$$

(where $\lfloor\cdot\rfloor$ denotes the largest integer less than the argument) then, by the classical central limit theorem

$$X_T^\epsilon := \epsilon^{1/2} \sum_{i=1}^{n} v_i \to X_T \tag{2.1.12}$$

as $\epsilon \to 0$, where X_T is a Gaussian random variable with variance $T\sigma^2$ (the convergence is in distribution). Thus the distribution of X_T has density

$$f_T(X) := \frac{1}{\sqrt{2\pi T\sigma^2}} \exp\left(-\frac{X^2}{2\sigma^2 T}\right). \tag{2.1.13}$$

From here it is easy to see that f_T satisfies the *heat equation* with diffusion coefficient σ^2:

$$\partial_T f_T(X) = \sigma^2 \partial_X^2 f_T(X). \tag{2.1.14}$$

Therefore the heuristic argument is that beginning from independent random variables, and applying the *diffusive scaling*, we arrive at a solution to the heat equation. The hope is then, by applying the same scaling to our particle systems, to achieve the same convergence to a solution of the heat equation. Hence the goal is to prove a central limit theorem for the path segments for the true process describing our particles. In fact, one can go further than the central limit theorem and prove the invariance principle (convergence to a Brownian motion).

For an excellent reference for the derivation of the linear Boltzmann and heat equations, and how this intuition has motivated some beautiful work in the quantum setting we recommend the lecture notes [Erd12].

2.1.3 Boltzmann-Grad Limit and the Invariance Principle

There are two central problems when considering certain particle systems, each concerning a different scaling regime.

First we can ask whether, in the diffusive limit, the law for the trajectory of a particle converges to that of a Brownian motion (invariance principle): *i.e* consider the position of a typical particle trajectory given by $t \mapsto X(t) \in \mathbb{R}^d$. The *diffusive limit* is given by

$$t \mapsto \frac{X(Tt)}{\sqrt{T}}, \qquad T \to \infty. \tag{2.1.15}$$

More generally, one scales the particle position $X(t)$ by the expected distance from the origin - $\sqrt{T}$ (in this instance). A fundamental question is then, in the limit $T \to \infty$, does the process $t \mapsto \frac{X(Tt)}{\sqrt{T}}$ converge to a Wiener process (*invariance principle*)? For example, this implies a central limit theorem for the random variable $\frac{X(T)}{\sqrt{T}}$.

The second scaling limit we consider is the so-called Boltzmann-Grad limit. If the interaction length is of order r, and the particles (or in some instances obstacles) have density ϱ, then the *Boltzmann-Grad* limit is:

$$r \to 0 \quad , \quad \varrho \to \infty, \qquad r^{-(d-1)}\varrho \to C. \tag{2.1.16}$$

In this limit the mean flight time between collisions is of constant order. Moreover, in this limit one expects that collisions become (in some sense) uncorrelated (this is because the mean free path length is much longer than the inter-particle distance). Hence, this should be the regime in which the particle density satisfies the Boltzmann equation. This intuition was put forth by H. Grad and later, Lanford [Lan75] (for the simplest interacting particle systems) and Gallavotti, Spohn and Boldrighini-Bunimovich-Sinai [Gal70, Spo78, BBS83] (for some non-interacting particle systems) showed that in this limit, if one begins with an initial distribution of particles $f_0 : \mathbb{R}^d \to \mathbb{R}$, then the distribution at time t, f_t is an exact solution to the (non-linear or linear respectively) Boltzmann equation.

2.2 The Lorentz Gas

In 1905, to model the motion of electrons through metals, Hendrik Lorentz proposed the following model [Lor05] - now called the *Lorentz gas*. Given an array of spherical scatterers arranged throughout space consider the motion of a point particle travelling in straight lines and reflecting symmetrically off of the scatterers. This model has been and continues to be a central topic in statistical mechanics owing to the fact that the model is mathematically tractable while still exhibiting complex phenomena.

More formally, let $\mathcal{P}$ be a point process on $\mathbb{R}^d$ ($d \geq 2$). Let $\mathcal{B}_r^d$ denote the d-dimensional ball of radius r. Then consider the array of 'scatterers' - $\mathcal{P} + \mathcal{B}_r^d$. We think of these balls as infinite mass, radius r obstacles. A Lorentz gas particle is a point particle moving in straight lines in the compliment $(\mathcal{P} + \mathcal{B}_r^d)^c$ and colliding reflectively off the boundary $\partial(\mathcal{P} + \mathcal{B}_r^d)$. We denote the position of such a Lorentz gas particle at time t - $X^r(t)$. There are typically two contexts in which this gas is studied: where the obstacles are centred on the points of a lattice, or where the obstacles are centred on a random point process (see Figure 2.1).

With regards the two questions described in the previous section, the Lorentz gas raises many open problems. In the diffusive limit, it is thought that the random Lorentz gas satisfies an invariance principle; while in the periodic setting it has been shown that the scaling in (2.1.15) is *sometimes* too slow and one requires additional factor of $\sqrt{\log T}$.

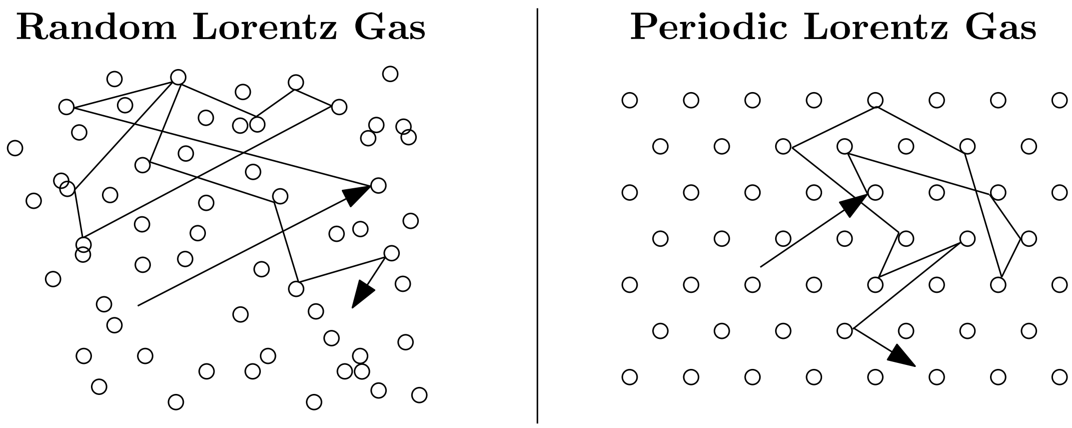

Figure 2.1: `A typical example of a periodic and a random Lorentz gas trajectory.`

Lorentz, in his original paper conjectured (using Boltzmann's heuristic argument) that (for a general class of scatterer configurations) if $f_{t,r} : T^1(\mathbb{R}^d) \to [0,1]$ (where $T^1(\mathbb{R}^d)$ represents the unit tangent bundle of $\mathbb{R}^d$) describes the particle density at a given point in the phase space. Then $\lim_{r\to 0} f_{t,r}$ satisfies a *linear* Boltzmann equation in the Boltzmann-Grad limit (2.1.16). Namely, if we consider the macroscopic coordinates

$$(q(t), v(t)) \mapsto (Q(t), V(t)) = (r^{d-1} q(r^{-(d-1)}t), v(r^{-(d-1)}t)) \tag{2.2.1}$$

and if we denote $f_t = \lim_{r\to 0} f_{t,r}$ then f_t is an exact solution to

$$(\partial_t + V \cdot \partial_Q) f_t(Q, V) = \int_{\mathbb{R}^d} [f_t(Q, V') - f_t(Q, V)]\, \sigma(V, V') dV', \tag{2.2.2}$$

where $\sigma(V, V')$ denotes the differential cross-section of a scatterer.

We next present some of the historical results towards these heuristics. As this topic has a long history we do not hope for completeness and point the interested reader to the following surveys [Det14, Mar14, Spo88a] and the monograph [Spo91].

2.2.1 Periodic Lorentz Gas - Diffusive Limit

The periodic Lorentz gas lends itself to analysis using tools from hyperbolic dynamics, and thus more has been rigorously proved in this context. Indeed, the periodic Lorentz gas is an example of a dispersing billiard table (for a detailed text on dispersive billiards see the monograph [CM06]). That is, because of the periodic structure, one can equivalently consider the motion of a particle on a torus with disjoint spherical holes (see Figure 2.2).

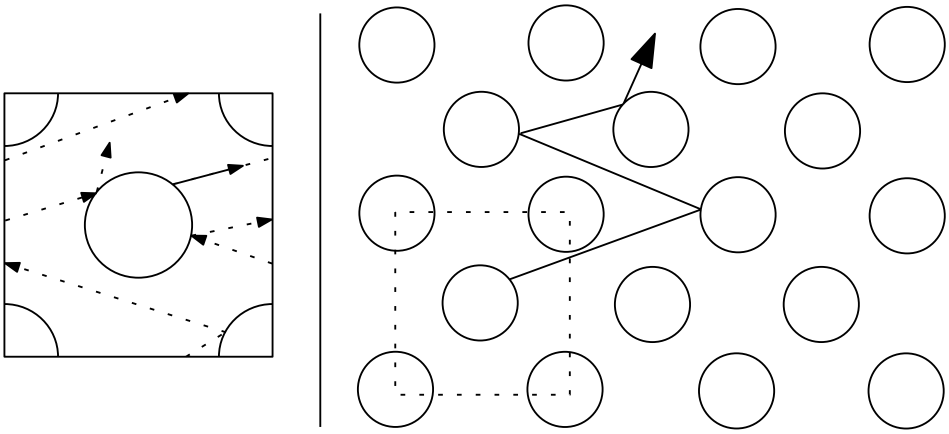

Figure 2.2: We show how the periodic Lorentz gas is equivalent to a dispersing billiard table (left). In both diagrams the same trajectory is pictured however on the left we consider motion on the torus and on the right we consider motion in a periodic array.

Thus, working in $d = 2$ for simplicity, we consider the torus $\mathbb{T}$ with n balls removed, $\{\mathcal{D}_i\}_{i=1}^n$. A point particle which has just hit one of these balls moves with velocity away from the surface (i.e if the point on the boundary is given by the vector $x \in S_1^1$ in the unit sphere and velocity is given by $v \in S_1^1$ then $x \cdot v > 0$). Therefore, the space

$$\mathscr{M} := \bigcup_i \left(\partial\mathcal{D}_i \times (-\pi/2, \pi/2)\right), \tag{2.2.3}$$

parameterises the set of collision points and exit velocities. Given a particle beginning at a point on the boundary $\bigcup_i \mathcal{D}_i$ and a velocity outwards, we consider the path drawn by this particle as it moves along straight lines colliding with the obstacles. This generates a set of points in $\mathscr{M}$ which describe the position and outgoing velocity of each collision. We call the map which sends one of these points to the next in the sequence the *Billiard map:*

$$T : \mathscr{M} \to \mathscr{M}. \tag{2.2.4}$$

Given a point in a connected component of $\mathscr{M}$ we write the point $(x, \varphi) \in S_1^1 \times (-\pi/2, \pi/2)$. The Liouville measure on the space, namely $d\mu = |\cos\varphi|\, dr d\varphi$ has been shown (see [CM06]) to be ergodic with respect to the map T.

In 1980 Bunimovich and Sinai [BS80] showed that some dispersive billiards admit a Markov partition. That is, the phase space $\mathscr{M}$ can be decomposed into stable and unstable curves and singular points corresponding to grazing collisions - collisions tangent to the obstacles. In a subsequent paper Bunimovich and Sinai [BS81] showed that this Markov partition can be used to estimate the decay of velocity correlations, which allowed them to prove an invariance principle (see Theorem 2.5.2 for an example of an invariance principle) for 2-dimensional periodic Lorentz gas particles with finite horizon (i.e where the length of any straight line, not intersecting a scatterer is bounded from above). In higher dimensions this result was extended in [Che94] by Chernov under an (as yet) unproved assumption on the singularities of the billiard flow.

If the periodic array has infinite horizon (therefore there exist trajectories with unbounded straight flight segments), as a result of these infinite channels, the free flight distribution of a particle flying in a uniformly sampled random direction has a heavy tail which results in a slower diffusion. Bleher [Ble92] suggested a super-diffusive scaling of $t \mapsto \frac{X(t)}{\sqrt{T \log T}}$. Subsequently, Szász and Varjú [SV07] showed that indeed a central limit theorem holds for this super-diffusive scaling in 2 dimensions. The 3 dimensional case remains open. Chernov and Dolgopyat [DC09] showed that this theorem has a continuous time analogue which implies an invariance principle (they also investigate the effect of an external field on the infinite horizon case therein).

2.2.2 Periodic Lorentz Gas - BG Limit

The periodic Lorentz gas in the Boltzmann-Grad limit can be understood in terms of the machinery of homogeneous dynamics. In so doing, the limiting behaviour of Lorentz gas trajectories can be understood in terms of equidistribution of expanding horospheres - see for example [Mar14, Section 6] for a summary of this connection.

Without entering too deeply into the history (which is summarised in [Gol06, Mar14]) we note that in a 2006 ICM address Golse [Gol06] discussed how (2.2.2) fails for general periodic configurations. However, in [CG10] (with an assumption valid only in 2 dimensions) and in [MS11] (in general dimensions) it was proved that, in the Boltzmann-Grad limit (2.1.16) for a fixed time interval $[0, T]$ the Lorentz gas converges weakly to a *non-Markovian* flight process which admits a full description in terms of a Markov chain. In particular the limiting stochastic process is a 'memory 2 Markov chain'. Marklof and Strömbergsson [MS11, Section 6] then showed that this stochastic process, when considered on an extended phase space satisfies a Boltzmann-like equation ([MS11, Theorem 6.4]).

Subsequently Marklof and Tóth [MT16] showed that, with a super-diffusive scaling of $\sqrt{T \log T}$, this limiting stochastic process satisfies an invariance principle (note that this result is not immediately implied by Donsker's invariance principle). An interesting open question analogous to the problem we study in Chapter 3 is to interpolate between this result and the aforementioned result of Chernov and Dolgopyat (discussed in section 2.2.1). That is, Marklof and Tóth show that if one first takes the Botlzmann-Grad limit then the super-diffusive limit one gets an invariance principle. While Chernov and Dolgopyat show that simply in the super-diffusive limit, the invariance principle holds. Thus one can ask what would happen in the intermediate regime where T is taken to go to ∞ as $r \to 0$?

2.2.3 Random Lorentz Gas - BG Limit

While the random Lorentz gas is of great importance, there have been fewer rigourous results proved than for the periodic case. The first seminal papers on the subject came when Gallavotti ([Gal69, Gal70]) showed that in the Boltzmann-Grad limit, for Poisson configurations, the Lorentz gas converges weakly to an exact solution of the linear Boltzmann equation (i.e a solution to (2.2.2)). To prove this result Gallavotti used classical methods, integrating over the space of trajectories and configurations. Spohn [Spo78] extended this result using far less-classical methods. Spohn used the BBGKY hierarchy (repeated application of Duhammel's formula) to estimate the decay of correlations for Lorentz gas trajectories. This allowed Spohn to show that the Lorentz gas process converges to a Markovian flight process on a fixed time window. For Poisson configurations and hard-spheres this result is implied by Gallavotti's work, however Spohn extended this to more general scattering potentials and configurations. While these PDE methods are very powerful and have been applied to numerous other settings (such as interacting particle systems [Spo91]), our main result in Chapter 3 will be to show that versatile probabilistic methods can be used to extend this result to a much longer time-scale.

Boldrighini-Bunimovich-Sinai [BBS83] followed Gallavotti and Spohn's results by showing that the convergence holds for *typical* realisations of the Poisson process (i.e *quenched*). Their argument, while returning to a probabilistic approach, is very technical and makes use of Bernstein-type estimates. In particular, in the quenched setting one needs to control the correlations between different trajectories all exploring the same physical space, making this a significantly more difficult problem.

2.2.4 Random Lorentz Gas - Weak Coupling Limit

Having discussed the Boltzmann-Grad and diffusive limits, there is one more limit we discuss in this introduction – the weak coupling limit. This limit is particularly relevant to our research since (before our result) it is the only regime in which the random Lorentz gas has been shown to converge to Brownian motion. The weak coupling limit is a physically different procedure and does not make sense for hard-spheres. Therefore we will repeat the usual set-up with the usual notation of the weak-coupling literature.

Let $\varepsilon \to 0$ be a scaling parameter and place infinite mass fixed scatterers on the points of a Poisson point process of density $\varrho = \varepsilon^{-d}$ in $\mathbb{R}^d$. However now we assume that the scattering potential, $\mathcal{U}$ is spherically symmetric, smooth, and supported in a ball of radius ε (rather than the hard-spheres considered earlier). So far the scaling corresponds to a linear spatial scaling by a factor ε. Therefore, with this scaling alone, the mean free path length would be ε^{-1}, we thus define the natural time-scale of the problem to be

$$T_{kin} := \varepsilon^{-1}. \tag{2.2.5}$$

In the weak coupling limit, rather than further scale down the radius of support, the strength of the potential is scaled. To that end, Newton's equations of motion for the kinetically scaled particle are

$$\dot{X}^\varepsilon(t) = V^\varepsilon(t), \qquad \dot{V}^\varepsilon(t) = -\nabla U^\varepsilon(X^\varepsilon(t))$$

in the potential field

$$U^\varepsilon(x) = \sum_{q\in\omega} \varepsilon^{1/2}\mathcal{U}(\varepsilon^{-1}(x-q)),$$

where ω is the realisation of the Poisson point process of intensity $\varrho = \varepsilon^{-d}$. In words, we apply a spatial scaling so that in one unit of time (T_{kin}) there are ε^{-1} collisions, however we also scale down the strength of the potential by a factor $\varepsilon^{1/2}$. Therefore there are significantly more velocity kicks than in the original model, however these kicks are much smaller.

From the work of Kesten and Papanicolaou [KP80] it follows that

$$V^\varepsilon(t) \Rightarrow \mathcal{V}(t), \qquad X^\varepsilon(t) \Rightarrow \mathcal{X}(t) := \int_0^t \mathcal{V}(s)ds, \tag{2.2.6}$$

where the limiting velocity process $\mathcal{V}(t)$ is a homogeneous diffusion (i.e. Brownian motion) on the surface of S_1^{d-1} and the weak convergence is meant in the space of continuous trajectories endowed with uniform topology on compact time intervals, (see [Bil68] or the survey [Spo88b]). Taking a second, diffusive limit, $T^{-1/2}\mathcal{X}(Tt) \to W(t)$, the displacement process converges to Brownian motion, as $T \to \infty$.

The simultaneous kinetic *and* diffusive limit in this context was done by Komorowski and Ryzhik in [KR06] where it is proved that in dimension $d \geq 3$, up to time scales

$$T = T(\varepsilon) = \varepsilon^{-\kappa}, \qquad \kappa \in (0, \kappa_0), \qquad \kappa_0 > 0, \tag{2.2.7}$$

the diffusive limit

$$T^{-1/2} X^{\varepsilon}(Tt) \Rightarrow W(t) \tag{2.2.8}$$

holds. In (2.2.7) κ_0 is small (possibly very small) and positive. Comparing these results with those for the random Lorentz gas, Kesten and Papanicolaou proved a result analogous to that of Gallavotti, Spohn, and Boldrighini-Bunimovich-Sinai - that is they prove convergence for time scales of the order T_{kin}. While Komorowski and Ryzhik go beyond the kinetic time scale. To our knowledge this was the first case for which the diffusive limit was rigourously established beyond the kinetic time scale in a context which includes the random Lorentz gas. The results in [KP80] and [KR06] are formulated in the more general context of spatially ergodic random potential fields with regularity conditions assumed. This covers weak coupling of the random Lorentz gas as a particular case.

The quantum Lorentz gas

The quantum versions of the weak coupling and low density limits for the random Lorentz gas were considered in Erdős-Yau [EY00], respectively, Eng-Erdős [EE05], where the long time evolution of a quantum particle interacting with a random potential is studied. They show that the phase-space density of the quantum evolution converges weakly to the linear Boltzmann (or Langevin) equation, with diffusive, respectively, hopping scattering kernels. These results are the quantum analogues of the classical (i.e. non-quantum) kinetic limits of [KP80] (for weak coupling), respectively, [Gal69, Gal70, Gal99, Spo78, Spo88b] (for low density).

In the weak coupling setup the simultaneous kinetic *and* diffusive scaling limit, formally analogous to [KR06] was done by Erdős-Salmhofer-Yau [ESY08, ESY07] where it is proved that under a scaling limit similar to (2.2.7) and (2.2.8) the time evolution of the spatial density of the quantum particle, weakly coupled with the fixed scatterers, converges to the solution of the heat equation. In this case the numerical value of the upper bound on the scaling exponent κ is specified in $d = 3$ as $\kappa_0 = 1/370$ (see [ESY08, Theorem 2.2]).

For a comprehensive survey of the kinetic and kinetic-diffusive limits in the quantum case see [Erd12].

2.3 The Wind-Tree Model

In 1912 Paul and Tatiana Ehrenfest [EE59] wrote a monograph exploring the history and some of the complications faced in the world of statistical mechanics. In an appendix to Section 5 of [EE59] they considered a simple model of a diffusive gas. Namely, given a d-dimensional cube parallel with the axes of side length r, $\mathcal{Q}_r$ and a Poisson point process $\mathcal{P}$ on $\mathbb{R}^d$ one considers the motion of a point particle in the compliment $(\mathcal{P} + \mathcal{Q}_r)^c$ which collides elastically with the sides of the cubes. In their monograph, Paul and Tatiana Ehrenfest used this *wind-tree* model to explain the return to equilibrium of the velocity distribution of a gas, as assumed by Boltzmann and Maxwell.

Subsequently, the wind-tree model has been the focused of a great deal of research. In particular the individual collisions of the wind-tree model are 'less defocusing' than the spherical Lorentz gas discussed in the previous section however the geometry of these collisions is simpler. As such it is interesting to ask how these particles diffuse. As with the spherical Lorentz gas, the wind-tree model is studied in both the random and periodic setting. This thesis is concerned with the random wind-tree model as presented by the Ehrenfests, however it is informative to compare some of the results in the periodic case as well.

2D Periodic Wind-Tree Model: In the periodic setting, one studies the wind-tree model as described above, with $\mathcal{P}$ replaced by the hypercubic lattice $\mathbb{Z}^2$. Moreover, rather than squares one can study the problem with rectangles parallel with the axes. This gives mathematicians two more parameters to play with. While the random model was the one introduced by the Ehrenfests, this periodic model is more extensively studied. This owes to the fact that the periodic wind-tree model (or Ehrenfest model) is an example of a parabolic dynamical system.

In particular the billiard flow (as discussed in Section 2.2.1) is parabolic - *i.e* close orbits diverge polynomially in time). Thus the standard approach is to use the so-called Katok-Zemliakov construction (see [Tab95]) to replace the billiard flow by linear flow on translation surfaces.

There have not yet been any theorems concerning the diffusive limit or an invariance principle for the periodic wind-tree process. However there have been a number of interesting and contrasting results concerned with the speed of diffusion and exceptional trajectories. Hardy and Weber [HW80] showed that some specific directions diffuse at a rate of $\log T \log\log T$. While Delecroix-Hubert-Lelièvre [DHL14] showed that typical (with respect to angle) trajectories satisfy the superdiffusive polynomial diffusion rate $T^{2/3}$. Additionally Delecroix [Del13] showed that for any rectangular scatterer, there is a set of diverging trajectories with positive Hausdorff measure. While Hubert-Lelièvre-Troubetzkoy [HLT11] and then Avila and Hubert [AH17] showed that the billiard flow is recurrent for almost every direction. Finally Frączek and Ulcigrai [FU14] proved that generically the billiard flow is not ergodic.

Random wind-tree model: The random wind-tree process is less well-studied. Gallavotti [Gal69] included the random wind-tree model when deriving the linear Boltzmann equation for the Lorentz gas model. But the subsequent work of Spohn and Boldrighini-Bunimovich-Sinai [Spo78, BBS83] on the Lorentz gas was restricted to spherical scatterers. That said, the model is of particular interest to those studying diffusion in gases. In particular, as evidenced by comparing the periodic wind-tree with the periodic Lorentz models, these square scatterers are less defocusing (i.e nearby parallel trajectories can stay together for longer in this model). As a consequence it is not evident that the random wind-tree and Lorentz processes would exhibit the same diffusive behaviour.

While the random wind-tree process with Poisson distributed scatterers was only previously treated by Gallavotti, there have been other efforts to understand the random setting. In a recent paper [MST18], Málaga Sabogal and Troubetzkoy consider a set of wind-tree configurations endowed with the Hausdorff topology. They show that in this topologically random setting, the wind-tree flow has infinite ergodic index in almost every direction. In particular, in that setting they are able to prove rigorously the Ansatz which motivated the Ehrenfests to propose this model. Namely by applying ergodic theorems they showed that the velocities of a cloud of initially parallel particles will decorrelate. As discussed in [MST18] Málaga Sabogal and Troubetzkoy have previously considered other random settings. However their results do not apply to the Poisson setting.

2.4 Interacting Particle Systems

The original work in this thesis concerns non-interacting particle systems. For completeness we give a short description of some of the work done in the interacting particle setting. As described in the introduction the central open problem in statistical mechanics is Hilbert's 6^{th} problem to develop "mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua". In the hard-sphere context the problem can be rephrased as follows: consider an infinite array of hard-spheres of fixed radius governed by Newtonian hard-sphere dynamics, can one, starting with the fine-scale Newtonian dynamics, derive large scale equations for the bulk (i.e the heat equation or Navier-Stokes equation)?

The heat and Navier-Stokes equations describe large scale flows in fluids, however deriving solutions to these equations from particle dynamics is a major open question. In the 1870s Boltzmann proposed the non-linear Boltzmann equation as an intermediary. That is, by looking at the particle density (mesoscopic scale) he proposed an intermediate scale to interpolate between the microscopic dynamics (hard-sphere interactions) and the macroscopic evolution (PDEs). Moving from the Boltzmann equation to fluid mechanics has been the focus of a great deal of research, going back to 1912 with the work of Hilbert [Hil12]. Utilising the methods developed by Hilbert and Chapman-Enskog (see [CC60]), Bardos-Golse-Levermore [BGL91, BGL93], in 1991 suggested a program for deriving the Navier-Stokes equation from (DiPerna-Lions) solutions to the Boltzmann equation. In the diffusive limit, Golse-Saint-Raymond achieved the result in 2004 ([GSR04, GSR09]). In words, this meant that one could move from solutions to the mesoscopic non-linear Boltzmann equation to the macroscopic heat or Navier-Stokes equations.

Lanford [Lan75] showed that, in the Boltzmann-Grad limit the hard-sphere model obeys a non-linear Boltzmann equation. Lanford proved his theorem by considering the marginals of the probability distribution describing the particle ensemble. Then he used the BBGKY hierarchy (i.e repeated iteration of the Duhamel formula) to prove a sort of independence result from which the validity of the Boltzmann equation follows. The problem is that Lanford's solutions are valid for a time-scale which goes to 0 in the diffusive scale, therefore one cannot use the work of Golse-Saint-Raymond to derive the macroscopic equations.

Recently in two instances ([BGSR16, BGSR16]), Lanford's method has been extended to longer times. For example in [BGSR16] the authors derive the linear heat equation from small-scale dynamics. The main result is to extend Lanford's result to infinite times (i.e macroscopic times). This gives a derivation of the (non-linear) Boltzmann equation for infinite times which the authors then show corresponds to solutions to the heat equation in the diffusive limit. As with our result for the Lorentz and wind-tree processes, the authors use probabilistic methods to classify the problematic events (here the problematic events correspond to recollisions between so-called collision trees) and show that these events occur with low probability. Therefore on their time scales these problematic trajectories do not cause a problem and a sort of independence result is achieved.

Again this problem has many variants and there are numerous results we have omitted, but to avoid over-extension we leave the discussion at these state-of-the-art results. We refer the interested reader to [Gal19] as a starting point.

2.5 Preliminaries

In this section we collect some of the preliminary facts and definitions needed in the subsequent chapters. All of these facts are classical and can be found elsewhere.

Probability moving particle is trapped:

Returning to the random Lorentz gas (and equivalently the random wind-tree model). In defining the models we assumed that the origin is not covered by a scatterer. Formally we say that if the origin is covered by a scatterer then the moving particle is stationary at the origin. More importantly, an invariance principle would not hold if the moving particle is trapped in a compact domain. Hence the following lemma, which is a consequence of several classical results from percolation theory [Gri99, Section 1.6] and a scaling argument, is needed before we proceed:

Lemma 2.5.1.

$$\mathbf{P}\,(\textit{the moving particle is not trapped in a compact domain}) = \vartheta_d(\varrho r^d), \tag{2.5.1}$$

where $\vartheta_d : \mathbb{R}_+ \to [0,1]$ *is a percolation probability which is (i) monotone non-increasing; (ii) continuous except for one possible jump at a positive and finite critical value* $u_c = u_c(d) \in (0,\infty)$*; (iii) vanishing for* $u \in (u_c,\infty)$ *and positive for* $u \in (0,u_c)$*; (iv)* $\lim_{u\to 0}\vartheta_d(u) = 1$.

In the Boltzmann-Grad limit considered here (see (2.1.16) above) we will have $\varrho r^d \to 0$. Therefore $u < u_c$ for r sufficiently small.

Proof. First of all, the property of the particle being trapped in a compact region is clearly invariant under spatial dilation. From here it follows that the function on the right hand side of (2.5.1) can only depend on ϱr^d.

Since, in a Poisson point process, the points are independently placed in $\mathbb{R}^d$, it immediately follows that ϑ_d is monotone non-increasing. That is, we can keep ϱ fixed and increase r, and clearly the probability of the particle being trapped is a non-increasing function of the obstacle radius.

The proof now follows from classical results about site percolation. We restrict to 2 dimensions for simplicity. Divide $\mathbb{R}^2$ into squares of side length $\frac{r}{2}$. The probability in (2.5.1) is bounded above by the probability that there exists a path of neighbouring squares from the origin to ∞, none of which contain a point of $\mathcal{P}$. The probability a square is empty is $p = e^{-\rho r^2}$. Therefore in the language of percolation, the probability in (2.5.1) is bounded above by the probability that the origin is connected to infinity in a site percolation on $\mathbb{Z}^2$ with probability p. A lower bound can also be similarly achieved.

From here (ii)-(iv) follow from rather classical results. The existence and boundedness of the critical value are given in [Gri99, Theorem 1.10]. The fact that percolation probability is continuous above the critical value is given in [Gri99, Theorem 8.8], and the fact that below the critical value the percolation probability vanishes follows from [Gri99, Theorem 6.1]. Finally, the limiting behaviour below the critical value is trivial and positivity follows from the definition of the critical value.

□

Annealed vs quenched convergence:

As mentioned in Section 2.2.3, when taking the limit as $T \to \infty$ (i.e the diffusive limit) there are two forms of convergence with which we are concerned:

(Q) *Quenched limit*: For almost all (i.e. typical) realisations of the underlying Poisson point process, with averaging over the random initial velocity of the particle.

(AQ) *Averaged-quenched* (a.k.a. *annealed*) *limit*: Averaging over the random initial velocity of the particle *and* the random placements of the scatterers.

Note that understanding the quenched limit is necessarily harder than the annealed limit as in the quenched limit one is averaging over a smaller state space. It is expected for the random Lorentz gas and wind-tree models that in the quenched setting, an invariance principle holds and the variance of the limiting Wiener process is deterministic (does not depend on the realisation of the Poisson process). As discussed in Section 2.2.3, there has been very little progress towards the resolution of either the annealed or the quenched problems. In this thesis we will be working in the *annealed* setting.

Wiener Process:

A one dimensional Wiener process on $\mathbb{R}$ (see for more information [Dur96]) (or standard Brownian motion) is a real valued stochastic process $t \mapsto W(t)$ satisfying:

(a) *Independent Increments:* If $t_1 < t_2 < \cdots < t_k$ then $W(t_1), W(t_2) - W(t_1), \ldots, W(t_k) - W(t_{k-1})$ are independent.

(b) *Gaussian Increments:* If $s, t \geq 0$ and A is a measurable set then

$$\mathbf{P}\left(B(s+t) - B(s) \in A\right) = \int_A (2\pi t)^{-1/2} \exp(-x^2/2t) dx. \tag{2.5.2}$$

That is, the increments are normally distributed with mean 0 and variance t.

(c) *Continuity:* $W(t)$ is almost surely continuous.

We say a Wiener process has variance σ if the variance of the normal distribution in (b) is σt (i.e this corresponds to $t \mapsto \sqrt{\sigma} W(t)$).

A Wiener process on $\mathbb{R}^d$ is a process $t \mapsto W(t)$ such that the projections onto each coordinate are independent one-dimensional Brownian motions. The variance is then given by a diagonal matrix with the variance of each coordinate along the diagonal.

Donsker's invariance principle:

Let $X_1, X_2, \ldots$ be i.i.d, mean 0 and variance 1 random variables. Let $S_n = X_1 + X_2 + \cdots + X_n$. Let

$$W_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0,1] \tag{2.5.3}$$

where $\lfloor \cdot \rfloor$ denote the nearest integer below the argument.

Theorem 2.5.2 (Donsker's invariance principle (see [Dur96, Section 7.6]))**.** *In the limit as $n \to \infty$, W_n converges in distribution to a one dimensional Wiener process with variance* 1.

In words, if one considers a random walk with i.i.d steps of mean 0 and variance 1 for very long times and zooms out with the appropriate scaling, then the resulting process is a standard Brownian motion.

Random Walk Estimate:

In what follows we will require a standard random walk estimate. However as we have not seen this written down elsewhere we give the proof here:

Let $\{v_i\}_{i \in \mathbb{N}} \subset S_1^{d-1}$ be i.i.d random velocities and $\xi_i \sim EXP(1)$ be an i.i.d sequence of flight times. We consider the random walk

$$Y_n := \sum_{i=1}^{n} \xi_i v_i \tag{2.5.4}$$

and define the occupation measures for a set $A \subset \mathbb{R}^d$

$$G(A) := \mathbf{E}\left(|\{1 \leq k < \infty : Y_k \in A\}|\right) \tag{2.5.5}$$

$$g(A) := \mathbf{P}\left(Y_1 \in A\right) \tag{2.5.6}$$

Proposition 2.5.3. *Let $d \geq 3$, then*

$$G(dx) \leq Cg(dx) + K(dx) \tag{2.5.7}$$

where $K(dx) := C\min\{|x|^{2-d}, 1\}dx$ *for some* $C < \infty$.

Proof. Since the individual steps of the walk are i.i.d, if we define $f_k(x)$ to be the density of the distribution of Y_k, the goal is to bound the sum

$$\sum_{i=1}^{\infty} f_k(x). \tag{2.5.8}$$

Consider the characteristic function

$$\psi(p) := \mathbf{E}\left(e^{ip\cdot Y_1}\right) = \mathbf{E}\left(e^{i(p\cdot v_1)\xi_1}\right) \tag{2.5.9}$$

then, by independence of the different steps and taking an inverse Fourier transform (see [Dur96, Section 2.3] for details of characteristic functions).

$$\sum_{k=1}^{\infty} f_k(x) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot p}\left(\sum_{k=1}^{\infty}\psi(p)^k\right)dp. \tag{2.5.10}$$

By Taylor expanding (2.5.9) and using $\mathbf{E}\left(\xi_1^k\right) = k!$,

$$\psi(p) = \mathbf{E}\left(\frac{1}{1+(p\cdot v_1)^2}\right). \tag{2.5.11}$$

Thus we aim to bound

$$\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty}\int_{\mathbb{R}^d} e^{-2\pi i x\cdot p}\mathbf{E}\left(\frac{1}{1+(p\cdot v_1)^2}\right)^k dp. \tag{2.5.12}$$

For $|x| \to 0$, the only contribution to the integral is for $|p| \to \infty$. Hence for $|x| \to 0$ (2.5.12) is dominated by the term $k = 1$, hence

$$G(dx) \le g(dx), \qquad \text{as } |x| \to 0. \tag{2.5.13}$$

Otherwise, using the exponential distribution of the fight times

$$\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{d} f_k(x) + \sum_{k=d+1}^{\infty} f_k(x) \le ce^{-Cx} + \sum_{k=d+1}^{\infty} f_k(x) \tag{2.5.14}$$

for some $c < \infty$, and $C > 0$ as $|x| \to \infty$. Now note, by (2.5.11)

$$\sum_{k=d+1}^{\infty}\psi(p) = \frac{\psi(p)^{d+1}}{1-\psi(p)} \le \frac{C}{|p|^2}, \qquad \text{as } |p| \to \infty. \tag{2.5.15}$$

Thus

$$\sum_{k=1}^{\infty} f_k(x) \leq ce^{-Cx} + C \int_{\mathbb{R}^d} e^{-2\pi i x \cdot p} \left|p\right|^{-2} dp \tag{2.5.16}$$

as $|x| \to \infty$. Finally since $|x| \to \infty$ we have that $|p| \leq |x|^{-1}$, hence

$$\sum_{k=1}^{\infty} f_k(x) \leq ce^{-Cx} + C \left|x\right|^{-(d-2)}, \qquad \text{as } |x| \to \infty. \tag{2.5.17}$$

Thus the proposition follows from (2.5.13) and (2.5.17). □

Chapter 3

Random Lorentz Gas

– Joint with Bálint Tóth –

3.1 Introduction

We consider the Lorentz gas with randomly placed spherical hard core scatterers in $\mathbb{R}^d$. That is, place spherical balls of radius r and infinite mass centred on the points of a Poisson point process of intensity ϱ in $\mathbb{R}^d$, where $r^d\varrho$ is sufficiently small so that with positive probability there is free passage out to infinity, and define $t \mapsto X^{r,\varrho}(t) \in \mathbb{R}^d$ to be the trajectory of a point particle starting with randomly oriented unit velocity, performing free flight in the complement of the scatterers and scattering elastically on them. As discussed in Chapter 2, a major problem in mathematical statistical physics is to understand the diffusive scaling limit of the particle trajectory

$$t \mapsto \frac{X^{r,\varrho}(Tt)}{\sqrt{T}}, \qquad \text{as} \qquad T \to \infty. \tag{3.1.1}$$

Indeed, the *Holy Grail* of this field of research would be to prove the invariance principle for the sequence of processes in (3.1.1) in either the quenched or annealed setting (see Chapter 2, Section 2.2.3).

Our main result (see Theorem 3.1.2 in Subsection 3.1.2) proves the invariance principle in the annealed setting if we take the *Boltzmann-Grad and diffusive limits simultaneously*. Thus while the diffusive limit (3.1.1) with fixed r and ϱ remains open, this is the first result proving convergence for infinite times in the setting of randomly placed scatterers, and hence it is a significant step towards the full resolution of the problem in the annealed setting.

3.1.1 The Boltzmann-Grad Limit

The Boltzmann-Grad limit (as introduced in Chapter 2, Section 2.1) is the following low (relative) density limit of the scatterer configuration:

$$r \to 0, \qquad \varrho \to \infty, \qquad \varrho r^{d-1} \to v_{d-1}, \tag{3.1.2}$$

where v_{d-1} is the area of the $(d-1)$-dimensional unit disc. In this limit the expected free path length between two successive collisions will be 1. Other choices of $\lim \varrho r^{d-1} \in (0, \infty)$ are equally legitimate and would change the limit only by a time (or space) scaling factor.

It is not difficult to see that in the averaged-quenched setting and under the Boltzmann-Grad limit (3.1.2) the distribution of the first free flight length starting at any deterministic time, converges to an $EXP(1)$ and the jump in velocity after the free flight happens in a Markovian way with transition kernel

$$\mathbf{P}\left(v_{\mathtt{out}} \in dv' \;\middle|\; v_{\mathtt{in}} = v\right) = \sigma(v, v')dv', \tag{3.1.3}$$

where dv' is the surface element on S_1^{d-1} and $\sigma : S_1^{d-1} \times S_1^{d-1} \to \mathbb{R}_+$ is the normalised *differential cross section* of a spherical hard core scatterer, computable as

$$\sigma(v, v') = \frac{1}{4v_{d-1}} \left|v - v'\right|^{3-d}. \tag{3.1.4}$$

Note that in 3-dimensions the transition probability (3.1.3) of velocity jumps is uniform. That is, the outgoing velocity $v_{\mathtt{out}}$ is uniformly distributed on S_1^2, independently of the incoming velocity $v_{\mathtt{in}}$.

It is intuitively compelling (but far from easy to prove) that under the Boltzmann-Grad limit (3.1.2)

$$\left\{t \mapsto X^{r,\varrho}(t)\right\} \Rightarrow \left\{t \mapsto Y(t)\right\}, \tag{3.1.5}$$

where the symbol $\Rightarrow$ stands for weak convergence (of probability measures) on the space of continuous trajectories in $\mathbb{R}^d$, see [Bil68]. The process $t \mapsto Y(t)$ on the right hand side is the Markovian random flight process consisting of independent free flights of $EXP(1)$-distributed length, with Markovian velocity changes according to the scattering transition kernel (3.1.3). A formal construction of the process $t \mapsto Y(t)$ is given in Section 3.2.1. The limit (3.1.5), valid in any compact time interval $t \in [0, T]$, $T < \infty$, is rigourously established in the averaged-quenched setting in [Gal69, Gal70, Gal99, Spo78, Spo88b], and in the quenched setting in [BBS83]. In [Spo78] more general point processes of the scatterer positions, with sufficiently strong mixing properties are considered.

The limiting Markovian flight process $t \mapsto Y(t)$ is a continuous time random walk. Therefore, by taking a second, diffusive limit *after* the Boltzmann-Grad limit (3.1.5), Donsker's theorem (see [Bil68]) yields indeed the invariance principle,

$$\left\{t \mapsto T^{-1/2}Y(Tt)\right\} \Rightarrow \left\{t \mapsto W(t)\right\}, \tag{3.1.6}$$

as $T \to \infty$, where $t \mapsto W(t)$ is the isotropic Wiener process in $\mathbb{R}^d$ of non-degenerate variance. The variance of the limiting Wiener process W can be explicitly computed but its concrete value has no importance.

The natural question arises whether one could somehow interpolate between the double limit of taking first the Boltzmann-Grad limit (3.1.5) and then the diffusive limit (3.1.6) and the plain diffusive limit for the Lorentz process, (3.1.1). Our main result, Theorem 3.1.2 formulated in Section 3.1.2 gives a positive partial answer in dimension 3. Since our results are proved in three-dimensions from now on we formulate all statements in $d = 3$ rather than in a general dimension.

3.1.2 Main Result

In the rest of the chapter we assume $\varrho = \varrho(r) = \pi r^{-2}$ and drop the superscript ϱ from the notation of the Lorentz process.

Our results (Theorems 3.1.1 and 3.1.2 formulated below) refer to a coupling – joint realisation on the same probability space – of the Markovian random flight process $t \mapsto Y(t)$, and the quenched-averaged (annealed) Lorentz process $t \mapsto X^r(t)$. The coupling is informally described later in this section and constructed with full formal rigour in Section 3.2.2.

The first theorem states that in our coupling, up to time $T \ll r^{-1}$, the Markovian flight and Lorentz exploration processes stay together.

Theorem 3.1.1. *Let $T = T(r)$ be such that $\lim_{r\to 0} T(r) = \infty$ and $\lim_{r\to 0} rT(r) = 0$. Then*

$$\lim_{r\to 0} \mathbf{P}\left(\inf\{t : X^r(t) \neq Y(t)\} \leq T\right) = 0. \tag{3.1.7}$$

Although, this result is subsumed by our main result, it shows the strength of the coupling method employed in this chapter. In particular, with some elementary arguments it provides a much stronger result than [Gal69, Gal70, Gal99, Spo78] discussed in Chapter 2, Subsection 2.2.3. On the other hand the proof of this "naïve" result sheds some light on the structure of proof of the more sophisticated Theorem 3.1.2, which is our main result.

Theorem 3.1.2. *Let $T = T(r)$ be such that $\lim_{r\to 0} T(r) = \infty$ and $\lim_{r\to 0} r^2 |\log r|^2 T(r) = 0$. Then, for any $\delta > 0$,*

$$\lim_{r\to 0} \mathbf{P}\left(\sup_{0\leq t\leq T} |X^r(t) - Y(t)| > \delta\sqrt{T}\right) = 0, \tag{3.1.8}$$

and hence

$$\left\{t \mapsto T^{-1/2} X^r(Tt)\right\} \Rightarrow \left\{t \mapsto W(t)\right\}, \tag{3.1.9}$$

as $r \to 0$, in the averaged-quenched sense. On the right hand side of (3.1.9) W is a standard Wiener process of variance 1 in $\mathbb{R}^3$.

Indeed, the invariance principle (3.1.9) readily follows from the invariance principle for the Markovian flight process, (3.1.6), and the closeness of the two processes quantified in (3.1.8). So, it remains to prove (3.1.8). This will be the content of Sections 3.4-3.7.

The point of Theorem 3.1.2 is that the Boltzmann-Grad limit of scatterer configuration (3.1.2) and the diffusive scaling of the trajectory are done simultaneously, and not consecutively. The memory effects due to recollisions and shading are controlled up to the time scale $T = T(r) = o(r^{-2} |\log r|^{-2})$.

Remarks on dimension:

1. Our proof is not valid in 2-dimensions for two different reasons:
   (a) Probabilistic estimates at the core of the proof are valid only in the transient dimensions of random walk, $d \geq 3$.
   (b) A subtle geometric argument which will show up in Sections 3.6.4-3.6.6 below, is valid only in $d \geq 3$, as well. This is unrelated to the recurrence/transience dichotomy and it is crucial in controlling the short range recollision and shading events in the Boltzmann-Grad limit (3.1.2).
2. The fact that in $d = 3$ the differential cross section of hard spherical scatterers is uniform on S_1^2 (see (3.1.3), (3.1.4)) facilitates our arguments, since, in this case, the successive velocities of the random flight process $Y(t)$ form an i.i.d. sequence. However, this is not of crucial importance. The same proofs could also be carried out for other differential cross sections, at the expense of more extensive arguments. We are not going to these generalisations here. Therefore the proofs presented in this chapter are valid *exactly in $d = 3$*.

Remark on time scales: Recall from Chapter 2, Section 2.2.4, that, in the weak coupling limit, similar results to Theorem 3.1.2 have been proved. In order to compare our time scale with the existing results on weak coupling diffusive limits ([KR06, ESY08, ESY07]), we define the *kinetic time scale* for our problem:

$$T_{\text{kin}} := \varrho^{1/d} = r^{-(d-1)/d}. \tag{3.1.10}$$

The previous results [Gal69, Gal70, Gal99, Spo78, BBS83, KP80, EY00, EE05], (discussed in Chapter 2, Section 2.2.3 and 2.2.4) when viewed as the scaling limit for a *microscopic* trajectory, hold on space-time scales of order T_{kin}. Thus, this time scale is the reference to which the time scale for the diffusive limit should be compared. In terms of this microscopic time *our* diffusive limit holds for time scales up to

$$T_{\text{diff}} = T_{\text{kin}} T \tag{3.1.11}$$

with

$$T = o\left(T_{\text{kin}}^{2d/(d-1)} (\log T_{\text{kin}})^{-2}\right) = o\left(T_{\text{kin}}^{3} (\log T_{\text{kin}})^{-2}\right). \tag{3.1.12}$$

The similar-in-spirit, 'infinite time', weak coupling results [KR06] and [ESY08, ESY07] should be compared to (3.1.12) (however we stress that our result is *not* in the weak coupling limit since the interactions with scatterers are not scaled).

The proof of Theorems 3.1.1 and 3.1.2 will be based on a coupling (that is: a joint realisation on the same probability space) of the Markovian flight process $t \mapsto Y(t)$ and the averaged-quenched realisation of the Lorentz process $t \mapsto X^r(t)$, such that the maximum distance of their positions up to time T be small order of $\sqrt{T}$. The Lorentz process $t \mapsto X^r(t)$ is realised as an *exploration* of the environment of scatterers. That is, as time goes on, more and more information is revealed about the position of the scatterers. As long as $X^r(t)$ traverses yet unexplored territories, it behaves just like the Markovian flight process $Y(t)$, discovering new, yet-unseen scatterers with rate 1 and scattering on them. However, unlike the Markovian flight process it has long memory, the discovered scatterers are placed forever and if the process $X^r(t)$ returns to these positions, recollisions occur. Likewise, the area swept in the past by the Lorentz exploration process $X^r(t)$ – that is: a tube of radius r around its past trajectory – is recorded as a domain where new collisions can not occur. For a formal definition of the coupling see Section 3.2.2. Let their velocity processes be $U(t) := \dot{Y}(t)$ and $V^r(t) := \dot{X}^r(t)$. These are almost surely piecewise constant jump processes. The coupling is realized in such a way, that

A) At the very beginning the two velocities coincide, $V^r(0) = U(0)$.

B) Occasionally, with typical frequency of order r mismatches of the two velocity processes occur. These mismatches are caused by two possible effects:

   ◦ *Recollisions* of the Lorentz exploration process with a scatterer placed in the past. This causes a collision event when $V^r(t)$ changes while $U(t)$ does not.

   ◦ Scatterings of the Markovian flight process $Y(t)$ in a moment when the Lorentz exploration process is in the explored tube, where it can not encounter a not-yet-seen new scatterer. In these moments the process $U(t)$ has a jump discontinuity, while the process $V^r(t)$ stays unchanged. We will call these events *shadowed scatterings* of the Markovian flight process.

C) However, shortly after the mismatch events described in item B) above, a new jointly realised scattering event of the two processes occurs, recoupling the two velocity processes to identical values. These recouplings occur typically at an $EXP(1)$-distributed time after the mismatches.

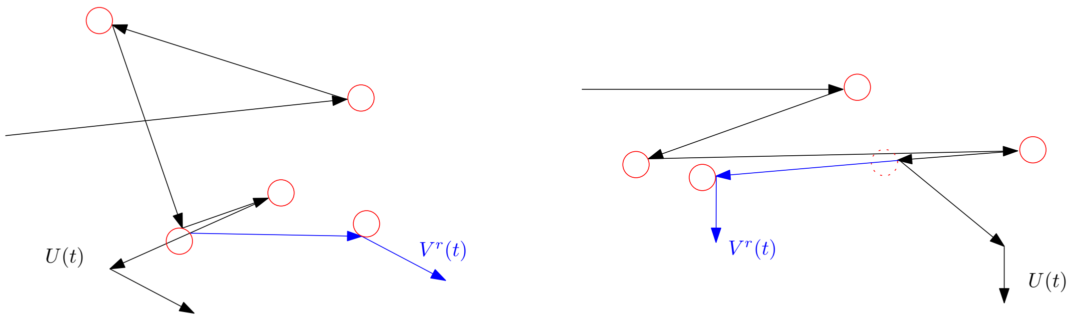

Figure 3.1: The above image shows a recollision (left) and a shadowing event (right). Note that after each event U and V^r are no longer coupled. However at the next scattering, if possible, the velocities are recoupled. On the right hand side the virtual scatterer drawn in dotted line is shadowed. That is: it is physically not present in the mechanical trajectory.

Summarising: The coupled velocity processes $t \mapsto (U(t), V^r(t))$ are realised in such a way that they assume the same values except for typical time intervals of length of order 1, separated by typical intervals of lengths of order r^{-1}. Other, more complicated mismatches of the two processes occur only at time scales of order $r^{-2} |\log r|^{-2}$. If all these are controlled (this will be the content of the proof) then the following hold:

Up to $T = T(r) = o(r^{-1})$, with high probability there is no mismatch whatsoever between $U(t)$ and $V^r(t)$. That is,

$$\lim_{r\to 0} \mathbf{P}\left(\inf\{t : V^r(t) \neq U(t)\} < T\right) = \lim_{r\to 0} \mathbf{P}\left(\inf\{t : X^r(t) \neq Y(t)\} < T\right) = 0. \tag{3.1.13}$$

In particular, the invariance principle (3.1.9) also follows, with $T = T(r) = o(r^{-1})$, rather than $T = T(r) = o(r^{-2} |\log r|^{-2})$. As a by-product of this argument a new and handier proof of the theorem (3.1.5) of [Gal69, Gal70, Gal99, Spo78, Spo88b] also drops out.

Going up to $T = T(r) = o(r^{-2} |\log r|^{-2})$ needs more argument. The ideas described in the outline A), B), C) above lead to the following chain of bounds:

$$\begin{aligned}\max_{0\le t\le 1}\left|\frac{X^r(Tt)}{\sqrt{T}} - \frac{Y(Tt)}{\sqrt{T}}\right| &= \frac{1}{\sqrt{T}} \max_{0\le t\le 1}\left|\int_0^{Tt} (V^r(s) - U(s))\, ds\right| \\ &\le \frac{1}{\sqrt{T}} \int_0^T |V^r(s) - U(s)|\, ds \asymp \frac{1}{\sqrt{T}} Tr = \sqrt{T} r.\end{aligned}$$

In the $\asymp$ step we use the arguments B) and C). Finally, choosing in the end $T = T(r) = o(r^{-2})$ we obtain a tightly close coupling of the diffusively scaled processes $t \mapsto X^r(Tt)/\sqrt{T}$ and $t \mapsto Y(Tt)/\sqrt{T}$, (3.1.8), and hence the invariance principle (3.1.9), for this longer time scale. This hand-waving argument should, however, be taken with a grain of salt: it does not show the logarithmic factor, which arises in the fine-tuning.

3.1.3 Structure of the Chapter

The rest of the chapter is devoted to the rigourous statement and proof of the arguments described in A), B), C) above. The overall structure is as follows:

- *Section 3.2:* We construct the Markovian flight and Lorentz exploration processes and thus lay out the coupling argument which is essential moving forward. Moreover, we will also introduce an auxiliary process, Z, a short-sighted or forgetful version of X which somehow interpolates between the processes Y and X.

– *Section 3.3:* We prove Theorem 3.1.1. We go through the proof of this statement as it is both informative for the dynamics, and the proof of Theorem 3.1.2 in its full strength will follow similar lines, however with substantial differences.

Sections 3.4-3.7 are fully devoted to the proof of Theorem 3.1.2, as follows:

– *Section 3.4:* We break up the process Z into independent legs of exponentially tight lengths. From here we state two propositions which are central to the proof. They state that
(i) with high probability the process X does not differ from Z in each leg;
(ii) with high probability, the different legs of the process Z do not interact (up to times of our time scales).

– *Section 3.5:* We prove the proposition concerning interactions between legs.

– *Section 3.6:* We prove the proposition concerning coincidence, with high probability, of the processes X and Z within a single leg. This section is longer than the others, due to the subtle geometric arguments and estimates needed in this proof.

– *Section 3.7:* We finish off the proof of Theorem 3.1.2.

3.2 Construction

3.2.1 Ingredients and the Markovian Flight Process

Let $\xi_j \in \mathbb{R}_+$ and $u_j \in \mathbb{R}^3$, $j = -2, -1, 0, 1, 2, \dots$, be completely independent random variables (defined on an unspecified probability space $(\Omega, \mathcal{F}, \mathbf{P})$) with distributions:

$$\xi_j \sim EXP(1), \qquad u_j \sim UNI(S^2), \tag{3.2.1}$$

and let

$$y_j := \xi_j u_j \in \mathbb{R}^3. \tag{3.2.2}$$

For later use we also introduce the sequence of indicators

$$\epsilon_j := \mathbb{1}\{\xi_j < 1\}, \tag{3.2.3}$$

and the corresponding conditional exponential distributions $EXP(1|1) := \text{distrib}(\xi \,|\, \epsilon = 1)$, respectively, $EXP(1|0) = \text{distrib}(\xi \,|\, \epsilon = 0)$, with distribution densities

$$(e-1)^{-1} e^{1-x} \mathbb{1}\{0 \le x < 1\}, \qquad \text{respectively,} \qquad e^{1-x} \mathbb{1}\{1 \le x < \infty\}.$$

We will also use the notation $\underline{\epsilon} := (\epsilon_j)_{j \ge 0}$ and call the sequence $\underline{\epsilon}$ the *signature* of the i.i.d. $EXP(1)$-sequence $(\xi_j)_{j \ge 0}$.

The variables ξ_j and u_j will be, respectively, the consecutive *flight length/flight times* and *flight velocities* of the Markovian flight process $t \mapsto Y(t) \in \mathbb{R}^3$ defined below.

Denote, for $n \in \mathbb{Z}_+$, $t \in \mathbb{R}_+$,

$$\tau_n := \sum_{j=1}^{n} \xi_j, \qquad \nu_t := \max\{n : \tau_n \leq t\}, \qquad \{t\} := t - \tau_{\nu_t}. \tag{3.2.4}$$

That is: τ_n denotes the consecutive scattering times of the flight process, ν_t is the number of scattering events of the flight process Y occurring in the time interval $(0, t]$, and $\{t\}$ is the length of the last free flight before time t.

Finally let

$$Y_n := \sum_{j=1}^{n} \xi_j u_j = \sum_{j=1}^{n} y_j, \qquad Y(t) := Y_{\nu_t} + \{t\} u_{\nu_t + 1}.$$

We shall refer to the process $t \mapsto Y(t)$ as the Markovian flight process. This will be our fundamental probabilistic object. All variables and processes will be defined in terms of this process, and adapted to the natural continuous time filtration $(\mathcal{F}_t)_{t\geq 0}$ of the flight process:

$$\mathcal{F}_t := \sigma(u_0, (Y(s))_{0\leq s\leq t}).$$

Note that the processes $n \mapsto Y_n$, $t \mapsto Y(t)$ and their respective natural filtrations $(\mathcal{F}_n)_{n\geq 0}$, $(\mathcal{F}_t)_{t\geq 0}$, do not depend on the parameter r.

We also define, for later use, the *virtual scatterers* of the flight process $t \mapsto Y(t)$. For $n \geq 0$, let

$$\begin{aligned} Y_k' &:= Y_k + r\frac{u_k - u_{k+1}}{|u_n - u_{k+1}|} = Y_k + r\frac{\dot{Y}(\tau_k^-) - \dot{Y}(\tau_k^+)}{\left|\dot{Y}(\tau_k^-) - \dot{Y}(\tau_k^+)\right|}, && k \geq 0, \\ \mathcal{S}_n^Y &:= \{Y_k' \in \mathbb{R}^3 : 0 \leq k \leq n\}, && n \geq 0. \end{aligned}$$

Here and throughout the chapter we use the notation $f(t^{\pm}) := \lim_{\varepsilon\downarrow 0} f(t \pm \varepsilon)$.
The points $Y_n' \in \mathbb{R}^3$ are the centres of virtual spherical scatterers of radius r which *would have caused* the nth scattering event of the flight process. They do not have any influence on the further trajectory of the flight process Y, but will play role in the forthcoming couplings.

3.2.2 The Lorentz Exploration Process

Let $r > 0$, and $\varrho = \varrho(r) = \pi r^{-2}$. We define the Lorentz exploration process $t \to X(t) = X^r(t) \in \mathbb{R}^3$, coupled with the flight process $t \mapsto Y(t)$, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The process $t \mapsto X(t)$ and all upcoming random variables related to it *do depend* on the choice of the parameter r (and ϱ), but from now on we will suppress explicit notation of dependence upon these parameters.

The construction goes inductively, on the successive time intervals $[\tau_{n-1}, \tau_n)$, $n = 1, 2, \ldots$. Start with Step 1: and then iterate indefinitely Step 2: and Step 3: below.

Step 1: Start with

$$X(0) = X_0 = 0, \qquad V(0^+) = u_1, \qquad X_0' := r\frac{u_0 - u_1}{|u_0 - u_1|} \qquad \mathcal{S}_0^X = \{X_0'\}.$$

Note that the trajectory of the exploration process X begins with a collision at time $t = 0$. This is not exactly as described previously but is of no consequence and aids the later exposition.

`Go to` Step 2:.

Step 2: This step starts with given $X(\tau_{n-1}) = X_{n-1} \in \mathbb{R}^3$, $V(\tau_{n-1}^+) \in S_1^2$ and $\mathcal{S}_{n-1}^X = \{X_k' : 0 \leq k \leq n-1\} \subset \mathbb{R}^3 \cup \{\bigstar\}$, where

- $\bigstar$ is a fictitious point at infinity, with $\inf_{x\in\mathbb{R}^3} |x - \bigstar| = \infty$, introduced for bookkeeping reasons;
- $|X_{n-1} - X_k'| \in (r, \infty]$ for $0 \leq k < n-1$, and $\left|X_{n-1} - X_{n-1}'\right| \in \{r, \infty\}$.

The trajectory $t \mapsto X(t)$, $t \in [\tau_{n-1}, \tau_n)$, is defined as free motion with elastic collisions on fixed spherical scatterers of radius r centred at the points in $\mathcal{S}_{n-1}^X$. At the end of this time interval the position and velocity of the Lorentz exploration process are $X(\tau_n) =: X_n$, respectively, $V(\tau_n^-)$.

`Go to Step 3:.`

Step 3: Let

$$X_n'' := X_n + r\frac{V(\tau_n^-) - u_{n+1}}{\left|V(\tau_n^-) - u_{n+1}\right|}, \qquad d_n := \min_{0\leq s<\tau_n} |X(s) - X_n''|.$$

Note that $d_n \leq r$.

- If $d_n < r$ then let $X_n' := \bigstar$, and $V(\tau_n^+) = V(\tau_n^-)$.
- If $d_n = r$ then let $X_n' := X_n''$,and $V(\tau_n^+) = u_{n+1}$.

Set $\mathcal{S}_n^X = \mathcal{S}_{n-1}^X \cup \{X_n'\}$.

`Go back to Step 2:.`

The process $t \mapsto X(t)$ is indeed adapted to the filtration $(\mathcal{F}_t)_{0\leq t<\infty}$ and indeed has the averaged-quenched distribution of the Lorentz process. This follows from the fact that the scatterers of the Lorentz process are centred on a Poisson point process and thus when sweeping not-yet-seen areas no information from the past interferes.

Our notation is fully consistent with the one used for the Markovian process Y: $X_n := X(\tau_n)$ and

$$X_k' := \begin{cases} X_k + r\dfrac{\dot{X}(\tau_k^-) - \dot{X}(\tau_k^+)}{\left|\dot{X}(\tau_k^-) - \dot{X}(\tau_k^+)\right|} & \text{if} \quad \dot{X}(\tau_k^-) \neq \dot{X}(\tau_k^+), \\ \bigstar & \text{if} \quad \dot{X}(\tau_k^-) = \dot{X}(\tau_k^+), \end{cases} \qquad k \geq 0,$$

$$\mathcal{S}_n^X := \{X_k' \in \mathbb{R}^3 : 0 \leq k \leq n\}, \qquad n \geq 0.$$

3.2.3 Mechanical Consistency and Compatibility of Piece-wise Linear Trajectories in $\mathbb{R}^3$

The key notion in the exploration construction of section 3.2.2 was mechanical r-consistency, and r-compatibility of finite segments of piece-wise linear trajectories in $\mathbb{R}^3$, which we formalise now for later reference.

Let

$$n \in \mathbb{N}, \qquad \tau_0 \in \mathbb{R}, \qquad Z_0 \in \mathbb{R}^3, \qquad v_0, \dots, v_{n+1} \in S^2 \qquad t_1, \dots, t_n \in \mathbb{R}_+,$$

be given and define for $j = 0, \dots, n$,

$$\tau_j := \tau_0 + \sum_{k=1}^{j} t_k, \qquad Z_j := Z_0 + \sum_{k=1}^{j} t_k v_k, \qquad Z_j' := \begin{cases} Z_j + r\dfrac{v_j - v_{j+1}}{|v_j - v_{j+1}|} & \text{if } v_j \neq v_{j+1}, \\ \bigstar & \text{if } v_j = v_{j+1}, \end{cases}$$

and for $t \in [\tau_j, \tau_{j+1}]$, $j = 0, \dots, n$,

$$Z(t) := Z_j + (t - \tau_j) v_{j+1}.$$

We call the piece-wise linear trajectory $\big(Z(t) : \tau_0^- < t < \tau_n^+\big)$ mechanically *r-consistent* or *r-inconsistent*, if

$$\min_{\tau_0 \le t \le \tau_n} \min_{0 \le j \le n} \left| Z(t) - Z'_j \right| = r, \qquad \text{respectively,} \qquad \min_{\tau_0 \le t \le \tau_n} \min_{0 \le j \le n} \left| Z(t) - Z'_j \right| < r \tag{3.2.5}$$

Note, that by formal definition the minimum distance on the left hand side can not be strictly larger than r.

Given two finite pieces of mechanically r-consistent trajectories $\big(Z_a(t) : \tau_{a,0}^- < t < \tau_{a,n_a}^+\big)$ and $\big(Z_b(t) : \tau_{b,0}^- < t < \tau_{b,n_b}^+\big)$, defined over non-overlapping time intervals: $[\tau_{a,0}, \tau_{a,n_a}] \cap [\tau_{b,0}, \tau_{b,n_b}] = \emptyset$, with $\tau_{a,n_a} \le \tau_{b,0}$, we will call them mechanically *r-compatible* or *r-incompatible* if

$$\begin{aligned} &\min\{\min_{\tau_{a,0} \le t \le \tau_{a,n_a}} \min_{0 < j \le n_b} \left| Z_a(t) - Z'_{b,j} \right|, \min_{\tau_{b,0} \le t \le \tau_{b,n_b}} \min_{0 \le j < n_a} \left| Z_b(t) - Z'_{a,j} \right|\} \ge r, \\ &\min\{\min_{\tau_{a,0} \le t \le \tau_{a,n_a}} \min_{0 < j \le n_b} \left| Z_a(t) - Z'_{b,j} \right|, \min_{\tau_{b,0} \le t \le \tau_{b,n_b}} \min_{0 \le j < n_a} \left| Z_b(t) - Z'_{a,j} \right|\} < r, \end{aligned} \tag{3.2.6}$$

respectively.

Given a mechanically r-consistent trajectory, any non-overlapping parts of it are pairwise mechanically r-compatible, and given a finite number of non-overlapping mechanically r-consistent pieces of trajectories which are also pair-wise mechanically r-compatible their concatenation (in the most natural way) is mechanically r-consistent.

3.2.4 An Auxiliary Process

It will be convenient to introduce a third, auxiliary process $t \mapsto Z(t) \in \mathbb{R}^3$, and consider the joint realisation of all three processes $t \mapsto (Y(t), X(t), Z(t))$ on the same probability space. This construction will not be needed until Section 3.4.

The process $t \mapsto Z(t)$ will be a *forgetful* (or short-sighted) version of the true physical process $t \mapsto X(t)$ in the sense that in its construction, only memory effects by the last seen scatterers are taken into account. That is: only direct recollisions with the last seen scatterer and shadowings by the last straight flight segment are incorporated, disregarding more complex memory effects. It will be shown that
(a) up to times $T = T(r) = o(r^{-2} |\log r|^{-2})$ the trajectories of the forgetful process $Z(t)$ and the true physical process $X(t)$ coincide, and
(b) the forgetful process $Z(t)$ and the Markovian process $Y(t)$ stay sufficiently close together with probability tending to 1 (as $r \to 0$). Thus, the invariance principle (3.1.6) can be transferred to the true physical process $X(t)$, thus yielding the invariance principle (3.1.9).

Define the following indicator variables:

$$\begin{aligned} \widehat{\eta}_j &= \widehat{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{ |y_{j-1}| < 1 \text{ and } \min_{0 \le t \le \xi_{j-2}} \left| y_{j-1} + r \frac{u_{j-1} - u_j}{|u_{j-1} - u_j|} + t u_{j-2} \right| < r \right\}, \\ \widetilde{\eta}_j &= \widetilde{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{ |y_{j-1}| < 1 \text{ and } \min_{0 \le t \le \xi_j} \left| y_{j-1} + r \frac{u_{j-1} - u_{j-2}}{|u_{j-1} - u_{j-2}|} + t u_j \right| < r \right\}, \\ \eta_j &:= \max\{\widehat{\eta}_j, \widetilde{\eta}_j\}. \end{aligned} \tag{3.2.7}$$

Before constructing the auxiliary process $t \mapsto Z(t)$ we prove the following

Lemma 3.2.1. *There exists a constant $C < \infty$ such that for any sequence of signatures $\underline{\epsilon} = (\epsilon_j)_{j\geq 1}$ the following bounds hold*

$$\mathbf{E}\left(\eta_j \mid \underline{\epsilon}\right) \leq Cr, \tag{3.2.8}$$

$$\mathbf{E}\left(\eta_j\eta_k \mid \underline{\epsilon}\right) \leq \begin{cases} Cr^2 \left|\log r\right| & \textit{if } |j-k| = 1, \\ Cr^2 & \textit{if } |j-k| > 1. \end{cases} \tag{3.2.9}$$

Proof of Lemma 3.2.1. Define the following auxiliary, and simpler, indicators:

$$\widehat{\eta}'_j := \mathbb{1}\left\{\angle(-u_{j-1}, u_{j-2}) < \frac{2r}{\xi_{j-1}}\right\}, \qquad \widetilde{\eta}'_j := \mathbb{1}\left\{\angle(-u_{j-1}, u_j) < \frac{2r}{\xi_{j-1}}\right\}.$$

Here, and in the rest of the chapter we use the notation

$$\angle : S_1^2 \times S_1^2 \to [0,\pi], \qquad \angle(u,v) := \arccos(u\cdot v).$$

Then, clearly,

$$\widetilde{\eta}_j \leq \widetilde{\eta}'_j, \qquad \widehat{\eta}_j \leq \widehat{\eta}'_j.$$

It is straightforward that the indicators $\left(\widehat{\eta}'_j : 1 \leq j < \infty\right)$, and likewise, the indicators $\left(\widetilde{\eta}'_j : 1 \leq j < \infty\right)$, are independent among themselves and one-dependent across the two sequences. This holds even if conditioned on the sequence of signatures $\underline{\epsilon}$.

Therefore, the following simple computations prove Lemma 3.2.1

$$\begin{aligned}
\mathbf{E}\left(\widehat{\eta}'_j \mid \underline{\epsilon}\right) &\leq \int_0^\infty e^{-y}\mathbf{P}\left(\angle(-u_{j-1}, u_{j-2}) < \frac{2r}{y}\right)dy \\
&\leq Cr^2 \int_0^\infty e^{-y}\min\{y^{-2}, r^{-2}\}dy \leq Cr \\
\mathbf{E}\left(\widetilde{\eta}'_j \mid \underline{\epsilon}\right) &\leq Cr^2 \int_0^\infty e^{-y}\min\{y^{-2}, r^{-2}\}dy \leq Cr, \\
\mathbf{E}\left(\widehat{\eta}'_{j+1}\widetilde{\eta}'_j \mid \underline{\epsilon}\right) &\leq Cr^2 \int_0^\infty \int_0^\infty e^{-y}e^{-z}\min\{y^{-2}, z^{-2}, r^{-2}\}dydz \leq Cr^2\left|\log r\right|.
\end{aligned}$$

We omit the elementary computational details. □

Lemma 3.2.1 assures that, as $r \to 0$, with probability tending to 1, up to time of order $T = T(r) = o(r^{-2}\left|\log r\right|^{-1})$ it will not occur that two neighbouring or next-neighbouring η-s happen to take the value 1 which would obscure the following construction.

The process $t \mapsto Z(t)$ is constructed on the successive intervals $[\tau_{j-1}, \tau_j)$, $j = 1, 2, \ldots,$ as follows:

- `(No interference with the past.)`
  If $\eta_j = 0$ then for $\tau_{j-1} \leq t \leq \tau_j$, $Z(t) = Z(\tau_{j-1}) + \{t\}u_j$.
- `(Direct shadowing.)`
  If $\widehat{\eta}_j = 1$, then for $\tau_{j-1} \leq t \leq \tau_j$, $Z(t) = Z(\tau_{j-1}) + \{t\}u_{j-1}$.
- `(Direct recollision with the last seen scatterer.)`
  If $\widehat{\eta}_j = 0$ and $\widetilde{\eta}_j = 1$ then, in the time interval $\tau_{j-1} \leq t \leq \tau_j$ the trajectory $t \mapsto Z(t)$ is defined as that of a mechanical particle starting with initial position $Z(\tau_{j-1})$, initial velocity $\dot{Z}(\tau_{j-1}^+) = u_j$

and colliding elastically with two infinite-mass spherical scatterers of radius r centred at the points

$$Z(\tau_{j-1}) + r\frac{u_{j-1}-u_j}{|u_{j-1}-u_j|}, \qquad \text{respectively} \qquad Z(\tau_{j-2}) - r\frac{u_{j-1}-u_{j-2}}{|u_{j-1}-u_{j-2}|}.$$

Consistently with the notations adopted for the processes $Y(t)$ and $X(t)$, we denote $Z_k := Z(\tau_k)$ for $k \geq 0$.

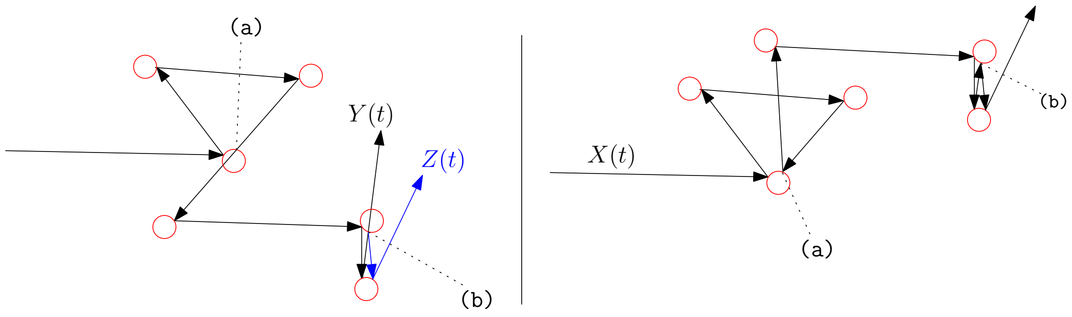

Figure 3.2: The above image shows a section of trajectory during which X, Y, and Z would all three differ. On the left we see Y and Z remain together until point (b), where a direct recollision is respected by Z. Note that Z ignores the mismatch at (a) as it is indirect. On the right, the process X is coupled to Y on the left. Note that X respects the indirect recollision at point (a) and the direct recollision at (b).

3.3 No Mismatches up to $T = o(r^{-1})$: Proof of Theorem 3.1.1

In this section we prove that the Markovian flight trajectory $Y(t)$, up to time scales of order $T = T(r) = o(r^{-1})$, is mechanically r-consistent with probability $1 - o(1)$, and therefore the coupling bound of Theorem 3.1.1 holds. On the way we establish various bounds to be used in later sections. This section uses only classical probabilistic tools. Moreover, presenting the proof in full will prepare the ideas (and notation) for Section 3.5 where a similar argument is exploited in a more complex form.

3.3.1 Interferences

Let $t \to Y(t)$ and $t \to Y^*(t)$ be two independent Markovian flight processes. Think about $Y(t)$ as running forward and $Y^*(t)$ as running backwards in time. (Note, that the Markovian flight process has invariant law under time reversal.) Define the following events

$$\begin{aligned}
\widehat{W}_j &:= \{\min\{|Y(t) - Y'_j| : 0 < t < \tau_{j-1}\} < r\}, \\
\widetilde{W}_j &:= \{\min\{|Y'_k - Y(t)| : 0 \leq k < j-1, \quad \tau_{j-1} < t < \tau_j\} < r\}, \\
\widehat{W}^*_j &:= \{\min\{|Y^*(t) - Y'_1| : 0 < t < \tau_{j-1}\} < r\}, \\
\widetilde{W}^{*\prime}_j &:= \{\min\{|Y^{*\prime}_k - Y(t)| : 0 < k \leq j-1, \quad 0 < t < \tau_1\} < r\}, \\
\widehat{W}^*_\infty &:= \{\min\{|Y^*(t) - Y'_1| : 0 < t < \infty\} < r\}, \\
\widetilde{W}^*_\infty &:= \{\min\{|Y^{*\prime}_k - Y(t)| : 0 < k < \infty, \quad 0 < t < \tau_1\} < r\},
\end{aligned}$$

In words $\widehat{W}_j$ is the event that the virtual collision at Y_j is *shadowed* by the past path. While $\widetilde{W}_j$ is the event that in the time interval (τ_{j-1}, τ_j) there is a *virtual recollision* with a past scatterer.

It is obvious that

$$
\begin{aligned}
\mathbf{P}\left(\widehat{W}_j\right) = \mathbf{P}\left(\widehat{W}_j^*\right) \le \mathbf{P}\left(\widehat{W}_{j+1}^*\right) \le \mathbf{P}\left(\widehat{W}_\infty^*\right), \\
\mathbf{P}\left(\widetilde{W}_j\right) = \mathbf{P}\left(\widetilde{W}_j^*\right) \le \mathbf{P}\left(\widetilde{W}_{j+1}^*\right) \le \mathbf{P}\left(\widetilde{W}_\infty^*\right).
\end{aligned}
\tag{3.3.1}
$$

On the other hand, by union bound and independence

$$
\begin{aligned}
\mathbf{P}\left(\widehat{W}_\infty^*\right) &\le \sum_{z\in\mathbb{Z}^3} \mathbf{P}\left(\{1<k<\infty : Y_k^* \in B_{zr,2r}\} \neq \emptyset\right) \mathbf{P}\left(\{0<t\le\xi : Y(t)\in B_{zr,2r}\}\neq\emptyset\right) \\
&\le \sum_{z\in\mathbb{Z}^3} (2r)^{-1} \mathbf{E}\left(|\{1<k<\infty : Y_k^*\in B_{zr,2r}\}|\right) \mathbf{E}\left(|\{0<t\le\xi : Y(t)\in B_{zr,3r}\}|\right) \\
\mathbf{P}\left(\widetilde{W}_\infty^*\right) &\le \sum_{z\in\mathbb{Z}^3} \mathbf{P}\left(\{0<t<\infty : Y^*(t)\in B_{zr,2r}\}\neq\emptyset\right) \mathbf{P}\left(Y_1\in B_{zr,2r}\right) \\
&\le \sum_{z\in\mathbb{Z}^3} (2r)^{-1} \mathbf{E}\left(|\{0<t<\infty : Y^*(t)\in B_{zr,3r}\}|\right) \mathbf{P}\left(Y_1\in B_{zr,2r}\right)
\end{aligned}
\tag{3.3.2}
$$

3.3.2 Occupation Measures (Green's Functions)

Define the following occupation measures (Green's functions): for $A \subset \mathbb{R}^3$

$$
\begin{aligned}
g(A) &:= \mathbf{P}\left(Y_1 \in A\right) \\
h(A) &:= \mathbf{E}\left(|\{0<t\le\xi_1 : Y(t)\in A\}|\right) \\
G(A) &:= \mathbf{E}\left(|\{1\le k<\infty : Y_k\in A\}|\right) \\
H(A) &:= \mathbf{E}\left(|\{0<t<\infty : Y(t)\in A\}|\right).
\end{aligned}
$$

Since the different steps of the Y-process are independent, we can express G and H as covolutions:

$$
\begin{aligned}
G(A) &= g(A) + \int_{\mathbb{R}^3} g(A-x)G(dx) \\
H(A) &= h(A) + \int_{\mathbb{R}^3} h(A-x)G(dx).
\end{aligned}
\tag{3.3.3}
$$

3.3.3 Bounds

Lemma 3.3.1. *The following identities and upper bounds hold:*

$$
h(dx) = g(dx) \le L(dx) \tag{3.3.4}
$$

$$
H(dx) = G(dx) \le K(dx) + L(dx) \tag{3.3.5}
$$

where

$$
K(dx) := C\min\{1, |x|^{-1}\}dx, \qquad L(dx) := Ce^{-c|x|}\,|x|^{-2}\,dx, \tag{3.3.6}
$$

with appropriately chosen $C<\infty$ and $c>0$.

Proof of Lemma 3.3.1. The identity $h=g$ is a direct consequence of the flight length ξ being $EXP(1)$-distributed. In polar coordinates with r on the radial direction and φ the solid angle

$$g(dx) = e^{-r} dr d\varphi$$
$$= |x|^{-2} e^{-|x|} dx,$$

where to go from the first line to the second we convert from polar to Cartesian coordinates. From here the the upper bound (3.3.4) follows.

(3.3.5) then follows from (3.3.3) and the standard Green's function estimate for a random walk with step distribution g outlined in Chapter 2, Section 2.5.

□

For later use we introduce the conditional versions – conditioned on the sequence $\underline{\epsilon}$ (see (3.2.3)) – of the bounds (3.3.4), (3.3.5). In this order we define the conditional versions of the Green's functions, given $\epsilon \in \{0,1\}$, respectively $\underline{\epsilon} \in \{0,1\}^{\mathbb{N}}$:

$$\begin{aligned}
g_\epsilon(A) &:= \mathbf{P}\left(Y_1 \in A \mid \epsilon\right) \\
h_\epsilon(A) &:= \mathbf{E}\left(|\{0 < t \leq \xi_1 : Y(t) \in A\}| \mid \epsilon\right) \\
G_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(|\{1 \leq k < \infty : Y_k \in A\}| \mid \underline{\epsilon}\right) \\
H_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(|\{0 < t < \infty : Y(t) \in A\}| \mid \underline{\epsilon}\right),
\end{aligned}$$

and state the conditional version of Lemma 3.3.1:

Lemma 3.3.2. *The following upper bounds hold uniformly in $\underline{\epsilon} \in \{0,1\}^{\mathbb{N}}$:*

$$g_\epsilon(dx) \leq L(dx), \qquad h_\epsilon(dx) \leq L(dx), \tag{3.3.7}$$
$$G_{\underline{\epsilon}}(dx) \leq K(dx) + L(dx), \qquad H_{\underline{\epsilon}}(dx) \leq K(dx) + L(dx), \tag{3.3.8}$$

with $K(x)$ and $L(x)$ as in (3.3.6), with appropriately chosen constants $C < \infty$ and $c > 0$.

Proof of Lemma 3.3.2. Noting that

$$g_\epsilon(dx) \leq C\,|x|^{-2} e^{-|x|} dx, \qquad h_\epsilon(dx) \leq C\,|x|^{-2} e^{-|x|} dx,$$

the proof of Lemma 3.3.2 follows very much the same lines as the proof of Lemma 3.3.1. We omit the details. □

3.3.4 Computation

According to (3.3.1), (3.3.2), for every $j = 1, 2, \ldots$

$$\mathbf{P}\left(\widehat{W}_j\right) \leq \mathbf{P}\left(\widehat{W}^*_\infty\right) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} G(B_{zr,2r}) h(B_{zr,3r}),$$
$$\mathbf{P}\left(\widetilde{W}_j\right) \leq \mathbf{P}\left(\widetilde{W}^*_\infty\right) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} H(B_{zr,3r}) g(B_{zr,2r}).$$

Moreover, straightforward computations yield

Lemma 3.3.3. *In dimension $d = 3$ the following bounds hold, with some $C < \infty$*

$$\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r}) L(B_{zr,2r}) \leq C r^3, \qquad \sum_{z \in \mathbb{Z}^3} L(B_{zr,3r}) L(B_{zr,2r}) \leq C r^2 \tag{3.3.9}$$

Proof of Lemma 3.3.3. The bounds (3.3.9) readily follow from explicit computations. First note

$$K(B_{zr,3r}) \le Cr^3, \tag{3.3.10}$$
$$L(B_{zr,3r}) \le \delta_{z,0}Cr + (1-\delta_{z,0})Cre^{-cr|z|}\,|z|^{-2} \tag{3.3.11}$$

Using (3.3.10) and (3.3.11) we can bound

$$\begin{aligned}\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})L(B_{zr,2r}) &\le C^2r^4 + C^2r^6 \sum_{0\ne z\in\mathbb{Z}^3} e^{-cr|z|}(r\,|z|)^{-2}\\ &\le C^2r^4 + C'r^3\int_{\mathbb{R}^3} e^{-c|y|}\,|y|^{-2}\,dy\\ &\le C''r^3,\end{aligned}$$

where to go from the first line to the second we approximate the sum by a Riemann integral. This then gives the left hand bound in (3.3.9).

Now, using (3.3.11)

$$\begin{aligned}\sum_{z\in\mathbb{Z}^3} L(B_{zr,3r})L(B_{zr,2r}) &\le C^2r^2 + C^2r^6 \sum_{0\ne z\in\mathbb{Z}^3} e^{-2cr|z|}(r\,|z|)^{-4}\\ &\le C^2r^2 + C^2r^2\sum_{0\ne z\in\mathbb{Z}^3} |z|^{-4}\\ &\le C'''r^2.\end{aligned}$$

Note that, in dimension 3, the sum in the second line converges.

□

We conclude this section with the following consequence of the above arguments and computations.

Corollary 3.3.4. *There exists a constant $C<\infty$ such that for any $j\ge 1$:*

$$\mathbf{P}\left(\widetilde{W}_j\right) \le Cr, \qquad \mathbf{P}\left(\widehat{W}_j\right) \le Cr. \tag{3.3.12}$$

3.3.5 No Mismatching – Up to $T \sim o(r^{-1})$

Define the stopping time

$$\sigma := \min\{j>0 : \max\{\mathbb{1}_{\widehat{W}_j}, \mathbb{1}_{\widetilde{W}_j}\} = 1\},$$

and note that by construction

$$\inf\{t>0 : X(t)\ne Y(t)\} \ge \tau_{\sigma-1}. \tag{3.3.13}$$

Lemma 3.3.5. *Let $T=T(r)$ be such that $\lim_{r\to 0} T(r)=\infty$ and $\lim_{r\to 0} rT(r) = 0$. Then*

$$\lim_{r\to 0}\mathbf{P}\left(\tau_{\sigma-1} < T\right) = 0. \tag{3.3.14}$$

Proof of Lemma 3.3.5.

$$\mathbf{P}\left(\tau_{\sigma-1}<T\right) \le \mathbf{P}\left(\sigma \le 2T\right) + \mathbf{P}\left(\sum_{j=1}^{2T-1}\xi_j < T\right) \le CrT + Ce^{-cT}, \tag{3.3.15}$$

where $C < \infty$ and $c > 0$. The first term in the middle expression of (3.3.15) is bounded by union bound and (3.3.12) of Corollary 3.3.4. In bounding the second term we use a large deviation upper bound for the sum of independent $EXP(1)$-distributed ξ_j-s.

Finally, (3.3.14) readily follows from (3.3.15). □

(3.1.7) now follows directly from (3.3.13) and (3.3.14). Thus, this concludes the proof of Theorem 3.1.1. □

3.4 Beyond the Naïve Coupling

The forthcoming sections rely on the joint realization (coupling) of the *three* processes $t \mapsto \big(Y(t), X(t), Z(t)\big)$ as described in Section 3.2. In particular, recall the construction of the process $t \mapsto Z(t)$ from Subection 3.2.4.

3.4.1 Breaking Z into Legs

Let $\Gamma_0 := 0$, $\Theta_0 = 0$ and for $n \geq 1$

$$\begin{aligned} &\Gamma_n := \min\{j \geq \Gamma_{n-1} + 2 : \min\{\xi_{j-1}, \xi_j, \xi_{j+1}, \xi_{j+2}\} > 1\}, && \gamma_n := \Gamma_n - \Gamma_{n-1}, \\ &\Theta_n := \tau_{\Gamma_n}, && \theta_n := \Theta_n - \Theta_{n-1}, \end{aligned} \tag{3.4.1}$$

and denote

$$\begin{aligned} &\xi_{n,j} := \xi_{\Gamma_{n-1}+j}, \quad u_{n,j} := u_{\Gamma_{n-1}+j}, \quad y_{n,j} := y_{\Gamma_{n-1}+j}, && 1 \leq j \leq \gamma_n, \\ &Y_n(t) := Y(\Theta_{n-1} + t) - Y(\Theta_{n-1}), && 0 \leq t \leq \theta_n, \\ &Z_n(t) := Z(\Theta_{n-1} + t) - Z(\Theta_{n-1}), && 0 \leq t \leq \theta_n. \end{aligned}$$

Then, it follows that the *packs of random variables*

$$\varpi_n := \left(\gamma_n; (\xi_{n,j}, u_{n,j}) : 1 \leq j \leq \gamma_n\right), \qquad n \geq 0, \tag{3.4.2}$$

are fully independent (for $n \geq 0$), and also identically distributed for $n \geq 1$. (The zeroth pack is deficient if $\min\{\xi_0, \xi_1\} < 1$.) Moreover the *legs* of the Markovian flight process

$$\left(\theta_n; Y_n(t) : 0 \leq t \leq \theta_n\right), \qquad n \geq 0,$$

are fully independent, and identically distributed for $n \geq 1$.

A *key observation is* that due to the rules of construction of the process $t \mapsto Z(t)$ exposed in Section 3.2.4, the legs

$$\left(\theta_n; Z_n(t) : 0 \leq t \leq \theta_n\right), \qquad n \geq 0, \tag{3.4.3}$$

of the auxiliary process $t \mapsto Z(t)$ *are also independently constructed* from the packs (3.4.2), following the rules in Section 3.2.4. Note, that the restrictions $|y_{j-1}| < 1$ in (3.2.7) were imposed exactly in order to ensure this independence of the legs (3.4.3). Therefore we will now construct the auxiliary process $t \mapsto Z(t)$ and its time reversal $t \mapsto Z^*(t)$ from an infinite sequence of independent packs (3.4.2). In order to reduce unnecessary complications of notation from now on we assume $\min\{\xi_0, \xi_1\} > 1$.

Remark: In order to break up the auxiliary process $t \mapsto Z(t)$ into *independent legs* the choice of simpler stopping times

$$\Gamma'_n := \min\{j \geq \Gamma_{n-1} + 1 : \min\{\xi_j, \xi_{j+1}\} > 1\},$$

would work. However, we need the slightly more complicated stoppings Γ_n, given in (3.4.1), for some other reasons which will become clear towards the end of Section 3.4.2 and in the statement and proof of Lemma 3.5.1.

3.4.2 One Leg

Let ξ_j, u_j, $j \geq 1$, be fully independent random variables with the distributions (3.2.1), conditioned to

$$\min\{\xi_1, \xi_2\} > 1.$$

and y_j as in (3.2.2). Let

$$\gamma := \min\{j \geq 2 : \min\{\xi_{j-1}, \xi_j, \xi_{j+1}, \xi_{j+2}\} > 1\} \in \{2\} \cup \{5, 6, \dots\}. \tag{3.4.4}$$

Note that γ can not assume the values $\{1, 3, 4\}$. Call

$$\varpi := (\gamma; (\xi_j, u_j) : 1 \leq j \leq \gamma) \tag{3.4.5}$$

a *pack*, and keep the notation $\tau_j := \sum_{k=1}^{j} \xi_k$, and $\theta := \tau_\gamma$.

The *forward leg*

$$(\theta; Z(t) : 0 \leq t \leq \theta)$$

is constructed from the pack ϖ according to the rules given in Section 3.2.4. We will also denote

$$Z_j := Z(\tau_j), \quad 0 \leq j \leq \gamma; \qquad\qquad \overline{Z} := Z_\gamma = Z(\theta).$$

These are the discrete steps, respectively, the terminal position of the leg.

It is easy to see that the distributions of γ and θ are exponentially tight: there exist constants $C < \infty$ and $c > 0$ such that for any $s \in [0, \infty)$

$$\mathbf{P}(\gamma > s) \leq Ce^{-cs}, \qquad\qquad \mathbf{P}(\theta > s) \leq Ce^{-cs}. \tag{3.4.6}$$

The left inequality is a consequence of Markov's inequality and moment generating functions (sometimes called Chernoff's inequality), while the second follows from the first and a large deviation principle for exponential random variables.

The *backwards leg*

$$(\theta; Z^*(t) : 0 \leq t \leq \theta)$$

is constructed from the pack ϖ as

$$Z^*(t, \varpi) := Z(\theta - t, \varpi^*) - \overline{Z}(\varpi^*),$$

where the backwards pack

$$\varpi^* := (\gamma; (\xi_{\gamma-j}, -u_{\gamma-j}) : 0 \le j \le \gamma)$$

is the time reversal of the pack ϖ. Note that the forward and backward packs, ϖ and ϖ^*, are identically distributed but the forward and backward processes $\big(t \mapsto Z(t) : 0 \le t \le \theta\big)$ and $\big(t \mapsto Z^*(t) : 0 \le t \le \theta\big)$ are not. The backwards process $t \mapsto Z^*(t)$ could also be defined in stepwise terms, similar (but not identical) to those in Section 3.2.4, but we will not rely on these step-wise rules and therefore omit their explicit formulation.

Consistent with the previous notation, we denote

$$Z_j^* := Z^*(\tau_j), \quad 0 \le j \le \gamma; \qquad \overline{Z}^* := Z_\gamma^* = Z^*(\theta) = -\overline{Z}.$$

Note, that due to the construction rules of the forward and backward legs, their beginning, middle and ending parts

$$\begin{aligned} &(\tau_1; Z(t) : 0 \le t \le \tau_1), \\ &(\tau_{\gamma-1} - \tau_1; Z(\tau_1 + t) - Z(\tau_1) : 0 \le t \le \tau_{\gamma-1} - \tau_1), \\ &(\tau_\gamma - \tau_{\gamma-1}; Z(\tau_{\gamma-1} + t) - Z(\tau_{\gamma-1}) : 0 \le t \le \tau_\gamma - \tau_{\gamma-1}), \end{aligned} \tag{3.4.7}$$

are *independent*, and likewise for the backwards process Z^*,

$$\begin{aligned} &(\tau_1; Z^*(t) : 0 \le t \le \tau_1), \\ &(\tau_{\gamma-1} - \tau_1; Z^*(\tau_1 + t) - Z^*(\tau_1) : 0 \le t \le \tau_{\gamma-1} - \tau_1), \\ &(\tau_\gamma - \tau_{\gamma-1}; Z^*(\tau_{\gamma-1} + t) - Z^*(\tau_{\gamma-1}) : 0 \le t \le \tau_\gamma - \tau_{\gamma-1}). \end{aligned} \tag{3.4.8}$$

This fact will be of crucial importance in the proof of Proposition 3.4.2, Section 3.5.2 below. This is the reason (alluded to in the remark at the end of Section 3.4.1) we chose the somewhat complicated stopping time as defined in (3.4.4).

3.4.3 Multi-Leg Concatenation

Let $\varpi_n = (\gamma_n;\ (\xi_{n,j}, u_{n,j}) : 1 \le j \le \gamma_n)$, $n \ge 1$, be a sequence of i.i.d *packs* (3.4.5), and denote θ_n, $(Z_n(t) : 0 \le t \le \theta_n)$, $(Z_{n,j} : 1 \le j \le \gamma_n)$, $(Z_n^*(t) : 0 \le t \le \theta_n)$, $(Z_{n,j}^* : 1 \le j \le \gamma_n)$, $\overline{Z}_n$, $\overline{Z}_n^*$ the various objects defined in Section 3.4.2, specified for the n-th independent leg.

In order to construct the concatenated forward and backward processes $t \mapsto Z(t)$, $t \mapsto Z^*(t)$, $0 \le t < \infty$, we first define for $n \in \mathbb{Z}_+$, respectively $t \in \mathbb{R}_+$

$$\begin{aligned} \Gamma_n &:= \sum_{k=1}^n \gamma_k, \qquad & \nu_n &:= \max\{m : \Gamma_m \le n\}, \qquad & \{n\} &:= n - \Gamma_{\nu_n}, \\ \Theta_n &:= \sum_{k=1}^n \theta_k, \qquad & \nu_t &:= \max\{m : \Theta_m < t\}, \qquad & \{t\} &:= t - \Theta_{\nu_t}. \end{aligned}$$

The concatenated (multi-leg) forward and backward Z-processes are

$$\begin{aligned} \Xi_n &:= \sum_{k=1}^n \overline{Z}_k, \qquad & Z_n &:= \Xi_{\nu_n} + Z_{\nu_n+1,\{n\}}, \qquad & Z(t) &:= \Xi_{\nu_t} + Z_{\nu_t+1}(\{t\}), \\ \Xi_n^* &:= \sum_{k=1}^n \overline{Z}_k^*, \qquad & Z_n^* &:= \Xi_{\nu_n}^* + Z_{\nu_n+1,\{n\}}^*, \qquad & Z^*(t) &:= \Xi_{\nu_t}^* + Z_{\nu_t+1}^*(\{t\}), \end{aligned} \tag{3.4.9}$$

Note that Ξ_n and Ξ_n^* are random walks with independent steps; $t \mapsto Z(t)$, $0 \le t < \infty$, is exactly the Z-process constructed in Section 3.2.4, with $Z_n = Z(\tau_n)$, $0 \le n < \infty$. Similarly, $t \mapsto Z^*(t)$, $0 \le t < \infty$, is the time reversal of the Z-process and $Z_n^* = Z^*(\tau_n)$, $0 \le n < \infty$.

Theorem 3.1.2 will follow from Propositions 3.4.1 and 3.4.2 of the next two sections.

3.4.4 Mismatches Within One Leg

Given a pack $\varpi = (\gamma; (\xi_j, u_j) : 1 \le j \le \gamma)$ (3.4.5), and *arbitrary* incoming and outgoing velocities $u_0, u_{\gamma+1} \in S^2$ let $\big((Y(t), \mathcal{X}(t), Z(t)) : 0^- < t < \theta^+\big)$, be the triplet of Markovian flight process, Lorentz exploration process and auxiliary Z-process jointly constructed with these data, as described in Sections 3.2.1, 3.2.2, respectively, 3.2.4. By $0^- < t < \theta^+$ we mean that the incoming velocities at 0^- are given as $\dot{Y}(0^-) = \dot{\mathcal{X}}(0^-) = \dot{Z}(0^-) = u_0$ and the outgoing velocities at θ^+ are $\dot{Y}(\theta^+) = \dot{Z}(\theta^+) = u_{\gamma+1}$, while $\dot{\mathcal{X}}(\theta^+)$ is determined by the construction from Section 3.2.2. That is, $\dot{\mathcal{X}}(\theta^+) = u_{\gamma+1}$ if this last scattering is not shadowed by the trajectory $\big(\mathcal{X}(t) : 0 \le t \le \theta\big)$ and $\dot{\mathcal{X}}(\theta^+) = \dot{\mathcal{X}}(\theta^-)$ if it is shadowed.

Proposition 3.4.1. *There exists a constant $C < \infty$ such that for any $u_0, u_{\gamma+1} \in S^2$*

$$\mathbf{P}\left(\mathcal{X}(t) \not\equiv Z(t) : 0^- < t < \theta^+\right) \le Cr^2 \left|\log r\right|^2. \tag{3.4.10}$$

The proof of this Proposition relies on controlling the geometry of mismatchings, and is postponed until Section 3.6.

3.4.5 Inter-Leg Mismatches

Let $t \to Z(t)$ be a forward Z-process built up as concatenation of legs, as described in Section 3.4.3 and define the following events

$$\begin{aligned}
\widehat{W}_j &:= \big\{\min\{|Z(t) - Z'_k| : \quad 0 < t < \Theta_{j-1}, \quad \Gamma_{j-1} < k \le \Gamma_j\} < r\big\}, \\
\widetilde{W}_j &:= \big\{\min\{|Z'_k - Z(t)| : \quad 0 \le k < \Gamma_{j-1}, \quad \Theta_{j-1} < t < \Theta_j\} < r\big\}.
\end{aligned} \tag{3.4.11}$$

In words $\widehat{W}_j$ is the event that a collision occuring in the j-th leg is *shadowed* by the past path. While $\widetilde{W}_j$ is the event that within the j-th leg the Z-trajectory bumps into a scatterer placed in an earlier leg. That is, $\widetilde{W}_j \cup \widehat{W}_j$ is precisely the event that the concatenated first $j-1$ legs and the j-th leg are mechanically r-incompatible (see Section 3.2.3).

The following proposition indicates that on our time scales there are no "inter-leg mismatches":

Proposition 3.4.2. *There exists a constant $C < \infty$ such that for all $j \ge 1$*

$$\mathbf{P}\left(\widetilde{W}_j\right) \le Cr^2, \qquad \mathbf{P}\left(\widehat{W}_j\right) \le Cr^2. \tag{3.4.12}$$

The proof of Proposition 3.4.2 is the content of Section 3.5.

3.5 Proof of Proposition 3.4.2

This section is purely probabilistic and is similar to Section 3.3. The notation used is also similar. However, similar is not identical. The various Green's functions used here, although denoted g, h, G, H, as in Section 3.3, are similar in their role but not the same. The estimates that will follow are also different.

3.5.1 Occupation Measures (Green's Functions)

Let now $t \mapsto Z^*(t)$, $0 \le t < \infty$, be a backward Z^*-process and $t \mapsto Z(t)$, $0 \le t \le \theta$, a forward one-leg Z-process, assumed independent. In analogy with the events $\widehat{W}_j$ and $\widetilde{W}_j$ defined in (3.4.11) we define

$$\begin{aligned}
\widehat{W}^*_j &:= \big\{ \min\{|Z^*(t) - Z'_k| : && 0 < t < \Theta_{j-1}, && 0 < k \le \gamma\} < r\big\},\\
\widetilde{W}^*_j &:= \big\{ \min\{|Z^{*\prime}_k - Z(t)| : && 0 < k \le \Gamma_{j-1}, && 0 < t < \theta\} < r\big\},\\
\widehat{W}^*_\infty &:= \big\{ \min\{|Z^*(t) - Z'_k| : && 0 < t < \infty, && 0 < k \le \gamma\} < r\big\},\\
\widetilde{W}^*_\infty &:= \big\{ \min\{|Z^{*\prime}_k - Z(t)| : && 0 < k < \infty, && 0 < t < \theta\} < r\big\}.
\end{aligned}$$

It follows from the definition that

$$\begin{aligned}
\mathbf{P}\left(\widehat{W}_j\right) = \mathbf{P}\left(\widehat{W}^*_j\right) \le \mathbf{P}\left(\widehat{W}^*_{j+1}\right) \le \mathbf{P}\left(\widehat{W}^*_\infty\right),\\
\mathbf{P}\left(\widetilde{W}_j\right) = \mathbf{P}\left(\widetilde{W}^*_j\right) \le \mathbf{P}\left(\widetilde{W}^*_{j+1}\right) \le \mathbf{P}\left(\widetilde{W}^*_\infty\right).
\end{aligned} \tag{3.5.1}$$

On the other hand, by the union bound and independence we have

$$\begin{aligned}
\mathbf{P}\left(\widehat{W}^*_\infty\right) &\le \sum_{z\in\mathbb{Z}^3} \mathbf{P}\left(\{0 < t < \infty : Z^*(t) \in B_{zr,2r}\} \neq \emptyset\right) \mathbf{P}\left(\{1 \le k \le \gamma : Z_k \in B_{zr,2r}\} \neq \emptyset\right)\\
&\le \sum_{z\in\mathbb{Z}^3} (2r)^{-1} \mathbf{E}\left(|\{0 < t < \infty : Z^*(t) \in B_{zr,3r}\}|\right) \mathbf{E}\left(|\{1 \le k \le \gamma : Z_k \in B_{zr,2r}\}|\right)\\
\mathbf{P}\left(\widetilde{W}^*_\infty\right) &\le \sum_{z\in\mathbb{Z}^3} \mathbf{P}\left(\{1 < k < \infty : Z^*_k \in B_{zr,2r}\} \neq \emptyset\right) \mathbf{P}\left(\{0 < t \le \theta : Z(t) \in B_{zr,2r}\} \neq \emptyset\right)\\
&\le \sum_{z\in\mathbb{Z}^3} (2r)^{-1} \mathbf{E}\left(|\{1 < k < \infty : Z^*_k \in B_{zr,2r}\}|\right) \mathbf{E}\left(|\{0 < t \le \theta : Z(t) \in B_{zr,3r}\}|\right)
\end{aligned} \tag{3.5.2}$$

Therefore, in view of (3.5.1) we have to control the mean occupation time measures appearing on the right hand side of (3.5.2).

Define the following mean occupation measures (Green's functions): for $A \subset \mathbb{R}^3$ let

$$\begin{aligned}
g(A) &:= \mathbf{E}\left(|\{1 \le k \le \gamma : Z_k \in A\}|\right),\\
g^*(A) &:= \mathbf{E}\left(|\{1 \le k \le \gamma : Z^*_k \in A\}|\right),\\
h(A) &:= \mathbf{E}\left(|\{0 < t \le \theta : Z(t) \in A\}|\right),\\
h^*(A) &:= \mathbf{E}\left(|\{0 < t \le \theta : Z^*(t) \in A\}|\right),\\
R^*(A) &:= \mathbf{E}\left(|\{1 \le n < \infty : \Xi^*_n \in A\}|\right),\\
G^*(A) &:= \mathbf{E}\left(|\{1 \le k < \infty : Z^*_k \in A\}|\right),\\
H^*(A) &:= \mathbf{E}\left(|\{0 < t < \infty : Z^*(t) \in A\}|\right).
\end{aligned}$$

Since the different legs are independent, we can express G^* and H^* as convolutions

$$\begin{aligned}
G^*(A) &= g^*(A) + \int_{\mathbb{R}^3} g^*(A - x) R^*(dx),\\
H^*(A) &= h^*(A) + \int_{\mathbb{R}^3} h^*(A - x) R^*(dx).
\end{aligned} \tag{3.5.3}$$

3.5.2 Bounds

Lemma 3.5.1. *The following upper bounds hold:*

$$\max\{g(dx), g^*(dx)\} \le M(dx), \qquad \max\{h(dx), h^*(dx)\} \le L(dx), \tag{3.5.4}$$

$$R^*(dx) \le K(dx), \tag{3.5.5}$$

$$G^*(dx) \le K(dx), \qquad H^*(dx) \le K(dx) + L(dx), \tag{3.5.6}$$

where

$$K(dx) := C\min\{1, |x|^{-1}\}dx, \qquad L(dx) := Ce^{-c|x|}\,|x|^{-2}\,dx, \qquad M(dx) := Ce^{-c|x|}dx,$$

with appropriately chosen $C < \infty$ and $c > 0$.

Proof of Lemma 3.5.1. The proof of the bounds (3.5.4) hinges on the decompositions (3.4.7) and (3.4.8) of the forward and backward legs into independent parts.

Let

$$\begin{aligned} g_1(A) &:= \mathbf{P}\left(Z_1 \in A\right) = \mathbf{P}\left(Z_1^* \in A\right) &&= C\int_A \mathbb{1}(|x| > 1)e^{-|x|}dx, \\ h_1(A) &:= \mathbf{E}\left(|\{t \le \tau_1 : Z(t) \in A\}|\right) = \mathbf{E}\left(|\{t \le \tau_1 : Z^*(t) \in A\}|\right) &&= C'\int_A |x|^{-2}\,e^{-\max\{1,|x|\}}dx, \end{aligned} \tag{3.5.7}$$

and

$$\begin{aligned} g_2(A) &:= \mathbf{E}\left(|\{1 \le k \le \gamma : Z_k - Z_1 \in A\}|\right), \\ g_2^*(A) &:= \mathbf{E}\left(|\{1 \le k \le \gamma : Z_k^* - Z_1^* \in A\}|\right), \\ h_2(A) &:= \mathbf{E}\left(|\{0 < t \le \theta - \tau_1 : Z(\tau_1 + t) - Z_1 \in A\}|\right), \\ h_2^*(A) &:= \mathbf{E}\left(|\{0 < t \le \theta - \tau_1 : Z^*(\tau_1 + t) - Z_1^* \in A\}|\right). \end{aligned}$$

Due to the exponential tail of the distribution of γ and θ, (3.4.6), there are constants $C < \infty$ and $c > 0$ such that for any $s < \infty$

$$\begin{aligned} &\max\{g_2(\{x : |x| > s\}), g_2^*(\{x : |x| > s\})\} \le Ce^{-cs}, \\ &\max\{h_2(\{x : |x| > s\}), h_2^*(\{x : |x| > s\})\} \le Ce^{-cs}, \end{aligned} \tag{3.5.8}$$

and furthermore,

$$\begin{aligned} g_2(\mathbb{R}^3) &= g_2^*(\mathbb{R}^3) = \mathbf{E}\left(\gamma\right) < \infty, \\ h_2(\mathbb{R}^3) &= h_2^*(\mathbb{R}^3) = \mathbf{E}\left(\theta - \tau_1\right) < \infty. \end{aligned} \tag{3.5.9}$$

From the independent decompositions (3.4.8) and (3.4.7) it follows that

$$\begin{aligned} g(A) &= \int_{\mathbb{R}^3} g_2(A-x)g_1(dx), & g^*(A) &= \int_{\mathbb{R}^3} g_2^*(A-x)g_1(dx), \\ h(A) &= \int_{\mathbb{R}^3} h_2(A-x)g_1(dx) + h_1(A), & h^*(A) &= \int_{\mathbb{R}^3} h_2^*(A-x)g_1(dx) + h_1(A). \end{aligned} \tag{3.5.10}$$

The bounds (3.5.4) readily follow from the explicit expressions (3.5.7), the convolutions (3.5.10) and the bounds (3.5.8) and (3.5.9).

The bound (3.5.5) is a straightforward Green's function bound for the the random walk Ξ_n^* defined in (3.4.9), by noting that the distribution of the i.i.d. steps $\overline{Z}_k^*$ of this random walk has bounded density and exponential tail decay (this follows the same lines as the bounds on the random walk distribution proved in Chapter 2, Section 2.5).

Finally, the bounds (3.5.6) follow from the convolutions (3.5.3) and the bounds (3.5.4), (3.5.5).

□

Remark: On the difference between Lemmas 3.3.1 and 3.5.1. Note the difference between the upper bounds for g in (3.3.4), respectively, (3.5.4), and on G in (3.3.5), respectively, (3.5.6). These are important and are due to the fact that the length of the first step in a Z- or Z^*-leg is distributed as $(\xi \mid \xi > 1) \sim EXP(1|0)$ rather than $\xi \sim EXP(1)$.

3.5.3 Computation

According to (3.5.2)

$$\begin{aligned} \mathbf{P}\left(\widetilde{W}_j\right) \le \mathbf{P}\left(\widetilde{W}_\infty^*\right) &\le (2r)^{-1} \sum_{z\in\mathbb{Z}^3} H^*(B_{zr,3r})g(B_{zr,2r}), \\ \mathbf{P}\left(\widehat{W}_j\right) \le \mathbf{P}\left(\widehat{W}_\infty^*\right) &\le (2r)^{-1} \sum_{z\in\mathbb{Z}^3} G^*(B_{zr,2r})h_r(B_{zr,3r}). \end{aligned} \tag{3.5.11}$$

Lemma 3.5.2. *In dimension $d = 3$ the following bounds hold, with some $C < \infty$*

$$\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})M(B_{zr,2r}) \le Cr^3, \qquad \sum_{z\in\mathbb{Z}^3} M(B_{zr,3r})L(B_{zr,2r}) \le Cr^3. \tag{3.5.12}$$

Proof of Lemma 3.5.2. The bounds (3.5.12) (similarly to the bounds (3.3.9)) readily follow from explicit computations which we omit. □

Proof of Proposition 3.4.2. Proposition 3.4.2 now follows by inserting the bounds (3.5.12) and one of the bounds in (3.3.9) into equations (3.5.11).

□

3.6 Proof of Proposition 3.4.1

Given a pack $\varpi = (\gamma; (\xi_j, u_j) : 1 \le j \le \gamma)$ (3.4.5), and arbitrary $u_0, u_{\gamma+1} \in S_1^2$, let $\big((Y(t), \mathcal{X}(t), Z(t)) : 0 \le t \le \theta\big)$ be the triplet of Markovian flight process, Lorentz exploration process and auxiliary Z-process jointly constructed with these data. We will prove the following bounds, stated in increasing order of difficulty/complexity.

$$\mathbf{P}\left(\{\mathcal{X}(t)\not\equiv Z(t):0^-\le t\le\theta^+\}\cap\{\sum_{j=1}^{\gamma}\eta_j>1\}\right)\le Cr^2\left|\log r\right|, \tag{3.6.1}$$

$$\mathbf{P}\left(\{\mathcal{X}(t)\not\equiv Z(t):0^-\le t\le\theta^+\}\cap\{\sum_{j=1}^{\gamma}\eta_j=0\}\right)\le Cr^2\left|\log r\right|, \tag{3.6.2}$$

$$\mathbf{P}\left(\{\mathcal{X}(t)\not\equiv Z(t):0^-\le t\le\theta^+\}\cap\{\sum_{j=1}^{\gamma}\eta_j=1\}\right)\le Cr^2\left|\log r\right|^2. \tag{3.6.3}$$

Note that by construction $\eta_1=\eta_2=\eta_3=\eta_\gamma=0$, so the sums on the left hand side go actually from 4 to $\gamma-1$. We stated and prove these bounds in their increasing order of complexity: (3.6.1) (proved in Section 3.6.1) and (3.6.2) (proved in Section 3.6.2) are of purely probabilistic nature while (3.6.3) (proved in Sections 3.6.3-3.6.7) also relies on the the finer geometric understanding of the mismatch events $\widehat{\eta}_j=1$ and $\widetilde{\eta}_j=1$.

3.6.1 Proof of (3.6.1)

This follows directly from Lemma 3.2.1. Indeed, given γ and $\underline{\epsilon}=(\epsilon_j)_{1\le j\le\gamma}$, due to (3.2.9),

$$\begin{aligned}\mathbf{P}\left(\sum_{j=1}^{\gamma}\eta_j>1\;\middle|\;\underline{\epsilon}\right)&\le\gamma\max_j\mathbf{P}\left(\eta_j=\eta_{j+1}=1\;\middle|\;\underline{\epsilon}\right)+\frac{\gamma^2}{2}\max_{j,k:|j-k|>1}\mathbf{P}\left(\eta_j=\eta_k=1\;\middle|\;\underline{\epsilon}\right)\\&\le C\gamma r^2\left|\log r\right|+C\gamma^2r^2,\end{aligned}$$

and hence, due to the exponential tail bound (3.4.6) we get

$$\mathbf{P}\left(\sum_{j=4}^{\gamma-1}\eta_j>1\right)=\mathbf{E}\left(\mathbf{P}\left(\sum_{j=4}^{\gamma-1}\eta_j>1\;\middle|\;\underline{\epsilon}\right)\right)\le Cr^2\left|\log r\right|.$$

which concludes the proof of (3.6.1). □

3.6.2 Proof of (3.6.2)

First note that by construction of the processes $\big((\mathcal{X}(t),Z(t)):0^-<t<\theta^+\big)$ the following identities hold:

$$\{\mathcal{X}(t)\not\equiv Z(t):0^-\le t\le\theta^+\}\cap\{\sum_{j=1}^{\gamma}\eta_j=0\}=\{\mathcal{X}(t)\not\equiv Y(t):0^-\le t\le\theta^+\}\cap\{\sum_{j=1}^{\gamma}\eta_j=0\}$$

$$\{\mathcal{X}(t)\not\equiv Y(t):0^-\le t\le\theta^+\}=\bigcup_{0<j<\gamma}\left\{\min_{\tau_j\le t\le\theta}\left|Y'_{j-1}-Y(t)\right|<r\right\}\cup\left\{\min_{0\le t\le\tau_j}\left|Y'_{j+1}-Y(t)\right|<r\right\}$$

And, hence

$$\{\mathcal{X}(t) \not\equiv Z(t) : 0^- \le t \le \theta^+\} \cap \{\sum_{j=1}^{\gamma} \eta_j = 0\} \tag{3.6.4}$$
$$= \bigcup_{0<j<\gamma} \left(\left\{ \min_{\tau_j \le t \le \tau_{j+1}} \left|Y'_{j-1} - Y(t)\right| < r \right\} \cup \left\{ \min_{\tau_{j-1} \le t \le \tau_j} \left|Y'_{j+1} - Y(t)\right| < r \right\} \right) \cap \{\xi_j > 1\}$$
$$\cup \bigcup_{0<j<\gamma} \left(\left\{ \min_{\tau_{j+1} \le t \le \theta} \left|Y'_{j-1} - Y(t)\right| < r \right\} \cup \left\{ \min_{0 \le t \le \tau_{j-1}} \left|Y'_{j+1} - Y(t)\right| < r \right\} \right)$$
$$\subset \bigcup_{0<j<\gamma} \left(\left\{ \min_{\tau_j \le t \le \tau_{j+1}} \left|Y_{j-1} - Y(t)\right| < 2r \right\} \cup \left\{ \min_{\tau_{j-1} \le t \le \tau_j} \left|Y_{j+1} - Y(t)\right| < 2r \right\} \right) \cap \{\xi_j > 1\}$$
$$\cup \bigcup_{0<j<\gamma} \left(\left\{ \min_{\tau_{j+1} \le t \le \theta} \left|Y_{j-1} - Y(t)\right| < 2r \right\} \cup \left\{ \min_{0 \le t \le \tau_{j-1}} \left|Y_{j+1} - Y(t)\right| < 2r \right\} \right)$$

By simple geometric inspection we see

$$\left\{ \min_{\tau_j \le t \le \tau_{j+1}} \left|Y_{j-1} - Y(t)\right| < 2r \right\} \cap \{\xi_j > 1\} \subset \{\angle(-u_{j-1}, u_j) < 4r\},$$
$$\left\{ \min_{\tau_{j-1} \le t \le \tau_j} \left|Y_{j+1} - Y(t)\right| < 2r \right\} \cap \{\xi_j > 1\} \subset \{\angle(-u_{j+1}, u_j) < 4r\}.$$

And therefore,

$$\max_{\underline{\epsilon}} \mathbf{P} \left(\left\{ \min_{\tau_j \le t \le \tau_{j+1}} \left|Y_{j-1} - Y(t)\right| < 2r \right\} \cap \{\xi_j > 1\} \mid \underline{\epsilon} \right) \le Cr^2$$
$$\max_{\underline{\epsilon}} \mathbf{P} \left(\left\{ \min_{\tau_{j-1} \le t \le \tau_j} \left|Y_{j+1} - Y(t)\right| < 2r \right\} \cap \{\xi_j > 1\} \mid \underline{\epsilon} \right) \le Cr^2. \tag{3.6.5}$$

On the other hand, from the conditional Green's function computations of section 3.3, in particular from Lemma 3.3.2, we get

$$\max_{\underline{\epsilon}} \mathbf{P} \left(\min_{\tau_{j+1} \le t \le \theta} \left|Y_{j-1} - Y(t)\right| < 2r \mid \underline{\epsilon} \right) \le \sup_{\underline{\epsilon}} \mathbf{P} \left(\min_{\tau_2 \le t < \infty} \left|Y(t)\right| < 2r \mid \underline{\epsilon} \right) \le Cr^2 \left|\log r\right|,$$
$$\max_{\underline{\epsilon}} \mathbf{P} \left(\min_{0 \le t \le \tau_{j-1}} \left|Y_{j+1} - Y(t)\right| < 2r \mid \underline{\epsilon} \right) \le \sup_{\underline{\epsilon}} \mathbf{P} \left(\min_{\tau_2 \le t < \infty} \left|Y(t)\right| < 2r \mid \underline{\epsilon} \right) \le Cr^2 \left|\log r\right|. \tag{3.6.6}$$

Putting (3.6.4), (3.6.5) and (3.6.6) together yields

$$\mathbf{P} \left(\{\mathcal{X}(t) \not\equiv Z(t) : 0^- \le t \le \theta^+\} \cap \{\sum_{j=4}^{\gamma-1} \eta_j = 0\} \mid \underline{\epsilon} \right) \le C\gamma r^2 \left|\log r\right|,$$

and hence, taking expectation over $\underline{\epsilon}$, we get (3.6.2).

3.6.3 Proof of (3.6.3) – Preparations

Let $\gamma \in \{2\} \cup \{5, 6, \dots\}$, and $\underline{\epsilon} = (\epsilon_j)_{1 \le j \le \gamma} \in \{0,1\}^\gamma$ compatible with the definition of a pack, and $3 < k < \gamma$ be fixed. Given a pack ϖ with signature $\underline{\epsilon}$ we define yet another auxiliary process $\left(Z^{(k)}(t) : 0^- < t < \theta^+\right)$ as follows:

- On $0^- < t \le \tau_{k-1}$, $Z^{(k)}(t) = Y(t)$.
- On $\tau_{k-1} < t \le \tau_k$, $Z^{(k)}(t)$ is constructed according to the rules of the Z-process, given in Section 3.2.4.

∘ On $\tau_k < t < \theta^+$, $Z^{(k)}(t) = Z^{(k)}(\tau_k) + Y(t) - Y(\tau_k)$.

Note that on the event $\{\eta_j = \delta_{j,k} : 1 \le j \le \gamma\}$ we have $Z^{(k)}(t) \equiv Z(t)$, $0^- < t < \theta^+$.

We will show that

$$\begin{aligned} \max_{\underline{\epsilon},k} \mathbf{P}\Big(\{\mathcal{X}(t) \not\equiv Z^{(k)}(t) : 0^- \le t \le \theta^+\} \cap \{\eta_j = \delta_{j,k} : 1 \le j \le \gamma\} \mid \underline{\epsilon}\Big) \\ \le \max_{\underline{\epsilon},k} \mathbf{P}\Big(\{\mathcal{X}(t) \not\equiv Z^{(k)}(t) : 0^- \le t \le \theta^+\} \cap \{\eta_k = 1\} \mid \underline{\epsilon}\Big) \\ \le C\gamma^2 r^2 \left|\log r\right|^2, \end{aligned} \tag{3.6.7}$$

and hence

$$\begin{aligned} \max_{\underline{\epsilon}} \mathbf{P}\left(\{\mathcal{X}(t) \not\equiv Z(t) : 0^- \le t \le \theta^+\} \cap \{\sum_{k=1}^{\gamma} \eta_k = 1\} \mid \underline{\epsilon}\right) \\ \le \gamma \max_{\underline{\epsilon},k} \mathbf{P}\big(\{\mathcal{X}(t) \not\equiv Z(t) : 0^- \le t \le \theta^+\} \cap \{\eta_j = \delta_{j,k} : 1 \le j \le \gamma\} \mid \underline{\epsilon}\big) \\ \le C\gamma^3 r^2 \left|\log r\right|^2. \end{aligned}$$

Then, taking expectation over $\underline{\epsilon}$ we get (3.6.3).

In order to prove (3.6.7) first write

$$\begin{aligned} \mathbf{P}\Big(\{\mathcal{X}(t) \not\equiv Z^{(k)}(t) : 0^- \le t \le \theta^+\} \cap \{\eta_j = \delta_{j,k} : 1 \le j \le \gamma\} \mid \underline{\epsilon}\Big) \\ \le \mathbf{P}\Big(\{\mathcal{X}(t) \not\equiv Z^{(k)}(t) : 0^- \le t \le \theta^+\} \cap \{\eta_k = 1\} \mid \underline{\epsilon}\Big) \\ = \mathbf{P}\Big(\{\mathcal{X}(t) \not\equiv Z^{(k)}(t) : 0^- \le t \le \theta^+\} \cap \{\widehat{\eta}_k = 1\} \mid \underline{\epsilon}\Big) + \\ \mathbf{P}\Big(\{\mathcal{X}(t) \not\equiv Z^{(k)}(t) : 0^- \le t \le \theta^+\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \mid \underline{\epsilon}\Big), \end{aligned}$$

and note that the three parts

$$\begin{aligned} &\big(Z^{(k)}(t) : 0^- < t < \tau_{k-3}\big) = \big(Y(t) : 0^- < t < \tau_{k-3}\big), \\ &\big(Z^{(k)}(\tau_{k-3} + t) - Z^{(k)}(\tau_{k-3}) : 0 \le t \le \tau_k - \tau_{k-3}\big), \\ &\big(Z^{(k)}(\tau_k) + t) - Z^{(k)}(\tau_k) : 0 \le t < \theta^+ - \tau_k\big) = \big(Y(\tau_k) + t) - Y(\tau_k) : 0 \le t < \theta^+ - \tau_k\big), \end{aligned} \tag{3.6.8}$$

are *independent* – even if the events $\{\widehat{\eta}_k = 1\}$, respectively, $\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\}$ are specified.

From the construction of the processes $\big((\mathcal{X}(t), Z^{(k)}(t)) : 0^- < t < \theta^+\big)$ it follows that if $\big(Z^{(k)}(t) : 0^- < t < \theta^+\big)$ is mechanically r-consistent then $\big(\mathcal{X}(t) \equiv Z^{(k)}(t) : 0^- < t < \theta^+\big)$.

Denote by $A^{(k)}_{a,a}$, $1 \le a \le 3$, the event that the a-th part of the decomposition (3.6.8) is mechanically r-*inconsistent*, and by $A_{a,b} = A_{b,a}$, $1 \le a, b \le 3$, $a \ne b$, the event that the a-th and b-th parts of the decomposition (3.6.8) are mechanically r-*incompatible* – in the sense of the definitions (3.2.5) and (3.2.6) in Section 3.2.3. In order to prove (3.6.7) we will have to prove appropriate upper bounds on the conditional probabilities

$$\begin{aligned} &\mathbf{P}\Big(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{a,b} \mid \underline{\epsilon}\Big), \\ &\mathbf{P}\Big(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{a,b} \mid \underline{\epsilon}\Big), \end{aligned} \qquad a, b = 1, 2, 3. \tag{3.6.9}$$

These are altogether 12 bounds. However, some of them are formally very similar.

$A^{(k)}_{1,1}$, $A^{(k)}_{3,3}$ and $A^{(k)}_{1,3}$ do not involve the middle part and therefore do not rely on the geometric arguments of the forthcoming Sections 3.6.4-3.6.6. Applying directly (3.2.8), (3.3.7), (3.3.9) and similar procedures as in Section 3.3.4, without any new effort we get

$$\begin{aligned} &\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{a,b} \mid \underline{\epsilon}\right) \le C\gamma^2 r^2, \\ &\mathbf{P}\left(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{a,b} \mid \underline{\epsilon}\right) \le C\gamma^2 r^2, \end{aligned} \qquad a, b = 1, 3. \tag{3.6.10}$$

We omit the repetition of these details.

The remaining six bounds rely on the geometric arguments of Sections 3.6.4-3.6.6 and, therefore, are postponed to Section 3.6.7

3.6.4 Geometric Estimates

We analyse the middle segment of the process $Z^{(k)}$, presented in (3.6.8), restricted to the events $\{\widehat{\eta}_k = 1\}$, respectively, $\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\}$. Since everything done in this analysis is invariant under time and space translations and also under rigid rotations of $\mathbb{R}^3$ it will be notationally convenient to place the origin of space-time at $(\tau_{k-2}, Z(\tau_{k-2}))$ and choose $u_{k-2} = e = (1, 0, 0)$, a fixed element of S^2_1. So, the ingredient random variables are $(\xi_-, u, \xi, v, \xi_+)$, fully independent and distributed as $\xi_- \sim EXP(1|\epsilon_{k-2})$, $\xi \sim EXP(1|\epsilon_{k-1}) = EXP(1|1)$, $\xi_+ \sim EXP(1|\epsilon_k)$, $u, v \sim UNI(S^2_1)$.

It will be enlightening to group the ingredient variables as $(\xi_-, (u, \xi, v), \xi_+)$, and accordingly write the sample space of this reduced context as $\mathbb{R}_+ \times \mathbb{D} \times \mathbb{R}_+$, where $\mathbb{D} := S^2_1 \times \mathbb{R}_+ \times S^2_1$, with the probability measure $EXP(1|\epsilon_{k-2}) \times \mu \times EXP(1|\epsilon_k)$ where, on $\mathbb{D}$,

$$\mu = UNI(S^2_1) \times EXP(1|1) \times UNI(S^2_1). \tag{3.6.11}$$

For $r < 1$, let $\widehat{\sigma}_r, \widetilde{\sigma}_r : \mathbb{D} \to \mathbb{R}_+ \cup \{\infty\}$ be

$$\begin{aligned} \widehat{\sigma}_r(u, \xi, v) &:= \inf\{t : \left|\xi u + r\frac{u - v}{|u - v|} + te\right| < r\}, \\ \widetilde{\sigma}_r(u, \xi, v) &:= \inf\{t : \left|\xi u + r\frac{u - e}{|u - e|} + tv\right| < r\}, \end{aligned}$$

(with the usual convention $\inf \emptyset = \infty$), and

$$\widehat{\mathbb{A}}_r := \{(u, \xi, v) \in \mathbb{D} : \widehat{\sigma}_r < \infty\}, \qquad \widetilde{\mathbb{A}}_r := \{(u, \xi, v) \in \mathbb{D} : \widetilde{\sigma}_r < \infty\}.$$

We define the process $\left(\widehat{Z}_r(t) : -\infty < t < \infty\right)$ and $\left(\widetilde{Z}_r(t) : -\infty < t < \infty\right)$ in terms of $(u, \xi, v) \in \widehat{\mathbb{A}}_r$, respectively, $(u, \xi, v) \in \widehat{\mathbb{A}}_r$ as follows. Strictly speaking, these are *deficient* processes, since $\mu(\widehat{\mathbb{A}}_r) < 1$, and $\mu(\widetilde{\mathbb{A}}_r) < 1$.

- On $-\infty < t \le 0$, $\widehat{Z}_r(t) = \widetilde{Z}_r(t) = te$.
- On $0 \le t \le \xi$, $\widehat{Z}_r(t) = \widetilde{Z}_r(t) = tu$,
- On $\xi \le t < \infty$,
  - ∘∘ $\widehat{Z}_r(t) = \widehat{Z}_r(\xi) + (t - \xi)u$,
  - ∘∘ $\widetilde{Z}_r(t)$ is the trajectory of a mechanical particle, with initial position $\widetilde{Z}_r(\xi)$ and initial velocity $\dot{\widetilde{Z}}_r(\xi^+) = v$, bouncing elastically between two infinite-mass spherical scatterers centred at $r\frac{e-u}{|e-u|}$, respectively, $\xi u + r\frac{u-v}{|u-v|}$, and, eventually, flying indefinitely with constant terminal velocity.

See Figure 3.3 for a reference to some of the labelling.

The *trapping time* $\widehat{\beta}_r, \widetilde{\beta}_r \in \mathbb{R}_+$ and *escape (terminal) velocity* $\widehat{w}_r, \widetilde{w}_r \in S_1^2$ of the process $\widehat{Z}_r(t)$, respectively, $\widetilde{Z}_r(t)$, are

$$\begin{aligned} &\widehat{\beta}_r := 0, && \widehat{w}_r := u, \\ &\widetilde{\beta}_r := \sup\{s < \infty : \dot{\widetilde{Z}}_r(\xi + s^+) \neq \dot{\widetilde{Z}}_r(\xi + s^-)\}, && \widetilde{w}_r := \dot{\widetilde{Z}}_r(\xi + \widetilde{\beta}_r^+). \end{aligned} \tag{3.6.12}$$

Note that $\widetilde{\beta}_r \geq \widetilde{\sigma}_r$.

The relation of the middle segment of (3.6.8) to $\widehat{Z}_r$ and $\widetilde{Z}_r$ is the following:

$$\begin{aligned} &\Big(\{\widehat{\eta}_k = 1\}, \big(Z^{(k)}(\tau_{k-2} + t) - Z^{(k)}(\tau_{k-2}) : -\xi_{k-2} \leq t \leq \xi_{k-1} + \xi_k\big)\Big) \sim \\ &\qquad\qquad \Big(\{\xi_- > \widehat{\sigma}_r\}, \big(\widehat{Z}_r(t) : -\xi_- \leq t \leq \xi + \xi_+\big)\Big), \\ &\Big(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\}, \big(Z^{(k)}(\tau_{k-2} + t) - Z^{(k)}(\tau_{k-2}) : -\xi_{k-2} \leq t \leq \xi_{k-1} + \xi_k\big)\Big) \sim \\ &\qquad\qquad \Big(\{\xi_- \leq \widehat{\sigma}_r\} \cap \{\xi_+ > \widetilde{\sigma}_r\}, \big(\widetilde{Z}_r(t) : -\xi_- \leq t \leq \xi + \xi_+\big)\Big), \end{aligned} \tag{3.6.13}$$

where $\sim$ stands for equality in distribution (in essence all we have done so far is isolate the middle segment and relabel). So, in order to prove (3.6.7) we have to prove some subtle estimates for the processes $\widetilde{Z}_r$ amd $\widetilde{Z}_r$. The main estimates are collected in Proposition 3.6.1 below.

Proposition 3.6.1. *There exists a constant $C < \infty$, such that for all $r < 1$ and $s \in (0, \infty)$, the following bounds hold:*

$$\mu\left((u,h,v) \in \widehat{\mathbb{A}}_r : \angle(-e, \widehat{w}_r) < s\right) \leq Cr \min\{s, 1\}, \tag{3.6.14}$$

$$\mu\left((u,h,v) \in \widetilde{\mathbb{A}}_r : \angle(-e, \widetilde{w}_r) < s\right) \leq Cr \min\{s(|\log s| \vee 1), 1\} \tag{3.6.15}$$

$$\mu\left((u,h,v) \in \widetilde{\mathbb{A}}_r : r^{-1}\widetilde{\beta}_r > s\right) \leq Cr \min\{s^{-1}(|\log s| \vee 1), 1\}. \tag{3.6.16}$$

Remarks: The bound (3.6.14) is sharp in the sense that a lower bound of the same order can be proved. In contrast, we think that the upper bound in (3.6.15) is not quite sharp. However, it is sufficient for our purposes so we do not strive for a better estimate.

The following consequence of Proposition 3.6.1 will be used to prove (3.6.3).

Corollary 3.6.2. *There exists a constant $C < \infty$ such that the following bounds hold:*

$$\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap \{\min_{\tau_{k-2} \leq t \leq \tau_k} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right| < s\} \mid \underline{\epsilon}\right) \leq Crs(|\log s| \vee 1), \tag{3.6.17}$$

$$\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap \{\min_{\tau_{k-3} \leq t \leq \tau_{k-1}} \left|Z^{(k)}(t) - Z^{(k)}(\tau_k)\right| < s\} \mid \underline{\epsilon}\right) \leq Crs(|\log s| \vee 1), \tag{3.6.18}$$

$$\begin{aligned} &\mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\min_{\tau_{k-2} \leq t \leq \tau_k} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right| < s\} \mid \underline{\epsilon}\right) \\ &\qquad\qquad \leq Cr \max\{s \left|\log s\right|^2, r \left|\log r\right|^2\} \end{aligned} \tag{3.6.19}$$

$$\begin{aligned} &\mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\min_{\tau_{k-3} \leq t \leq \tau_{k-1} + \widetilde{\beta}} \left|Z^{(k)}(t) - Z^{(k)}(\tau_k)\right| < s\} \mid \underline{\epsilon}\right) \\ &\qquad\qquad \leq Cr \max\{s \left|\log s\right|^2, r \left|\log r\right|^2\} \end{aligned} \tag{3.6.20}$$

Proposition 3.6.1 and its Corollary 3.6.2 are proved in Sections 3.6.5, respectively, 3.6.6.

3.6.5 Geometric Estimates Ctd: Proof of Proposition 3.6.1

Preparations

Beside the probability measure μ (see (3.6.11)) we will also need the flat Lebesgue measure on $\mathbb{D}$,

$$\lambda = UNI(S_1^2) \times LEB(\mathbb{R}_+) \times UNI(S_1^2),$$

so that

$$d\mu(u,h,v) = \frac{e^{1-h}}{e-1}\mathbb{1}\{0 \le h < 1\} d\lambda(u,h,v).$$

For $r > 0$ we define the *dilation map* $D_r : \mathbb{D} \to \mathbb{D}$ as

$$D_r(u,h,v) = (u, rh, v),$$

and note that

$$\widehat{\mathbb{A}}_r = D_r \widehat{\mathbb{A}}_1 \qquad\qquad \widetilde{\mathbb{A}}_r = D_r \widetilde{\mathbb{A}}_1.$$

In the forthcoming steps all events in $\widehat{\mathbb{A}}_r$ and $\widetilde{\mathbb{A}}_r$ will be mapped by the inverse dilation $D_r^{-1} = D_{r^{-1}}$ into $\widehat{\mathbb{A}}_1$, respectively, $\widetilde{\mathbb{A}}_1$. Therefore, in order to simplify notation we will use $\widehat{\mathbb{A}} := \widehat{\mathbb{A}}_1$ and $\widetilde{\mathbb{A}} := \widetilde{\mathbb{A}}_1$.

The dilation D_r transforms the measures μ as follows. Given an event $E \subset \mathbb{D}$,

$$\mu(D_r E) = \int_{D_r E} \frac{e^{1-h}}{e-1}\mathbb{1}\{0 \le h \le 1\} d\lambda(u,h,v) = r \int_E \frac{e^{1-rh}}{e-1}\mathbb{1}\{0 \le h \le r^{-1}\} d\lambda(u,h,v), \qquad (3.6.21)$$

and hence, for any event $E \subset \mathbb{D}$ and any $\bar{h} < \infty$

$$\frac{e^{1-r\bar{h}}}{e-1} r\lambda(E \cap \{h \le \bar{h}\}) \le \mu(D_r E) \le \frac{e}{e-1} r\lambda(E). \qquad (3.6.22)$$

The following simple observation is of paramount importance in the forthcoming arguments:

Proposition 3.6.3. *In dimension* 3 *(and more)*

$$\lambda(\widehat{\mathbb{A}}) = \lambda(\widetilde{\mathbb{A}}) < \infty. \qquad (3.6.23)$$

Proof of Proposition 3.6.3. Note that, by some simple geometric inspection,

$$\widehat{\mathbb{A}} \subset \widehat{\mathbb{A}}' := \{(u,h,v) \in \mathbb{D} : \angle(-e,u) \le 2h^{-1}\},$$
$$\widetilde{\mathbb{A}} \subset \widetilde{\mathbb{A}}' := \{(u,h,v) \in \mathbb{D} : \angle(-u,v) \le 2h^{-1}\}.$$

Since, in dimension 3,

$$\left|\{(u,v) \in S^2 \times S^2 : \angle(-e,u) < 2h^{-1}\}\right| =$$
$$\left|\{(u,v) \in S^2 \times S^2 : \angle(-u,v) < 2h^{-1}\}\right| \le C \min\{h^{-2}, 1\},$$

the claim follows by integrating over $h \in \mathbb{R}_+$. □

Remark: In 2-dimension, the corresponding sets $\widehat{\mathbb{A}}$, $\widetilde{\mathbb{A}}$ have infinite Lebesgue measure and, therefore, a similar proof would fail.

Due to (3.6.23) in 3-dimensions the following *conditional probability measures* make sense

$$\lambda_{\widehat{\mathbb{A}}}(\cdot) = \lambda(\cdot \,|\widehat{\mathbb{A}}) := \frac{\lambda(\cdot \cap \widehat{\mathbb{A}})}{\lambda(\widehat{\mathbb{A}})}, \qquad \lambda_{\widetilde{\mathbb{A}}}(\cdot) = \lambda(\cdot \,|\widetilde{\mathbb{A}}) := \frac{\lambda(\cdot \cap \widetilde{\mathbb{A}})}{\lambda(\widetilde{\mathbb{A}})},$$

and, moreover, due to (3.6.22) and (3.6.23), for any event $E \in \mathbb{D}$

$$\lim_{r\to 0} \mu(D_r E \,|\, \widehat{\mathbb{A}}_r) = \lambda_{\widehat{\mathbb{A}}}(E), \qquad \lim_{r\to 0} \mu(D_r E \,|\, \widetilde{\mathbb{A}}_r) = \lambda_{\widetilde{\mathbb{A}}}(E),$$

In a technical sense, we will only use the upper bound in (3.6.22), and (3.6.23).

In view of the upper bound in (3.6.22), in order to prove (3.6.14), (3.6.15) and (3.6.16) we need, in turn,

$$\lambda\Big((u,h,v) \in \widehat{\mathbb{A}} : \angle(-e, \widehat{w}) \le s\Big) \le C \min\{s, 1\}, \tag{3.6.24}$$

$$\lambda\Big((u,h,v) \in \widetilde{\mathbb{A}} : \angle(-e, \widetilde{w}) \le s\Big) \le C \min\{s(|\log s| \vee 1), 1\}, \tag{3.6.25}$$

$$\lambda\Big((u,h,v) \in \widetilde{\mathbb{A}} : \widetilde{\beta} > s\Big) \le C \min\{s^{-1}(|\log s| \vee 1), 1\}. \tag{3.6.26}$$

Here, and in the rest of this section, we use the simplified notation $\widehat{w} := \widehat{w}_1$, $\widetilde{w} := \widetilde{w}_1$, $\widetilde{\beta} := \widetilde{\beta}_1$.

Proof of (3.6.24)

Proof. This is straightforward. Recall (3.6.12): $\widehat{w}(u,h,v) = u$. For easing notation let

$$\vartheta := \angle(-e, u)$$

and note that for any $t \in \mathbb{R}_+$

$$\big|\{u \in S_1^2 : 0 \le \vartheta \le t\}\big| \le C \min\{t^2, 1\},$$

with some explicit $C < \infty$.

Then,

$$\begin{aligned}
\lambda\Big((u,h,v) \in \widehat{\mathbb{A}} : \angle(-e, \widehat{w})) \le s\Big) &\le \lambda\Big((u,h,v) \in \widehat{\mathbb{A}}' : \vartheta \le s\Big) \\
&\le \lambda\left((u,h,v) \in \mathbb{D} : \vartheta \le \min\{s, 2h^{-1}\}\right) \\
&= \lambda\left((u,h,v) \in \mathbb{D} : \{h \le 2s^{-1}\} \cap \{\vartheta \le s\}\right) + \lambda\left((u,h,v) \in \mathbb{D} : \{h \ge 2s^{-1}\} \cap \{\vartheta \le 2h^{-1}\}\right) \\
&\le Cs.
\end{aligned}$$

□

Proof of (3.6.25) **and** (3.6.26)

Figure 3 aides in understanding this subsection.

Let a and b be the vectors in $\mathbb{R}^3$ pointing from the origin to the centre of the spherical scatterers of radius 1, on which the first, respectively, the second collisions occur:

$$a = \frac{e-u}{|e-u|}, \qquad b = hu + \frac{u-v}{|u-v|},$$

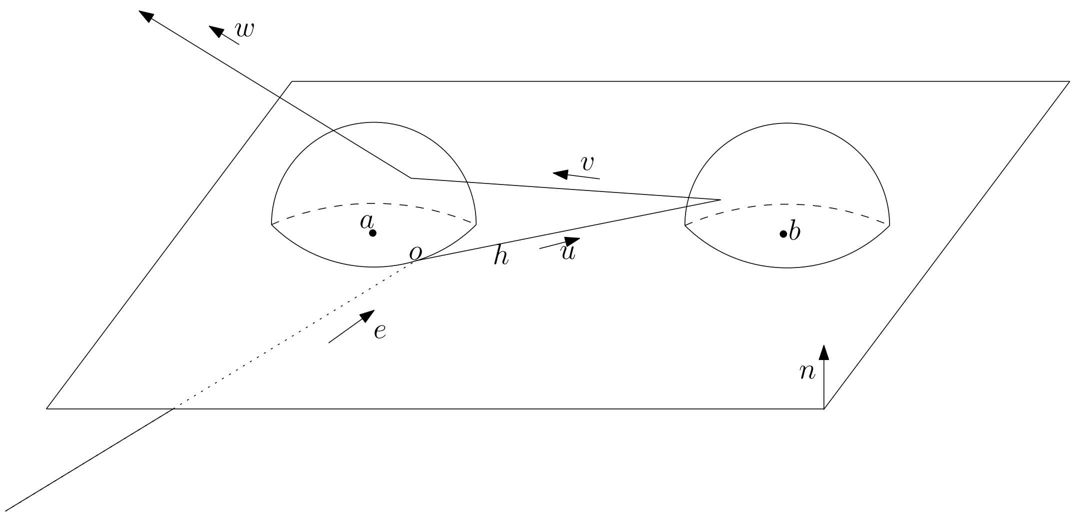

Figure 3.3: Above we show a 3 dimensional example of the geometric labelling used in this section. The Z trajectory enters with velocity e from beneath the relevant plane (the dotted line represents motion below the plane). After which the particle remains above the plane.

and n the unit vector orthogonal to the plane determined by a and b, pointing so, that $e \cdot n > 0$:

$$n := \frac{a \times b}{|a|\,|b| \sin(\angle(a,b))},$$

with

$$a \times b = (h + \frac{1}{|u-v|})\frac{1}{|e-u|}\; e \times u - \frac{1}{|e-u|\,|u-v|}\; e \times v + \frac{1}{|e-u|\,|u-v|}\; u \times v, \tag{3.6.27}$$

$$|a| = 1, \qquad h - 1 \le |b| \le h + 1, \qquad 0 \le \sin(\angle(a,b)) \le 1. \tag{3.6.28}$$

Assume there are altogether $\nu \ge 3$ collisions (which occur alternatively, on the first and second scatterer) before escape. Let $w_0 = e$ and w_j, $1 \le j \le \nu$, the outgoing velocity after the j-th scattering. So, $w_1 = u, w_2 = v, \ldots, w_\nu = \widetilde{w}$.

The proof of (3.6.25) and (3.6.26) relies on the following observations:

(a) The n-projection of the velocity of the moving particle does not decrease. More precisely, for $1 \le j \le \nu$, $0 \le w_{j-1} \cdot n \le w_j \cdot n$. This is due to the choice of the plane determined by the centres of the two scatterers and the first impact point.

(b) Since $e \cdot n > 0$ and $w_j \cdot n > 0$, for all $1 \le j \le \nu$ we have $\angle(-e, w_j) > \frac{\pi}{2} - \angle(n, w_j)$.

(c) The trapping time $\widetilde{\beta}$ is certainly not longer than the time the moving particle spends in the slab $\{x \in \mathbb{R}^3 : 0 \le x \cdot n \le 1\}$. In particular, it follows that

$$\widetilde{\beta} \le h + |v \cdot n|^{-1} \le |u \cdot n|^{-1} = |e \cdot n|^{-1}. \tag{3.6.29}$$

Proof of (3.6.25). Without loss of generality we may assume $s \le \frac{\pi}{2}$.

From the arguments (a) and (b) above it follows, in particular, that

$$\angle(-e, \widetilde{w}) = \angle(-e, w_\nu) \ge \frac{\pi}{2} - \angle(n, w_\nu) \ge \frac{\pi}{2} - \angle(n, w_2) = \frac{\pi}{2} - \angle(n, v),$$

and hence

$$\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}:\angle(-e,\widetilde{w})<s\Big)\leq\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}':|n\cdot w|<2s\Big). \tag{3.6.30}$$

Note that due to (3.6.27) and (3.6.28)

$$|v\cdot n|\geq\frac{1}{2}\,|v\cdot(e\times u)|\,,$$

and thus

$$\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}':|v\cdot n|<2s\Big)\leq\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}':|e\cdot(u\times v)|<4s\Big). \tag{3.6.31}$$

Next, if u and v are i.i.d. $UNI(S_1^2)$-distributed then

$$w:=\frac{u\times v}{|u\times v|},\qquad\text{and}\qquad\vartheta:=|u\times v|=\sin(\angle(u,v))$$

are independent and distributed as

$$w\sim UNI(S_1^2),\qquad\qquad\vartheta\sim\mathbb{1}_{\{0\leq t\leq 1\}}(1-t^2)^{-1/2}tdt.$$

Therefore,

$$\begin{aligned}\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}':|e\cdot(u\times v)|<4s\Big)&=\int_0^\infty dh\int_{S^2}dw\int_0^{\min\{2/h,1\}}(1-t^2)^{-1/2}tdt\mathbb{1}\{|e\cdot w|\leq\frac{4s}{t}\}\\&=\int_0^\infty dh\int_0^{\min\{2/h,1\}}(1-t^2)^{-1/2}dt\min\{4s,t\}\\&\leq C\min\{s\,|\log s|\vee 1),1\}.\end{aligned} \tag{3.6.32}$$

The last step follows from explicit computations which we omit.

Finally, (3.6.30), (3.6.31) and (3.6.32) yield (3.6.25). □

Proof of (3.6.26). We proceed with the first (sharper) bound in (3.6.29) (the second (weaker) bound would yield only upper bound of order $s^{-1/2}$ on the right hand side of (3.6.25)):

$$\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}:\widetilde{\beta}>s\Big)\leq\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}':h>\frac{s}{2}\Big)+\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}':|v\cdot n|<\frac{2}{s}\Big). \tag{3.6.33}$$

Bounding the first term on the right hand side of (3.6.33) is straightforward:

$$\begin{aligned}\lambda\Big((u,h,v)\in\widetilde{\mathbb{A}}':h>\frac{s}{2}\Big)&=\int_{s/2}^\infty\big|\{(u,v)\in S_1^2\times S_1^2:\angle(-u,v)<2h^{-1}\}\big|\,dh\\&\leq C\int_{s/2}^\infty\min\{h^{-2},1\}dh\leq C\min\{s^{-1},1\}.\end{aligned} \tag{3.6.34}$$

Concerning the second term on the right hand side of (3.6.33), this has exactly been done in the proof of (3.6.25) above, ending in (3.6.32) – with the rôle of s and s^{-1} swapped.

(3.6.33), (3.6.34) and (3.6.32) yield (3.6.16).

□

3.6.6 Geometric Estimates Ctd: Proof of Corollary 3.6.2

We start with the following straightforward geometric fact.

Lemma 3.6.4. *Let $e, w \in S_1^2$ and $x \in \mathbb{R}^3$. Then*

$$\left|\{t' > 0 : \min_{t\geq 0} |x + t'w + te| < s\}\right| = \left|\{t' > 0 : \min_{t\geq 0} |x + tw + t'e| < s\}\right| \leq \frac{4s}{\angle(-e, w)}. \tag{3.6.35}$$

Proof of Lemma 3.6.4. This is elementary 3-dimensional geometry. We omit the details. □

Proof of (3.6.17) *and* (3.6.18). On $\{\widehat{\eta}_k = 1\}$

$$\begin{aligned}\min_{\tau_{k-2}\leq t\leq \tau_k} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right| &\geq \min_{t\geq 0} |tu_{k-1} + \xi_{k-2}u_{k-2}| \\ \min_{\tau_{k-3}\leq t\leq \tau_{k-1}} \left|Z^{(k)}(t) - Z^{(k)}(\tau_k)\right| &\geq \min\{\min_{t\geq 0} |\xi_{k-1}u_{k-1} + tu_{k-2} + \xi_k u_{k-1}|, \xi_k\}.\end{aligned} \tag{3.6.36}$$

The bounds in (3.6.17) and (3.6.18) follow from applying (3.6.35) and (3.6.14), bearing in mind that the distribution density of ξ_{k-2} and ξ_k is bounded. Since these are very similar we will only prove (3.6.17) here.

$$\begin{aligned}&\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap \{\min_{\tau_{k-2}\leq t\leq \tau_k} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right| < s\}\right) \\ &\qquad\qquad \leq \mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap \{\min_{t\geq 0} |tu_{k-1} + \xi_{k-2}u_{k-2}| < s\}\right) \\ &\qquad\qquad = \int_{\widehat{\mathbb{A}}_r} \mathbf{P}\left(\xi_- \in \{t' : \min_{t\geq 0} |tu + t'e| < s\}\right) d\mu(u, h, v) \\ &\qquad\qquad \leq C \int_{\widehat{\mathbb{A}}_r} \min\{\frac{s}{\angle(-e, u)}, 1\} d\mu(u, h, v) \\ &\qquad\qquad \leq Crs(|\log s| \vee 1).\end{aligned}$$

In the first step we used (3.6.36). The second step follows from the representation (3.6.13). The third step relies on (3.6.35) and on uniform boundedness of the distribution density of ξ_- (which is either $EXP(1|1)$ or $EXP(1|0)$, depending on the value of ϵ_{k-2}). Finally, the last calculation is based on (3.6.14).

□

Proof of (3.6.19).

$$\begin{aligned}&\min_{\tau_{k-2}\leq t\leq \tau_k} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right| \\ &\qquad = \min\left\{\min_{\tau_{k-2}\leq t\leq \tau_{k-1}+\widetilde{\beta}} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right|, \min_{\tau_{k-1}+\widetilde{\beta}\leq t\leq \tau_k} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right|\right\}.\end{aligned} \tag{3.6.37}$$

Here, and in the rest of this proof, $\widetilde{\beta}$ and $\widetilde{w}$ denote the trapping time and escape direction of the recollision sequence:

$$\widetilde{\beta} := \max\{s \leq \xi_k : \dot{Z}^{(k)}(\tau_{k-1} + s^-) \neq \dot{Z}^{(k)}(\tau_{k-1} + s^+)\} \qquad \widetilde{w} := \dot{Z}^{(k)}(\tau_{k-1} + \widetilde{\beta}^+).$$

To bound the first expression on the right hand side of (3.6.37) we first observe that by the triangle inequality

$$\min_{\tau_{k-2}\leq t\leq \tau_{k-1}+\widetilde{\beta}} \left|Z^{(k)}(t) - Z^{(k)}(\tau_{k-3})\right| \geq \xi_{k-2} - \xi_{k-1} - 4r. \tag{3.6.38}$$

Applying the representation and bounds developed in sections 3.6.4, 3.6.5,

$$\mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\min_{\tau_{k-2} \le t \le \tau_{k-1} + \widetilde{\beta}} \left| Z^{(k)}(t) - Z^{(k)}(\tau_{k-3}) \right| < s\}\right)$$
$$\begin{aligned} &\le \mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\xi_{k-2} \le \xi_{k-1} + 4r + s\}\right) \\ &= \int_{\widetilde{\mathbb{A}}_r} \mathbf{P}\left(\xi_- < h + 4r + s\right) d\mu(u,h,v) \\ &\le C \int_{\widetilde{\mathbb{A}}_r} (\min\{h,1\} + 4r + s) d\mu(u,h,v) \\ &\le Cr^2 + Crs + Cr^2 \left|\log r\right|. \end{aligned} \tag{3.6.39}$$

In the first step we used (3.6.38). The second step follows from the representation (3.6.13). The third step relies on on uniform boundedness of the distribution density of ξ_- (which is either $EXP(1|1)$ or $EXP(1|0)$, depending on the value of ϵ_{k-2}). Finally, the last step follows from explicit calculation, using (3.6.22).

To bound the second term on the right hands side of (3.6.37) we proceed as in the proof of (3.6.17) above. First note that

$$\min_{\tau_{k-1} + \widetilde{\beta} \le t \le \tau_k} \left| Z^{(k)}(t) - Z^{(k)}(\tau_{k-3}) \right| \ge \min_{0 \le t} \left| (Z^{(k)}(\tau_{k-2}) - Z^{(k)}(\tau_{k-1} + \widetilde{\beta})) + t\widetilde{w} + \xi_{k-2} u_{k-2} \right|. \tag{3.6.40}$$

Using in turn (3.6.40), (3.6.13), (3.6.35) and uniform boundedness of the distribution density of ξ_- (which is either $EXP(1|1)$ or $EXP(1|0)$, depending on the value of ϵ_{k-2}), and finally (3.6.15), we obtain:

$$\mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \min_{\tau_{k-1} + \widetilde{\beta} \le t \le \tau_k} \left| Z^{(k)}(t) - Z^{(k)}(\tau_{k-3}) \right| < s\right)$$
$$\begin{aligned} &\le \mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\min_{0 \le t} \left| (Z^{(k)}(\tau_{k-2}) - Z^{(k)}(\tau_{k-1} + \widetilde{\beta})) + t\widetilde{w} + \xi_{k-2} u_{k-2} \right| < s\}\right) \\ &= \int_{\widetilde{\mathbb{A}}_r} \mathbf{P}\left(\xi_- \in \{t' : \min_{0 \le t} \left| \widetilde{Z}_r(\widetilde{\beta}_r) + t\widetilde{w}_r + t' e \right| < s\}\right) d\mu(u,h,v) \\ &\le C \int_{\widetilde{\mathbb{A}}_r} \min\{\frac{s}{\angle(-e, \widetilde{w}_r)}, 1\} d\mu(u,h,v) \\ &\le Crs(\left|\log s\right|^2 \vee 1). \end{aligned} \tag{3.6.41}$$

From (3.6.37), (3.6.39) and (3.6.41) we obtain (3.6.19). □

Proof of (3.6.20). We proceed very similarly as in the proof of (3.6.19).

$$\begin{aligned} &\min_{\tau_{k-3} \le t \le \tau_{k-1} + \widetilde{\beta}} \left| Z^{(k)}(t) - Z^{(k)}(\tau_k) \right| \\ &\qquad \ge \min\left\{ \min_{\tau_{k-2} \le t \le \tau_{k-1} + \widetilde{\beta}} \left| Z^{(k)}(t) - Z^{(k)}(\tau_k) \right|, \min_{\tau_{k-3} \le t \le \tau_{k-2}} \left| Z^{(k)}(t) - Z^{(k)}(\tau_k) \right| \right\}. \end{aligned} \tag{3.6.42}$$

To bound the first expression on the right hand side of (3.6.42) we first observe that by the triangle inequality

$$\min_{\tau_{k-2} \le t \le \tau_{k-1} + \widetilde{\beta}} \left| Z^{(k)}(t) - Z^{(k)}(\tau_k) \right| \ge \xi_k - 2\widetilde{\beta} - 4r \tag{3.6.43}$$

Using in turn (3.6.43), (3.6.13), (3.6.16) and explicit computation based on uniform boundedness of the distribution density of ξ_+ (which is either $EXP(1|1)$ or $EXP(1|0)$, depending on the value of ϵ_k) we write

$$
\begin{aligned}
&\mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{ \min_{\tau_{k-2} \le t \le \tau_{k-1}+\widetilde{\beta}} \left| Z^{(k)}(t) - Z^{(k)}(\tau_k) \right| < s\}\right) \\
&\qquad \le \mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\xi_k < 8r + 2s\}\right) + \mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \{\xi_k < 4\widetilde{\beta}\}\right) \\
&\qquad = \mathbf{P}\left(\xi_+ < 8r + 2s\right) \mu(\widetilde{\mathbb{A}}_r) + \mathbf{E}\left(\mu((u,h,v) \in \widetilde{\mathbb{A}}_r : \xi_+ \le 4\widetilde{\beta}_r)\right) \\
&\qquad \le Cr(r+s) + Cr\mathbf{E}\left(\min\{\left(\frac{\xi_+}{2r}\right)^{-1}\left(\left|\log \frac{\xi_+}{2r}\right| \vee 1\right), 1\}\right) \\
&\qquad \le Cr^2 + Crs + Cr^2 \left|\log r\right|^2 .
\end{aligned} \tag{3.6.44}
$$

The second term on the right hand side of (3.6.42) is bounded in a very similar way as the analogous second term on the right hand side of (3.6.37), see (3.6.40)-(3.6.41). Without repeating these details we state that

$$
\mathbf{P}\left(\{\widehat{\eta}_k = 0\} \cap \{\widetilde{\eta}_k = 1\} \cap \min_{\tau_{k-2} \le t \le \tau_{k-1}} \left| Z^{(k)}(t) - Z^{(k)}(\tau_k) \right| < s\right) \le Crs \left|\log s\right|^2 . \tag{3.6.45}
$$

Eventually, from (3.6.42), (3.6.44) and (3.6.45) we obtain (3.6.20).

□

3.6.7 Proof of (3.6.3) – Concluded

Recall the events $A^{(k)}_{a,b}$, $a, b \in \{1, 2, 3\}$ from the end of section 3.6.3.

The bounds (3.6.17), (3.6.18), respectively, (3.6.19), (3.6.20), with $s = r$, directly imply

$$
\begin{aligned}
&\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{2,2} \mid \underline{\epsilon}\right) \le C\gamma r^2 \left|\log r\right| , \\
&\mathbf{P}\left(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{2,2} \mid \underline{\epsilon}\right) \le C\gamma r^2 \left|\log r\right|^2 .
\end{aligned} \tag{3.6.46}
$$

It remains to prove

$$
\left.
\begin{aligned}
&\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{b,2} \mid \underline{\epsilon}\right) \le C\gamma r^2 \left|\log r\right| , \\
&\mathbf{P}\left(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{b,2} \mid \underline{\epsilon}\right) \le C\gamma r^2 \left|\log r\right|^2 ,
\end{aligned}
\right. \qquad b = 1, 3. \tag{3.6.47}
$$

Since the cases $b = 1$ and $b = 3$ are formally identical we will go through the steps of proof with $b = 3$ only. In order to do this we first define the necessary occupation time measures (Green's functions). For $A \subset \mathbb{R}^3$, define the following occupation time measures for the last part of (3.6.8)

$$
\begin{aligned}
G^{(k)}_{\underline{\epsilon}}(A) :=& \mathbf{E}\left(\#\{1 \le j \le \gamma - k : Y(\tau_j) \in A\} \mid \epsilon_{k+j} : 1 \le j \le \gamma - k\right) \\
=& \mathbf{E}\left(\#\{k+1 \le j \le \gamma : Z^{(k)}(\tau_j) - Z^{(k)}(\tau_k) \in A\} \mid \underline{\epsilon} \cap \{\widehat{\eta}_k = 1\}\right) \\
=& \mathbf{E}\left(\#\{k+1 \le j \le \gamma : Z^{(k)}(\tau_j) - Z^{(k)}(\tau_k) \in A\} \mid \underline{\epsilon} \cap \{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\}\right) , \\
H^{(k)}_{\underline{\epsilon}}(A) :=& \mathbf{E}\left(\left|\{0 \le t \le \tau_{\gamma-k} : Y(t) \in A\}\right| \mid \epsilon_{k+j} : 1 \le j \le \gamma - k\right) \\
=& \mathbf{E}\left(\left|\{\tau_k \le t \le \theta : Z^{(k)}(t) - Z^{(k)}(\tau_k) \in A\}\right| \mid \underline{\epsilon} \cap \{\widehat{\eta}_k = 1\}\right) \\
=& \mathbf{E}\left(\left|\{\tau_k \le t \le \theta : Z^{(k)}(t) - Z^{(k)}(\tau_k) \in A\}\right| \mid \underline{\epsilon} \cap \{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\}\right) .
\end{aligned}
$$

Similarly, define the following occupation time measures for the middle part of (3.6.8)

$$\begin{aligned}
\widehat{G}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\#\{1 \le j \le 3 : Z^{(k)}(\tau_{k-j}) - Z^{(k)}(\tau_k) \in A\} \cdot \widehat{\eta}_k \mid \underline{\epsilon}\right)\\
\widehat{H}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\left|\{\tau_{k-3} \le t \le \tau_k : Z^{(k)}(t) - Z^{(k)}(\tau_k) \in A\}\right| \cdot \widehat{\eta}_k \mid \underline{\epsilon}\right)\\
\widetilde{G}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\#\{1 \le j \le 3 : Z^{(k)}(\tau_{k-j}) - Z^{(k)}(\tau_k) \in A\} \cdot \widetilde{\eta}_k \cdot (1-\widehat{\eta}_k) \mid \underline{\epsilon}\right)\\
\widetilde{H}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\left|\{\tau_{k-3} \le t \le \tau_k : Z^{(k)}(t) - Z^{(k)}(\tau_k) \in A\}\right| \cdot \widetilde{\eta}_k \cdot (1-\widehat{\eta}_k) \mid \underline{\epsilon}\right).
\end{aligned}$$

Using the independence of the middle and last parts in the decomposition (3.6.8), similarly as (3.3.2) or (3.5.2), following bounds are obtained

$$\begin{aligned}
&\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{3,2} \mid \underline{\epsilon}\right) \le Cr^{-1}\int_{\mathbb{R}^3} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widehat{H}^{(k)}_{\underline{\epsilon}}(dx) + Cr^{-1}\int_{\mathbb{R}^3} H^{(k)}_{\underline{\epsilon}}(B_{x,3r})\widehat{G}^{(k)}_{\underline{\epsilon}}(dx)\\
&\mathbf{P}\left(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{3,2} \mid \underline{\epsilon}\right) \le \\
&\qquad \le Cr^{-1}\int_{\mathbb{R}^3} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widetilde{H}^{(k)}_{\underline{\epsilon}}(dx) + Cr^{-1}\int_{\mathbb{R}^3} H^{(k)}_{\underline{\epsilon}}(B_{x,3r})\widetilde{G}^{(k)}_{\underline{\epsilon}}(dx).
\end{aligned} \tag{3.6.48}$$

Due to (3.3.8) of Lemma 3.3.2 by direct computations the following upper bounds hold

$$G^{(k)}_{\underline{\epsilon}}(B_{x,2r}) \le CF(|x|), \qquad H^{(k)}_{\underline{\epsilon}}(B_{x,3r}) \le CF(|x|), \tag{3.6.49}$$

where $C < \infty$ is an appropriately chosen constant and $F : \mathbb{R}_+ \to \mathbb{R}$,

$$F(u) := r\mathbb{1}\{0 \le u < r\} + \frac{r^3}{u^2}\mathbb{1}\{r \le u < 1\} + \frac{r^3}{u}\mathbb{1}\{1 \le u < \infty\}.$$

On the other hand, from (3.6.17), (3.6.18), (3.6.19), (3.6.20) of Corollary 3.6.2 follows that

$$\begin{aligned}
&\widehat{G}^{(k)}_{\underline{\epsilon}}(B_{0,s}) \le Crs(|\log s| \vee 1), && \widehat{H}^{(k)}_{\underline{\epsilon}}(B_{0,s}) \le Crs(|\log s| \vee 1),\\
&\widetilde{G}^{(k)}_{\underline{\epsilon}}(B_{0,s}) \le Cr\max\{s\left|\log s\right|^2, r\left|\log r\right|^2\} && \widetilde{H}^{(k)}_{\underline{\epsilon}}(B_{0,s}) \le Cr\max\{s\left|\log s\right|^2, r\left|\log r\right|^2\}.
\end{aligned} \tag{3.6.50}$$

Finally, we also have the global bounds

$$\begin{aligned}
&\widehat{G}^{(k)}_{\underline{\epsilon}}(\mathbb{R}^3) = 3\mathbf{E}\left(\widehat{\eta}_k \mid \underline{\epsilon}\right) \le Cr, && \widehat{H}^{(k)}_{\underline{\epsilon}}(\mathbb{R}^3) = \mathbf{E}\left(\widehat{\eta}_k \cdot \sum_{j=k-2}^{k} \xi_j \mid \underline{\epsilon}\right) \le Cr,\\
&\widetilde{G}^{(k)}_{\underline{\epsilon}}(\mathbb{R}^3) = 3\mathbf{E}\left(\widetilde{\eta}_k \cdot (1-\widehat{\eta}_k) \mid \underline{\epsilon}\right) \le Cr, && \widetilde{H}^{(k)}_{\underline{\epsilon}}(\mathbb{R}^3) = \mathbf{E}\left(\widetilde{\eta}_k \cdot (1-\widehat{\eta}_k) \cdot \sum_{j=k-2}^{k} \xi_j \mid \underline{\epsilon}\right) \le Cr.
\end{aligned} \tag{3.6.51}$$

We will prove the upper bound (3.6.47) for the first term on the right hand side of the first line in (3.6.48). The other four terms are done in very similar way.

First we split the integral as

$$\int_{\mathbb{R}^3} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widehat{H}^{(k)}_{\underline{\epsilon}}(dx) = \int_{|x|<1} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widehat{H}^{(k)}_{\underline{\epsilon}}(dx) + \int_{|x|\ge 1} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widehat{H}^{(k)}_{\underline{\epsilon}}(dx) \tag{3.6.52}$$

and note that due to (3.6.49) and (3.6.51) the second term on the right hand side is bounded as

$$\int_{|x|\ge 1} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widehat{H}^{(k)}_{\underline{\epsilon}}(dx) \le Cr^4. \tag{3.6.53}$$

To bound the first term on the right hand side of (3.6.52) we proceed as follows

$$\begin{aligned}\int_{|x|<1} G_{\underline{\epsilon}}^{(k)}(B_{x,2r})\widehat{H}_{\underline{\epsilon}}^{(k)}(dx) &\leq C\int_0^1 F(u)d\widehat{H}_{\underline{\epsilon}}^{(k)}(B_{0,u}) \\ &= Cr^3\widehat{H}_{\underline{\epsilon}}^{(k)}(B_{0,1}) - C\int_0^1 \widehat{H}_{\underline{\epsilon}}^{(k)}(B_{0,u})F'(u)du \\ &\leq Cr^4 + Cr^4\int_r^1 u^{-2}\left|\log u\right| du \\ &\leq Cr^4 + Cr^3\left|\log r\right|. \end{aligned} \tag{3.6.54}$$

In the first step we have used (3.6.49). The second step is an integration by parts. In the third step we use (3.6.50), (3.6.51) and the explicit form of the function F. The last step is explicit integration.

Finally, (3.6.52), (3.6.53), (3.6.54) and identical comoputations for the second term on the right hand side of the first line in (3.6.48) yield the first inequality in (3.6.47). The second line of (3.6.47) for $b = 3$ is proved in an identical way, which we omit to repeat. The cases $b = 1$ is done in a formally identical way.

Finally, (3.6.3) follows from (3.6.10), (3.6.46) and (3.6.47).

□

3.7 Proof of Theorem 3.1.2 – Concluded

As in section 3.4.3 let $\varpi_n = (\gamma_n;\ (\xi_{n,j}, u_{n,j}) : 1 \leq j \leq \gamma_n)$, $n \geq 1$, be a sequence of i.i.d *packs*. Denote θ_n, $((Y_n(t), Z_n(t)) : 0 \leq t \leq \theta_n)$ the pair of Y and (forward) Z-processes constructed from them and

$$Y(t) = \sum_{k=1}^{\nu_t} Y(\theta_n) + Y_{\nu_t+1}(\{t\}), \qquad Z(t) = \sum_{k=1}^{\nu_t} Z(\theta_n) + Z_{\nu_t+1}(\{t\}).$$

Beside these two we now define yet another auxiliary process $t \mapsto \mathcal{X}(t)$ as follows:
$(\mathcal{X}_n(t) : 0 \leq t \leq \theta_n)$ is the Lorentz exploration process constructed with data from
$(Y_n(t) : 0 \leq t \leq \theta_n)$ *and* incoming velocity

$$u_{n,0} = \begin{cases} u_0 & \text{if } n = 1, \\ \dot{\mathcal{X}}_{n-1}(\theta_{n-1}^-) & \text{if } n > 1. \end{cases}$$

Finally, from these legs concatenate

$$\mathcal{X}(t) = \sum_{k=1}^{\nu_t} \mathcal{X}(\theta_n) + \mathcal{X}_{\nu_t+1}(\{t\}).$$

Note that the auxiliary process $(\mathcal{X}(t) : 0 \leq t < \infty)$ is not identical with the Lorentz exploration process $(X(t) : 0 \leq t < \infty)$, constructed with data from $(Y(t) : 0 \leq t \leq \infty)$ and initial incoming velocity u_0, since the former one does not takes into account memory effects caused by earlier legs. However, based on Propositions 3.4.1 and 3.4.2, we will prove that until time $T = T(r) = o(r^{-2}\left|\log r\right|^{-2})$ the processes $t \mapsto X(t)$, $t \mapsto \mathcal{X}(t)$, and $t \mapsto Z(t)$ coincide with high probability.

For this, we define the (discrete) stopping times

$$\begin{aligned}\rho &:= \min\{n : \mathcal{X}_n(t) \not\equiv Z_n(t), 0 \leq t \leq \theta_n\} \\ \sigma &:= \min\{n : \max\{\mathbb{1}_{\widetilde{W}_n}, \mathbb{1}_{\widehat{W}_n} > 0\} = 1\},\end{aligned}$$

and note that by construction

$$\inf\{t : Z(t) \neq X(t)\} \geq \Theta_{\min\{\rho,\sigma\}-1}.$$

Lemma 3.7.1. *Let $T = T(r)$ such that $\lim_{r\to\infty} T(r) = \infty$ and $\lim_{r\to\infty} r^2 |\log r| T(r) = 0$. Then*

$$\lim_{r\to 0} \mathbf{P}\left(\Theta_{\min\{\rho,\sigma\}-1} < T\right) = 0. \tag{3.7.1}$$

Lemma 3.7.2. *Let $T = T(r)$ such that $\lim_{r\to\infty} T(r) = \infty$ and $\lim_{r\to\infty} r^2 T(r) = 0$. Then for any $\delta > 0$*

$$\lim_{r\to 0} \mathbf{P}\left(\max_{0\leq t\leq T} |Y(t) - Z(t)| > \delta\sqrt{T}\right) = 0. \tag{3.7.2}$$

Remark: Actually, (3.7.2) holds under the much weaker condition $\lim_{r\to\infty} r \log\log T = 0$. This can be achieved by applying the law of iterated logarithm rather than a weak law of large numbers type of argument to bound $\max_{0\leq t\leq T} |Y(t) - Z(t)|$ in the proof of Lemma 3.7.2, below. However, since the condition of Lemma 3.7.1 can not be much relaxed, in the end we would not gain much with the extra effort.

Proof of Lemma 3.7.1.

$$\begin{aligned}
\mathbf{P}\left(\Theta_{\min\{\rho,\sigma\}-1} < T\right) &\leq \mathbf{P}\left(\rho \leq 2\mathbf{E}(\theta)^{-1} T\right) + \mathbf{P}\left(\sigma \leq 2\mathbf{E}(\theta)^{-1} T\right) + \mathbf{P}\left(\sum_{j=1}^{2\mathbf{E}(\theta)^{-1}T} \theta_j < T\right) \\
&\leq Cr^2 |\log r| T + Cr^2 T + Ce^{-cT},
\end{aligned} \tag{3.7.3}$$

where $C < \infty$ and $c > 0$. The first term on the right hand side of (3.7.3) is bounded by union bound and (3.4.10) from Proposition 3.4.1. Likewise, the second term is bounded by union bound and (3.4.12) of Propositions 3.4.2. In bounding the third term we use a large deviation upper bound for the sum of independent θ_j-s.

Finally, (3.7.1) readily follows from (3.7.3). □

Proof of Lemma 3.7.2. Note first that

$$\max_{0\leq t\leq T} |Y(t) - Z(t)| \leq \sum_{j=1}^{\nu_T+1} \eta_j \xi_j,$$

with ν_T and η_j defined in (3.2.4), respectively, (3.2.7). Hence,

$$\begin{aligned}
\mathbf{P}\left(\max_{0\leq t\leq T} |Y(t) - Z(t)| > \delta\sqrt{T}\right) &\leq \mathbf{P}\left(\sum_{j=1}^{2T} \eta_j \xi_j > \delta\sqrt{T}\right) + \mathbf{P}(\nu_T > 2T) \\
&\leq C\delta^{-1}\sqrt{T} r + e^{-cT},
\end{aligned} \tag{3.7.4}$$

with $C < \infty$ and $c > 0$. The first term on the right hand side of (3.7.4) is bounded by Markov's inequality and the straightforward bound

$$\mathbf{E}(\eta_j \xi_j) \leq Cr.$$

The bound on the second term follows from a straightforward large deviation estimate on $\nu_T \sim POI(T)$.

Finally, (3.7.2) readily follows from (3.7.4).

□

(3.1.8) is direct consequence of Lemmas 3.7.1 and 3.7.2 and this concludes the proof of Theorem 3.1.2. □

Chapter 4

Random Wind-Tree

– Joint with Bálint Tóth –

4.1 Introduction

In this chapter we return to the *random wind-tree* model discussed in Chapter 2 Section 2.3. That is, we consider the motion of a point particle through an array of randomly placed, identically oriented cubes in $\mathbb{R}^3$ [EE59]. In the previous chapter we showed that the random Lorentz gas satisfies an invariance principle in a scaling limit intermediate between the kinetic and diffusive scalings. In this chapter we will show that the wind-tree process satisfies a similar invariance principle in the same intermediate regime. The proof follows the same strategy, and the central ideas are present in the previous chapter (for completeness we will repeat some of the details). However there are two key differences: in the Lorentz gas, after collision with a randomly placed scatterer (in 3 dimensions) the velocity is redistributed independently of the initial velocity while for the wind-tree process the velocities form a genuine Markov chain. On the other hand as the collisions are simpler in the wind-tree setting the necessary geometric estimates follow with less effort.

Formally let $\mathcal{P}$ be a Poisson point process of intensity $\varrho > 0$ in $\mathbb{R}^3$. Our results hold for general dimension $d \geq 3$, however to reduce notation we restrict to $d = 3$. In dimension $d = 2$ because of the recurrence of the random walk the proof does not directly apply. Let $\mathcal{Q}_r$ be a cube of side length r oriented parallel with the axes and let $\mathcal{P} + \mathcal{Q}_r$ be an array of *obstacles/scatterers*. We consider the trajectory of a point particle $X^{r,\varrho}(t)$ starting at the origin ($X^{r,\varrho}(0) = 0$) with a fixed initial velocity of unit length. The particle then flies in straight lines, reflecting elastically off of the obstacles. In this setting the origin is in $(\mathcal{P} + \mathcal{Q}_r)^c$ with probability tending to 1, hence such a trajectory is well-defined (see Chapter 2 Lemma 2.5.1).

A fundamental open problem for both the random wind-tree model and the random Lorentz gas is to prove an invariance principle in the diffusive limit. That is, in the limit

$$\frac{X^{r,\varrho}(Tt)}{\sqrt{T}}, \qquad T \to \infty, \tag{4.1.1}$$

does the scaled process converge weakly to a Wiener process? In this chapter we show that the wind-tree process satisfies an invariance principle in the limit (4.1.1) if we *simultaneously* take the low-density limit in a particular scaling limit.

4.1.1 Scaling and Main Result

Fix a probability vector $\mathbf{p} = (p_1, p_2, p_3)$ with $p_i > 0$ for all i, and let $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. The state-space of velocities for the wind-tree process is then

$$\Omega := \left\{ v \in S_1^2 : |v_i| = \frac{p_i}{|\mathbf{p}|} \right\} \tag{4.1.2}$$

Fix the initial velocity $\dot{X}^{r,\varrho}(0^+) \in \Omega$. We study the process $t \mapsto X^{r,\varrho}(t)$ on $[0,T]$ in the joint Boltzmann-Grad and diffusive scaling limit (as in the previous chapter):

$$\begin{gathered} r \to 0 \quad , \quad r^2\varrho \to |\mathbf{p}|^{-1} \quad , \quad T(r) \to \infty \\ t \mapsto \frac{X(tT)}{\sqrt{T}}, \end{gathered} \tag{4.1.3}$$

note that $|\mathbf{p}|^{-1}$ is the cross-sectional area of the cube as viewed by the particle, and we have dropped the dependence on r and ϱ in the notation (thus $X^{r,\varrho}(t) = X(t)$). With that, the main result of this chapter is the following invariance principle:

Theorem 4.1.1. *Consider the intermediate scaling limit (4.1.3) such that* $\lim_{r\to 0} T(r)r^2 = 0$ *then*

$$\left\{ t \mapsto T^{-1/2}X(tT) \right\} \Longrightarrow \{t \mapsto W(t)\} \tag{4.1.4}$$

as $r \to 0$ *in the averaged-quenched sense (with the initial velocity chosen from a finite set). Where* $W(t)$ *is a Wiener process with covariance matrix* $M = \mathrm{diag}(v_1^2, v_2^2, v_3^2)$ *in* $\mathbb{R}^3$.

The proof follows from a joint construction of $t \mapsto X(t)$ and a second Markovian process which we introduce in Section 4.2.2. In Section 4.2.4 we state and outline the proof of the main technical theorem of the chapter (Theorem 4.2.2). Theorem 4.1.1 is then a straightforward corollary of that theorem.

Remark. For the Lorentz gas we proved the same theorem with the asymptotic constraint $\lim_{r\to 0} T(r)r^2 |\log r|^2 = 0$ (see Chapter 3 Theorem 3.1.2). The reasons for this logarithmic correction are those collisions for which the angle between incoming and outgoing velocities is small. In the wind-tree model the velocity of the point particle is restricted to a fixed discrete set, hence the log factor can be removed.

4.2 Coupling Construction

4.2.1 State-Space and Notation

Returning now to the random wind-tree model, for the rest of the chapter we assume the initial velocity is fixed to be $v_0 \in \Omega$. This will aid in the exposition but can be assumed without loss of generality, since the time taken to reach this velocity is exponentially bounded.

At each collision one component of the velocity changes sign. Let $\vartheta_i : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $\vartheta_i(v)_j = (-1)^{\delta_{i,j}} v_j$ for $j = 1, 2, 3$. During a collision the probability $\mathbf{P}(v \mapsto \vartheta_i(v)) = p_i$. For any $v \in \Omega$ let Ω_v denote the set of accessible velocities after one collision starting from v, namely

$$\Omega_v = \{w \in \Omega : w = \vartheta_i(v) \text{ for some } 1 \le i \le 3\}. \tag{4.2.1}$$

Let m_v denote the measure on Ω_v which selects $\vartheta_i(v)$ with probability p_i. Moreover, for $v \in \Omega$ and $w \in \Omega_v$ let $B(v,w)$ be the face of the cube $\mathcal{Q}_r$ such that a particle travelling with velocity v colliding with that face would adopt the velocity w. Formally, for $v \in \Omega$ and $w = \vartheta_k(v)$

$$B(v,w) = \left\{ b \in \partial\mathcal{Q}_r : b_k = -\frac{v_k}{|v_k|} r \right\}. \tag{4.2.2}$$

4.2.2 Markovian Flight Process

Let $\{u_n\}_{n=0}^{\infty}$ be a realisation of the following Markov chain on Ω: $u_1 = v_0$ and then for all $i \geq 1$, u_{i+1} are independently selected from Ω_{u_i} according to the measure m_{u_i}. For later use let $u_0 \in \Omega_{v_0}$. Let

$$\{\xi_n\}_{n=1}^{\infty} \sim EXP(1) \tag{4.2.3}$$

be i.i.d exponentially distributed *flight times* and let

$$Y_n := \sum_{i=1}^{n} y_i \quad , \quad y_n := \xi_n u_n \tag{4.2.4}$$

denote the *discrete Markovian Flight Process*. To define the continuous process, for $t \in \mathbb{R}$ let

$$\tau_n := \sum_{i=1}^{n} \xi_i \quad , \quad \nu_t := \max\{n : \tau_n \leq t\} \quad , \quad \{t\} := t - \tau_{\nu_t}, \tag{4.2.5}$$

that is τ_n are the scattering times, ν_t is the label of the most recent scattering, and $\{t\}$ is the time since the previous scattering, at time t. Now define

$$Y(t) := Y_{\nu_t} + u_{\nu_t+1}\{t\} \tag{4.2.6}$$

to be the *(continuous) Markovian Flight Process*. Note that the processes $t \mapsto Y(t)$ and $\{Y_n\}_{n=1}^{\infty}$ do not depend on r.

For later use we introduce the following *virtual scatterers*:

$$\begin{aligned} Y'_k &:= Y_k + \beta_k \quad , \quad \beta_k \sim UNI(-B(u_k, u_{k+1})) \quad , \quad k \geq 0 \\ \mathcal{S}_n^Y &:= \{Y'_k \in \mathbb{R}^3, \quad 0 \leq k \leq n\} \quad , \quad n \geq 0. \end{aligned} \tag{4.2.7}$$

In words Y'_k is the position of a scatterer *if it had caused* the k^{th} collision (of course Y is independent of any scatterers, thus the term virtual). Note also that we assume there is a virtual collision at time 0, this has no effect on the definition of the model however will ease the notation. One difference with the random Lorentz gas is that the position of a scatterer associated to a velocity jump is not uniquely determined. Therefore we select from among the possible virtual scatterers uniformly.

For later use we introduce the sequence of indicators $\epsilon_j = \mathbb{1}\{\xi_j < 1\}$ and the corresponding distributions $EXP(1|1) := \mathrm{distrib}(\xi|\epsilon = 1)$ and similarly $EXP(1|0) = \mathrm{distrib}(\xi|\epsilon = 0)$. We refer to $\underline{\epsilon} := (\epsilon_j)_{j\geq 0}$ as the *signature* of the sequence $(\xi_j)_{j\geq 0}$.

4.2.3 Joint Construction

Our goal for this section is to construct the physical wind-tree and Markovian processes on the same probability space. We construct the wind-tree process as an *exploration process*: in that the process explores its environment as time moves forward. For convenience for what follows we will also construct a third *auxiliary process*, $\{t \mapsto Z(t)\}$, coupled to the X and Y processes. The auxiliary process, which we call either the *forgetful* or *myopic* process, is only used in Sections 4.4 - 4.6. Hence some readers may wish to ignore it until later. Indeed if we only wanted to prove Theorem 4.1.1 for times of order $o(r^{-1})$ (we do this in Section 4.3) then this myopic process does not play a role and can be ignored.

The construction will proceed inductively on certain (as yet unspecified) time intervals. To simplify the explanation, first we will explain how the processes X and Z are constructed on a given time interval, given certain random data. Then, we will explain how the random data is generated to enable the coupling to $\{t \mapsto Y(t)\}$ and we will explain on which time intervals these processes are defined.

Throughout the construction we label the velocity of $\dot{X}(t) =: V(t)$, $\dot{Y}(t) =: U(t)$ and $\dot{Z}(t) = W(t)$.

Building X on $[\widehat{\tau}_{n-1}, \widehat{\tau}_n)$

We label the intervals of construction of X by $[\widehat{\tau}_{n-1}, \widehat{\tau}_n)$. In Subsection 4.2.3 we will make precise what these $\widehat{\tau}$ are.

To construct X on an interval $[\widehat{\tau}_{n-1}, \widehat{\tau}_n)$, given a position $X(\widehat{\tau}_{n-1}) = X_{n-1} \in \mathbb{R}^3$, a velocity $V(\widehat{\tau}_{n-1}^+) \in \Omega$ and $\mathcal{S}_{n-1}^X \subset \mathbb{R}^{n-1} \cup \{\bigstar\}$ a finite set of points (where $\bigstar$ is a fictitious point at infinity with $\inf_{x\in\mathbb{R}^3} |x - \bigstar| = \infty$ which will aid in the exposition) perform the following steps:

Step 1: Mechanical flight on $\mathcal{S}_{n-1}^X$ in $[\widehat{\tau}_{n-1}, \widehat{\tau}_n)$: The trajectory $t \mapsto X(t)$ on $t \in [\widehat{\tau}_{n-1}, \widehat{\tau}_n)$ is defined to be free motion, with initial position X_{n-1} and velocity $V(\widehat{\tau}_{n-1}^+)$, and with reflective collisions on $\mathcal{Q}_r + \mathcal{S}_{n-1}^X$.

Step 2: Attempt Fresh Collision: Suppose, we are given a velocity $\widehat{w}_{n+1} \in \Omega_{V(\widehat{\tau}_n^-)}$ and an impact parameter $\widehat{\beta}_n \in -B(V(\widehat{\tau}_n^-), \widehat{w}_{n+1})$. Set

$$X'' := X(\widehat{\tau}_n) + \widehat{\beta}_n \tag{4.2.8}$$

Now

- If $\exists 0 < s \leq \widehat{\tau}_{n-1} : X(s) \in X'' + \mathcal{Q}_r$ then let $X'_n := \bigstar$, and $V(\widehat{\tau}_n^+) = V(\widehat{\tau}_n^-)$.
- If not, then $X'_n := X''$, and $V(\widehat{\tau}_n^+) = \widehat{w}_{n+1}$.

Now set $\mathcal{S}_n^X = \mathcal{S}_{n-1}^X \cup \{X'_n\}$.

We say: on the interval $[\widehat{\tau}_{n-1}, \widehat{\tau}_n)$ the process $\{t \mapsto X(t)\}$ *attempts* a *fresh collision* at $\widehat{\tau}_n$ with *data* $(\widehat{w}_{n+1}, \widehat{\beta}_n)$.

We will make precise the distributions of $\widehat{w}_{n+1}$ and $\widehat{\beta}_n$ in the construction below. Note that if, given a $\widehat{w}_{n+1}$ and a $\widehat{\beta}_n$, we build X on the interval $[\widehat{\tau}_{n-1}, \widehat{\tau}_n)$ then, after the construction we have sufficient information to build X on the interval $[\widehat{\tau}_n, \widehat{\tau}_{n+1})$ (provided we are given another pair $\widehat{w}_{n+2}, \widehat{\beta}_{n+1}$).

Building Z on $[\widetilde{\tau}_{n-1}, \widetilde{\tau}_n)$

We call the process $\{t \mapsto Z(t)\}$ forgetful in that the process only respects *direct mismatches* (see Figure 4.1 for a diagram). That is, recollisions with the immediately preceding scatterer, or shadowed events where the scattering is shadowed by the immediately preceding path segment.

Suppose that we are given a time interval $[\widetilde{\tau}_{n-1}, \widetilde{\tau}_n)$. Assume further, we are given a position $Z(\widetilde{\tau}_{n-1}) = Z_{n-1}$, velocity $W(\widetilde{\tau}_{n-1}^+) \in \Omega$, and a pair $\mathcal{S}_{n-1}^Z = \{Z'_{n-1}, Z'_{n-2}\} \subset \mathbb{R}^3 \cup \{\bigstar\}$.

Step 1: Mechanical flight on $\mathcal{S}_{n-1}^Z$ in $[\widetilde{\tau}_{n-1}, \widetilde{\tau}_n)$: The trajectory $t \mapsto Z(t)$ on $t \in [\widetilde{\tau}_{n-1}, \widetilde{\tau}_n)$ is defined to be free motion starting at position $Z(\widetilde{\tau}_{n-1})$ and with velocity $W(\widetilde{\tau}_{n-1}^+)$ with reflective collisions on $\mathcal{Q}_r + \mathcal{S}_{n-1}^Z$.

Step 2: Attempt Fresh Collision: Suppose that we are given a velocity $\widetilde{w}_{n+1} \in \Omega_{W(\widetilde{\tau}_n^-)}$ and an impact parameter $\widetilde{\beta}_n \in -B(W(\widetilde{\tau}_n^-), \widetilde{w}_{n+1})$. Set

$$Z'' := Z(\widetilde{\tau}_n) + \widetilde{\beta}_n \tag{4.2.9}$$

Now

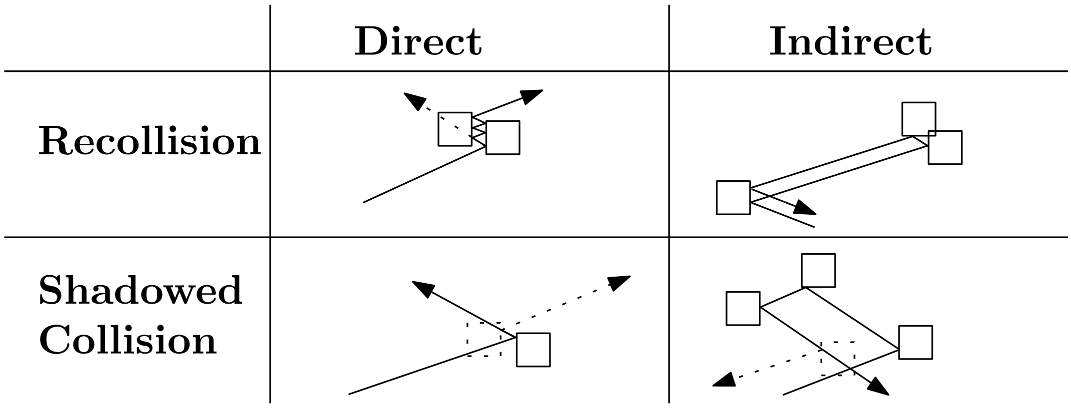

Figure 4.1: `In the above diagram we show examples of direct and indirect, recollisions and shadowed events. In each case the path of the Markovian process is in dotted line while the wind-tree process is in solid line. Additionally, virtual scatterers are in dotted line while actual scatterers for the` X `process are in solid line.`

- If there exists an $s \in (\widetilde{\tau}_{n-2}, \widetilde{\tau}_{n-1}] : Z(s) \in Z'' + \mathcal{Q}_r$ then let $Z'_n := \bigstar$, and $W(\widetilde{\tau}_n^+) = W(\widetilde{\tau}_n^-)$.
- If not, then $Z'_n := Z''$, and $Z(\widetilde{\tau}_n^+) = \widetilde{w}_{n+1}$.

Now set $\mathcal{S}_n^Z = \{Z'_n, Z'_{n-1}\}$.

Similarly we say that on the interval $[\widetilde{\tau}_{n-1}, \widetilde{\tau}_n)$ the process $\{t \mapsto Z(t)\}$ *attempts* a *fresh collision* at $\widetilde{\tau}_n$ with *data* $(\widetilde{w}_{n+1}, \widetilde{\beta}_n)$.

Parity

Consider just the processes $\{t \mapsto Y(t)\}$ and $\{t \mapsto X(t)\}$, the idea behind the coupling is the following:

- $X(0) = Y(0)$ and the velocities are initially parallel.
- X and Y then run parallel until one of two possible *mismatches* occurs:
  - A *recollision*, which corresponds to a collision with a previously placed scatterer during **Step 1:** of Subsection 4.2.3.
  - A *shadowed collision*, which corresponds to $X'_n = \bigstar$ in **Step 2:** of Subsection 4.2.3.
- After a mismatch the two velocity processes proceed independently.
- When the two velocities happen to coincide we recouple the two processes and they run parallel until the next mismatch.

However there is a problem with this setup as we have described it. Note that there are two *parity classes*: $(\mathbf{v}, (\vartheta_i(\vartheta_j(\mathbf{v})))_{i \neq j})$ and $(-\mathbf{v}, (\vartheta_i(\mathbf{v}))_{i=1,2,3})$. The Markov process $(u_n)_{n \in \mathbb{N}}$ alternates between these two classes. The problem is that if there is a parity mismatch between $V(t)$ and $U(t)$ at a given time, then as long as the two processes experience fresh collisions at the same times, only another mismatch can restore the parity. This is too long to wait. Therefore we need to alter the sequence of collision times to restore parity. For this we will make use of Lemma 4.2.1. For future use, we define the equivalence relation $u \overset{p}{\sim} v$ if u and v are in the same parity class.

Lemma 4.2.1. *Let* $(\tau_j)_{j \geq 1}$ *be the points of a Poisson point process of intensity 1 on* $\mathbb{R}_+$*. Form a new sequence as follows: sample* $\xi' \sim EXP(1)$*, independently of the sequence* $(\tau_j)_{j \geq 1}$*. Let the new sequence* $(\tau'_j)_{j \geq 1}$ *be as follows:*

- *If $\xi' < \tau_1$ then $\tau_1' = \xi'$, and $\tau_j' = \tau_{j-1}$ for $j \geq 2$.(That is: insert $\xi' < \tau_1$ as the first point and leave the rest as they are.)*
- *If $\xi' > \tau_1$ then $\tau_j' = \tau_{j+1}$ for $j \geq 1$. (That is: delete the first point τ_1 and leave the rest as they are.)*

Proof. Consider the distribution of τ_1'

$$\begin{aligned}\mathbf{P}(\tau_1' > t) &= \mathbf{P}(\xi > t,\ \ \xi < \tau_1) + \mathbf{P}(\tau_2 > t,\ \ \xi > \tau_1) \\ &= e^{-t}\mathbf{P}\left(\tau_1 > \xi \mid \xi > t\right) + \mathbf{P}(\xi > \tau_1)\,\mathbf{P}\left(\tau_2 > t \mid \xi > \tau_1\right)\end{aligned}$$

where we have used the definition of conditional probability and the fact that ξ is exponentially distribution. Now note that $\mathbf{P}\left(\tau_2 > t \mid \xi > \tau_1\right) = \mathbf{P}\left(\xi > t \mid \xi > \tau_1\right)$ since τ_2 and ξ are both exponentially distributed conditioned to be larger that τ_1. Therefore

$$\begin{aligned}\mathbf{P}(\tau_1' > t) &= e^{-t}\mathbf{P}\left(\tau_1 > \xi \mid \xi > t\right) + \mathbf{P}(\xi > \tau_1)\,\mathbf{P}\left(\xi > t \mid \xi > \tau_1\right) \\ &= e^{-t}\mathbf{P}\left(\tau_1 > \xi \mid \xi > t\right) + e^{-t}\mathbf{P}\left(\xi > \tau_1 \mid \xi > t\right) \\ &= e^{-t}\mathbf{P}\left(\tau_1 > \xi \mid \xi > t\right) + e^{-t}(1 - \mathbf{P}\left(\tau_1 > \xi \mid \xi > t\right)) \\ &= e^{-t}.\end{aligned}$$

Turning now to the distribution $\tau_2' - \tau_1'$ (all the other increments are clearly i.i.d exponentially distributed)

$$\begin{aligned}\mathbf{P}(\tau_2' - \tau_1' > t) &= \mathbf{P}(\tau_1 - \xi > t, \tau_1 > \xi) + \mathbf{P}(\tau_3 - \tau_2 > t, \tau_1 < \xi) \\ &= e^{-t}\mathbf{P}(\tau_1 > \xi) + e^{-t}\mathbf{P}(\tau_1 < \xi) = e^{-t}.\end{aligned}$$

Finally, we look at the joint distribution

$$\mathbf{P}(\tau_1' > t, \tau_2' - \tau_1' > s) = \mathbf{P}(\xi > t, \tau_1 - \xi > s) + \mathbf{P}(\xi > \tau_1, \tau_2 > t, \tau_3 - \tau_2 > s)\,.$$

By construction $\tau_3 - \tau_2$ is exponentially distributed and independent of τ_1, τ_2, ξ, thus

$$\begin{aligned}\mathbf{P}(\tau_1' > t, \tau_2' - \tau_1' > s) &= \mathbf{P}(\xi > t, \tau_1 - \xi > s) + \mathbf{P}(\xi > \tau_1, \tau_2 > t)\,e^{-s} \\ &= \mathbf{P}\left(\xi > t, \tau_1 - \xi > s \mid \xi < \tau_1\right)\mathbf{P}(\xi < \tau_1) + \mathbf{P}(\xi > \tau_1, \tau_2 > t)\,e^{-s}.\end{aligned}$$

Conditioned on $\xi < \tau_1$, $\tau_1 - \xi$ is exponentially distributed independently of ξ. Thus

$$\begin{aligned}\mathbf{P}(\tau_1' > t, \tau_2' - \tau_1' > s) &= e^{-s}\mathbf{P}\left(\xi > t \mid \xi < \tau_1\right)\mathbf{P}(\xi < \tau_1) + \mathbf{P}(\xi > \tau_1, \tau_2 > t)\,e^{-s} \\ &= e^{-s}\mathbf{P}(\xi > t, \xi < \tau_1) + \mathbf{P}(\xi > \tau_1, \tau_2 > t)\,e^{-s} \\ &= e^{-s}\mathbf{P}(\tau_1' > t) \\ &= e^{-s}e^{-t}.\end{aligned}$$

□

Joint Coupling

Assume $\{t \mapsto Y(t)\}$ is constructed as in Subsection 4.2.2. We will construct the X and Z processes inductively on the intervals $[\tau_{2n}, \tau_{2n+2})$ as follows: First set

$$\begin{array}{llll} X(0) = X_0 = 0 \quad , & V(0^+) = u_1 \quad , & X_0' = \widehat{\beta}_0 = \beta_0 \quad , & \mathcal{S}_0^X = \{X_0'\} \\ Z(0) = Z_0 = 0 \quad , & W(0^+) = u_1 \quad , & W_0' = \widetilde{\beta}_0 = \beta_0 \quad , & \mathcal{S}_0^Z = \{Z_0', Z_{-1}'\} \end{array} \tag{4.2.10}$$

where $Z_{-1}' = \bigstar$. Let $n \in \mathbb{N}$ and sample an exponential time $\zeta_n \sim EXP(1)$ independent of the entire history up to this point. In which case there are 7 possible situations arranged and labelled in the following table:

Parity at time τ_{2n}^+	$\zeta_n \leq \xi_{2n+1}$	$\zeta_n > \xi_{2n+1}$
$U \overset{p}{\sim} V \overset{p}{\sim} W$	A	
$U \overset{p}{\nsim} V \overset{p}{\sim} W$	B	C
$U \overset{p}{\sim} V \overset{p}{\nsim} W$	D	E
$U \overset{p}{\sim} W \overset{p}{\nsim} V$	F	G

For completeness of the construction we define all of these cases, however on our time scales we will (w.h.p) only see situations A, B, and C.

On the interval $[\tau_{2n}, \tau_{2n+2})$ the X and Z processes attempt fresh collisions at the following times:

Situation	X	Z
A	τ_{2n+1}, τ_{2n+2}	τ_{2n+1}, τ_{2n+2}
B	$\tau_{2n} + \zeta_n$, τ_{2n+1}, τ_{2n+2}	$\tau_{2n} + \zeta_n$, τ_{2n+1}, τ_{2n+2}
C	τ_{2n+2}	τ_{2n+2}
D	τ_{2n+1}, τ_{2n+2}	$\tau_{2n} + \zeta_n$, τ_{2n+1}, τ_{2n+2}
E	τ_{2n+1}, τ_{2n+2}	τ_{2n+2}
F	$\tau_{2n} + \zeta_n$, τ_{2n+1}, τ_{2n+2}	τ_{2n+1}, τ_{2n+2}
G	τ_{2n+2}	τ_{2n+1}, τ_{2n+2}

In what follows the following **coupling rule** will dictate the random variables $\widehat{\beta}_n, \widehat{w}_n, \widetilde{\beta}_n, \widetilde{w}_n$ used in the attempted fresh collisions.

For the Z-process: If the Z-process is to attempt a fresh collision at time t_a, sample $\widetilde{w}$ from $\Omega_{W(t_a^-)}$ according to the measure $\mathrm{m}_{W(t_a^-)}$ and sample $\widetilde{\beta}$ from $-B(W(t_a^-), \widetilde{w})$ both independent of the past. We now attempt to couple W with U at t_a:

- **Couple W to U:** If $W(t_a^-) = U(t_a^-)$ and $t_a = \tau_n$ for some n, attempt a fresh collision at $Z(t_a)$ using data (β_n, u_{n+1}).
- **W is independent of U:** Otherwise attempt a fresh collision at $Z(t_a)$ using data $(\widetilde{\beta}, \widetilde{w})$.

For the X-process: If the X-process is to attempt a fresh collision at time t_a, sample $\widehat{w}$ from $\Omega_{V(t_a^-)}$ according to the measure $\mathrm{m}_{V(t_a^-)}$ and sample $\widehat{\beta}$ from $-B(V(t_a^-), \widehat{w})$ both independent of the past. We now couple V to either U *and/or* W if possible:

- **Couple V to U:** If $V(t_a^-) = U(t_a^-)$ and $t_a = \tau_n$ for some n attempt a fresh collision at $X(t_a)$ using (β_n, u_{n+1}).

- <u>**Couple V to W:**</u> If $V(t_a^-) = W(t_a^-)$ and the Z process also attempts a fresh collision *independent of* U at time t_a, attempt a fresh collision at $X(t_a)$ using $(\widetilde{\beta}, \widetilde{w})$.
- <u>**V is independent of U and W:**</u> Otherwise attempt a fresh collision at $X(t_a)$ using $(\widehat{\beta}, \widehat{w})$.

After this construction we have generated two processes. For the wind-tree exploration process $\{t \mapsto X(t)\}$, the *attempted fresh collision* times are $\{\widehat{\tau}_n\}_{n\in\mathbb{N}}$, by Lemma 4.2.1 these form a (temporal) Poisson point process on $\mathbb{R}_+$; the scatterers are placed at positions $\{X_n'\} \subset \mathbb{R}^3 \cup \{\bigstar\}$; and the impact parameters are $\{\widehat{\beta}_n\}_{n\in\mathbb{N}}$. Moreover, the *attempted* velocities after collisions are $\{\widehat{w}_n\}_{n\in\mathbb{N}}$, these velocities are attempted since, in **Step 2:** the attempted collision may be rejected (i.e $X_n' = \bigstar$). Because of the Poisson distribution of the scatterers in $\mathbb{R}^3$ this process is distributed like the original wind-tree model as described in the introduction.

For the process $\{t \mapsto Z(t)\}$, the *attempted fresh collision* times are $\{\widetilde{\tau}_n\}_{n\in\mathbb{N}}$, which by Lemma 4.2.1 form a (temporal) Poisson point process on $\mathbb{R}_+$; the scatterers are placed at positions $\{Z_n'\} \subset \mathbb{R}^3 \cup \{\bigstar\}$; and the impact parameters are $\{\widetilde{\beta}_n\}_{n\in\mathbb{N}}$. The *attempted* velocities for the Z-process are $\{\widetilde{w}_n\}_{n\in\mathbb{N}}$.

4.2.4 Main Technical Result and Method Proof

The main result we prove is the following

Theorem 4.2.2. *Let $T = T(r)$ be such that $\lim_{r\to 0} T(r) = \infty$ and $\lim_{r\to 0} r^2 T(r) = 0$. Then for any $\delta > 0$*

$$\lim_{r\to 0} \mathbf{P}\left(\sup_{0\le t\le T} |X(t) - Y(t)| > \delta\sqrt{T} \right) = 0. \tag{4.2.11}$$

From here Theorem 4.1.1 follows as a consequence of the classical Donsker's invariance principle (Chapter 2 Theorem 2.5.2): that is, the process $t \mapsto Y(t)$ is a true Markov process, hence Donsker's original invariance principle does not apply directly, however in what follows we will show how to separate Y into i.i.d mean 0 pieces with finite second moment. Thus Donsker's principle will imply that $t \mapsto \frac{Y(tT)}{\sqrt{T}}$ converges to a Wiener process in the diffusive scaling. Therefore the process $t \mapsto X(t)$ does as well. We omit the details of this final step and the rest of the chapter is devoted to proving Theorem 4.2.2.

The strategy of proof is the same as in Chapter 3. We begin with the joint realization of the Markovian flight process and the wind-tree exploration process described above. During the two mismatch events (recollisions and shadowed scatterings) the two velocity processes diverge. In either case the two processes are decoupled until recoupling is possible. At which point the two processes are recoupled and proceed parallel to each other until the next mismatch.

The proof then follows two steps. In Section 4.3 we show that such mismatches occur only on time scales of order r^{-1}. Hence until such times both process are (w.h.p) in the the same position and Theorem 4.2.2 follows immediately for $T = o(r^{-1})$. Note that this intermediate result is a statement about the Markovian flight process. During the rest of the chapter we show that on time scales of order $o(r^{-2})$ only (geometrically) simple mismatches occur. During such mismatches the separation between X and Y is of order $\mathcal{O}(1)$. Hence on the time scales of Theorem 4.2.2 there are $o(Tr)$ mismatches. During each mismatch the two processes separate by a distance of order $\mathcal{O}(1)$, hence up to $T = o(r^{-2})$, $\frac{|X(T(r))-Y(T(r))|}{\sqrt{T}} \to 0$, thus proving (4.2.11). Sections 4.4-4.6 are devoted to formalizing this argument.

The reason for introducing the forgetful process $\{t \mapsto Z(t)\}$ is that the forgetful process will satisfy additional independence properties exploited in the proof. Thus during the second stage of the prove, we will in fact show that the forgetful and Markovian processes do not diverge too much. Then we show that with high probability the wind-tree and forgetful processes are in fact the same on these time scales (i.e we show that with probability tending to 1 as $r \to 0$, the direct mismatches defining the Z-process are the only ones seen by the X-process).

Remark on dimension: As with the Lorentz gas, because of the recurrence of the random walk the same proof does not yield the result in 2 dimensions. For the Lorentz gas the geometry of mismatches imposed another reason that the proof cannot be extended to 2 dimensions. However for the wind-tree model the mismatches have a far simpler geometry and thus this obstruction is not present in 2 dimensions.

4.2.5 r-consistency and r-compatibility

The proof will hinge on two definitions which we present now for a general process (i.e this could be a segment of any of the above mentioned processes). Let

$$n \in \mathbb{N}, \qquad \tau_0 \in \mathbb{R}, \qquad \mathcal{Z}_0 \in \mathbb{R}^3, \qquad U_0, \dots, U_{n+1} \in \Omega \qquad t_1, \dots, t_n \in \mathbb{R}_+,$$

be given, such that either $U_{i+1} \in \Omega_{U_i}$ or $U_{i+1} = U_i$ for all $0 \leq i \leq n$. Moreover fix a set of vectors $\beta_j \in B(U_j, U_{j+1})$ (if $U_j = U_{j+1}$ we set $\beta_j = \bigstar$) and define for $j = 0, \dots, n$,

$$\tau_j := \tau_0 + \sum_{k=1}^{j} t_k, \qquad \mathcal{Z}_j := \mathcal{Z}_0 + \sum_{k=1}^{j} t_k U_k, \qquad \mathcal{Z}_j' := \mathcal{Z}_j + \beta_j$$

and for $t \in [\tau_j, \tau_{j+1}]$, $j = 0, \dots, n$,

$$\mathcal{Z}(t) := \mathcal{Z}_j + (t - \tau_j) U_{j+1}.$$

We call the piece-wise linear trajectory $\left(\mathcal{Z}(t) : \tau_0^- < t < \tau_n^+\right)$ mechanically *r-consistent* if

$$\nexists\, t \in [\tau_0, \tau_n],\ j \in \{0, \dots, n\} : \mathcal{Z}(t) - \mathcal{Z}_j' \in \mathcal{Q}_r^o \tag{4.2.12}$$

($\mathcal{Q}_r^o$ denotes the interior) and *r-inconsistent* if (4.2.12) fails.

Given two finite pieces of mechanically r-consistent trajectories $\left(\mathcal{Z}_a(t) : \tau_{a,0}^- < t < \tau_{a,n_a}^+\right)$ and $\left(\mathcal{Z}_b(t) : \tau_{b,0}^- < t < \tau_{b,n_b}^+\right)$, defined over non-overlapping time intervals: $[\tau_{a,0}, \tau_{a,n_a}] \cap [\tau_{b,0}, \tau_{b,n_b}] = \emptyset$ with $\tau_{a,n_a} \leq \tau_{b,0}$, we will call them mechanically *r-compatible* if

$$\begin{aligned} &\nexists\, t \in [\tau_{a,0}, \tau_{a,n_a}], j \in \{0, \dots n_b\} : \mathcal{Z}_a(t) - \mathcal{Z}_{b,j}' \in \mathcal{Q}_r^o, \\ \text{and} \quad &\nexists\, t \in [\tau_{b,0}, \tau_{b,n_b}], j \in \{0, \dots n_a\} : \mathcal{Z}_b(t) - \mathcal{Z}_{a,j}' \in \mathcal{Q}_r^o \end{aligned} \tag{4.2.13}$$

mechanical trajectories are *r-incompatible* if (4.2.13) fails. Note that these definitions, while similar, are not the same as those in Chapter 3 Section 3.2.3.

4.3 No Mismatches Till $T = o(r^{-1})$

4.3.1 Excursions

Unlike in the 3-dimensional Lorentz gas case the directions of path segments of the Markovian flight process are not independent. To decompose the process $t \mapsto Y(t)$ into i.i.d segments we introduce *excursions.* Let

$$\gamma := \min\{i > 1 : u_{i+1} = v_0\} \tag{4.3.1}$$

and define a *pack* to be a collection

$$\varpi := (\gamma; \{u_i\}_{i=1}^{\gamma}, \{\beta_i\}_{i=1}^{\gamma}, \{\xi_i\}_{i=1}^{\gamma}),$$

$u_\gamma \in \Omega_{v_0}$, and for all $i > 1$, $u_i \neq v_0$ and $u_{i-1} \in \Omega_{u_i}$. Given a pack we consider the process $t \mapsto Y(t)$ associated to it via the rules set forth in Section 4.2.2 - call the process built from such a pack, *an excursion.*

4.3.2 Concatenation

For $n = 1, 2, 3, \ldots$ consider infinitely many independent packs:

$$\varpi_n = (\gamma_n, \{u_{n,i}\}_{i=1}^{\gamma_n}, \{\beta_{n,i}\}_{i=1}^{\gamma_n}, \{\xi_{n,i}\}_{i=1}^{\gamma_n}).$$

For each pack define the associated flight process $t \mapsto Y_n(t)$ together with the discrete process $\{Y_{n,i}\}_{i=0}^{\gamma_n}$. Denote

$$\theta_n := \sum_{i=1}^{\gamma_n} \xi_{n,i}, \qquad \overline{Y_n} := Y_{n,\gamma_n}.$$

Define the following variables

$$\Gamma_0 = 0, \qquad \Gamma_n = \Gamma_{n-1} + \gamma_n, \quad \text{for } n \geq 1$$
$$\nu_n := \max\{m : \Gamma_n \leq n\}, \qquad \{n\} := n - \Gamma_{\nu_n}.$$

Likewise

$$\Theta_0 = 0, \qquad \Theta_n = \Theta_{n-1} + \theta_n, \quad \text{for } n \geq 1$$
$$\nu_t := \max\{m : \Theta_n \leq t\}, \qquad \{t\} := t - \Theta_{\nu_t}.$$

Now define the following three processes: the *end-point process* with $\Xi_0 = 0$

$$\Xi_n := \sum_{k=1}^{n} \overline{Y_k},$$

the *concatenated discrete Markovian flight process* with $Y_0 = 0$

$$Y_n := \Xi_{\nu_n} + Y_{\nu_n+1,\{n\}},$$

and the continuous *concatenated Markovian flight process* with $Y(0) = 0$

$$Y(t) := \Xi_{\nu_t} + Y_{\nu_t+1}(\{t\}).$$

The advantage of this decomposition is that the different excursions making up the process Y are i.i.d steps with exponentially decaying tails.

4.3.3 Occupation Measures

Define the following occupation measures for a set $A \subset \mathbb{R}^3$

$$G(A) := \mathbf{E}\left(|\{1 \le k < \infty : Y_k \in A\}|\right), \qquad H(A) := \mathbf{E}\left(|\{0 < t < \infty : Y(t) \in A\}|\right),$$
$$g(A) := \mathbf{E}\left(|\{1 \le k \le \gamma_1 : Y_k \in A\}|\right), \qquad h(A) := \mathbf{E}\left(|\{0 < t < \Theta_1 : Y(t) \in A\}|\right),$$
$$R(A) := \mathbf{E}\left(|\{1 \le k < \infty : \Xi_k \in A\}|\right).$$

Lemma 4.3.1. *The following upper bounds hold for any measurable set $A \subset \mathbb{R}^3$*

$$R(A) \le K(A) + L_{v_0}(A), \tag{4.3.2}$$
$$g(A) \le M(A) + L_{v_0}(A), \qquad h(A) \le M(A) + L_{v_0}(A), \tag{4.3.3}$$
$$G(A) \le K(A) + L_{v_0}(A), \qquad H(A) \le K(A) + L_{v_0}(A), \tag{4.3.4}$$

where

$$K(dx) := C \min\{1, |x|^{-1}\}dx \quad , \quad M(dx) := Ce^{-c|x|}dx$$
$$L_{v_0}(A) := C \int_0^\infty \mathbb{1}\{tv_0 \in A\} e^{-ct} dt$$

This Lemma is slightly different from the Lorentz gas case as L_{v_0} takes into account the discrete state-space of velocities. However the end result (Proposition 4.3.3) remains the same.

Proof. To bound $g(A)$ let

$$g_1(A) := \mathbf{P}\left(Y_1 \in A\right) = C \int_0^\infty \mathbb{1}\{tv_0 \in A\} e^{-t} dt.$$

We have fixed the initial velocity to be $u_1 = v_0$, therefore the points $\{Y_k - Y_1\}_{k=1}^{\gamma_1}$ are independent of the initial step Y_1. Therefore write

$$g_2(A) := \mathbf{E}\left(|\{1 \le k \le \gamma_1 : Y_k - Y_1 \in A\}|\right),$$

and note that

$$g(A) = \int_{\mathbb{R}^3} g_2(A - x) g_1(dx). \tag{4.3.5}$$

Similarly we can write

$$h_1(A) := \mathbf{E}\left(|\{t \le \tau_1 : Y(t) \in A\}|\right) = C \int_0^\infty \mathbb{1}\{tv_0 \in A\} e^{-\max\{1,t\}} dt,$$
$$h_2(A) := \mathbf{E}\left(|\{\tau_1 \le t \le \Theta_1 : Y(t) - Y_1 \in A\}|\right)$$
$$h(A) = \int_{\mathbb{R}^3} h_2(A - x) g_1(dx) + h_1(A). \tag{4.3.6}$$

Now the bounds (4.3.3) follow by inserting the bounds:

$$\begin{aligned} &g_2(\{x : |x| > s\}) \le Ce^{-cs}, \qquad h_2\left(\{x : |x| > s\}\right) \le Ce^{-cs} \\ &g_2(\mathbb{R}^3) = \mathbf{E}\left(\gamma_1\right) < \infty, \qquad h_2(\mathbb{R}^3) = \mathbf{E}\left(\Theta_1 - \tau_1\right) < \infty \end{aligned} \tag{4.3.7}$$

into (4.3.5) and (4.3.6). That is,

$$g(A) \le \int_{A^c} g_2(\{y : |y| > |x|\})dx + C \int_A g_1(dx) \le M(A) + L_{v_0}(A) \tag{4.3.8}$$

and likewise for $h(A)$.

Now, to achieve (4.3.2) note that since $\gamma_1 > 1$

$$\mathbf{P}\left(\Xi_1 \in A\right) \le \mathbf{E}\left(|\{2 \le k \le \gamma_1 : Y_k \in A\}|\right) \le g(A) \tag{4.3.9}$$

Hence the density of distribution of Ξ_1 is bounded by the density of g. Moreover, because $\mathbf{P}\left(\theta_1 > s\right) \le Ce^{-cs}$ for some $C < \infty$ and $c > 0$, we know that the density of distribution of Ξ_1 has exponentially decaying tails. Therefore Ξ is a random walk, with i.i.d steps, and step distribution bounded by g with exponentially decaying tails. Hence a standard random walk argument implies (4.3.2).

(4.3.4) then follows by writing (using the fact that the different excursions are i.i.d)

$$G(A) = g(A) + \int_{\mathbb{R}^3} g(A - x)R(dx), \qquad H(A) = h(A) + \int_{\mathbb{R}^3} h(A - x)R(dx)$$

and inserting (4.3.2) and (4.3.3). □

4.3.4 Inter-Excursion Mismatches

Let $t \to Y^*(t)$ denote a Markovian flight process with associated virtual scatterers $Y^{*\prime} \in \mathcal{S}^{Y^*}$ and initial velocity $u_1^* \in -\Omega_{v_0}$. Let $t \to Y(t)$ be a second Markovian flight process with associated virtual scatterers $\mathcal{S}^Y$, and initial velocity v_0.

We think of Y^* as the process run backwards in time. Define the events

$$\begin{aligned}
\widehat{W}_j &:= \big\{\{Y(t) - Y_k' : && 0 < t < \Theta_{j-1}, && \Gamma_{j-1} < k \le \Gamma_j\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widetilde{W}_j &:= \big\{\{Y_k' - Y(t) : && 0 \le k < \Gamma_{j-1}, && \Theta_{j-1} < t < \Theta_j\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widehat{W}_j^* &:= \big\{\{Y^*(t) - Y_k' : && 0 < t < \Theta_{j-1}, && 0 < k \le \gamma\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widetilde{W}_j^* &:= \big\{\{Y_k^{*\prime} - Y(t) : && 0 < k \le \Gamma_{j-1}, && 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widehat{W}_\infty^* &:= \big\{\{Y^*(t) - Y_k' : && 0 < t < \infty, && 0 < k \le \gamma\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widetilde{W}_\infty^* &:= \big\{\{Y_k^{*\prime} - Y(t) : && 0 < k < \infty, && 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset\big\}.
\end{aligned}$$

In words $\widehat{W}_j$ is the event that during the $(j-1)^{th}$ excursion, a collision of Y is (virtually) *shadowed* by a previous excursion. And $\widetilde{W}_j$ is the event that during the $(j-1)^{th}$ excursion the process (virtually) *recollides* with a scatterer from an earlier excursion.

It readily follows that

$$\begin{aligned}
\mathbf{P}\left(\widehat{W}_j\right) &= \mathbf{P}\left(\widehat{W}_j^*\right) \le \mathbf{P}\left(\widehat{W}_{j+1}^*\right) \le \mathbf{P}\left(\widehat{W}_\infty^*\right),\\
\mathbf{P}\left(\widetilde{W}_j\right) &= \mathbf{P}\left(\widetilde{W}_j^*\right) \le \mathbf{P}\left(\widetilde{W}_{j+1}^*\right) \le \mathbf{P}\left(\widetilde{W}_\infty^*\right).
\end{aligned} \tag{4.3.10}$$

By the union bound

$$
\begin{aligned}
\mathbf{P}\left(\widehat{W}^*_\infty\right) &\le \sum_{z\in\mathbb{Z}^3} \mathbf{P}\left(\{1<k<\infty : Y^*_k \in B_{zr,2r}\} \neq \emptyset\right) \mathbf{P}\left(\{0<t\le\theta : Y(t)\in B_{zr,2r}\}\neq\emptyset\right) \\
&\le \sum_{z\in\mathbb{Z}^3} (2r)^{-1}\mathbf{E}\left(|\{1<k<\infty : Y^*_k \in B_{zr,2r}\}|\right)\cdot \mathbf{E}\left(|\{0<t\le\theta : Y(t)\in B_{zr,3r}\}|\right) \\
\mathbf{P}\left(\widetilde{W}^*_\infty\right) &\le \sum_{z\in\mathbb{Z}^3} \mathbf{P}\left(\{0<t<\infty : Y^*(t) \in B_{zr,2r}\} \neq \emptyset\right) \mathbf{P}\left(\{1\le j\le\gamma : Y_j\in B_{zr,2r}\}\neq\emptyset\right) \\
&\le \sum_{z\in\mathbb{Z}^3} (2r)^{-1}\mathbf{E}\left(|\{0<t<\infty : Y^*(t) \in B_{zr,3r}\}|\right)\cdot \mathbf{E}\left(|\{1\le j\le\gamma : Y_j\in B_{zr,2r}\}|\right)
\end{aligned} \tag{4.3.11}
$$

4.3.5 Computations

(4.3.11) implies that

$$
\begin{aligned}
\mathbf{P}\left(\widetilde{W}^*_\infty\right) &\le (2r)^{-1}\sum_{z\in\mathbb{Z}^3} H^*(B_{zr,3r})g(B_{zr,2r}) \\
\mathbf{P}\left(\widehat{W}^*_\infty\right) &\le (2r)^{-1}\sum_{z\in\mathbb{Z}^3} G^*(B_{zr,3r})h(B_{zr,2r}).
\end{aligned} \tag{4.3.12}
$$

where G^* is defined like G, except that in this instance the initial velocity is chosen from $-\Omega_{v_0}$ rather than fixed to be v_0.

Lemma 4.3.2. *The following bounds hold for some $C<\infty$ and any $v\in\Omega$*

$$
\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})M(B_{zr,2r}) \le Cr^3, \qquad \sum_{z\in\mathbb{Z}^3} L_v(B_{zr,3r})M(B_{zr,2r}) \le Cr^3
$$

$$
\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})L_v(B_{zr,2r}) \le Cr^3, \qquad \sum_{z\in\mathbb{Z}^3} L_v(B_{zr,3r})L_w(B_{zr,2r}) \le Cr^2.
$$

for $v\neq w\in\Omega$

Proof. The following bounds follow immediately from the definitions of K, M, and L_v

$$
\begin{aligned}
&K(B_{zr,3r}) \le Cr^3, \\
&M(B_{zr,3r}) \le Cr^3 e^{-cr|z|}, \\
&L_v(B_{zr,3r}) \le Cr^3\delta_{0,z} + Cr\mathbb{1}\{\exists t>0 : vt\cap B_{zr,3r}\}(1-\delta_{0,z})e^{-cr|z|}.
\end{aligned} \tag{4.3.13}
$$

From here

$$
\begin{aligned}
\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})M(B_{zr,2r}) &\le Cr^6 \sum_{z\in\mathbb{Z}^3} e^{-cr|z|} \\
&\le Cr^3\int_{\mathbb{R}^3} e^{-c|z|}dz \le Cr^3
\end{aligned}
$$

where we use a Riemann integral to go from the first line to the second. Likewise

$$
\begin{aligned}
\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})L_v(B_{zr,2r}) &\le Cr^6 + Cr^4 \sum_{z\in(\mathbb{Z}^3)^*} \mathbb{1}\{\exists t>0 : vt\cap B_{zr,3r}\} e^{-cr|z|} \\
&\le Cr^6 + C'r^4 \sum_{z=1}^{\infty} e^{-cr|vz|} \\
&\le Cr^6 + Cr^3 \int_0^{\infty} e^{-c|vt|}dt \le Cr^3
\end{aligned} \tag{4.3.14}
$$

where from the first line to the second we approximate the points $zr \in r\mathbb{Z}^3$ close to the line vt by the points rvz for $z \in \mathbb{Z}$.

Similarly

$$
\sum_{z\in\mathbb{Z}^3} L_v(B_{zr,3r})M(B_{zr,2r}) \le Cr^6 + Cr^4 \sum_{z\in(\mathbb{Z}^3)^*} \mathbb{1}\{\exists t>0 : vt\cap B_{zr,3r}\} e^{-2cr|z|}
$$

the bound then follows as it did in (4.3.14).

Finally,

$$
\begin{aligned}
\sum_{z\in\mathbb{Z}^3} L_v(B_{zr,3r})L_w(B_{zr,2r}) &\le Cr^2 \sum_{z\in(\mathbb{Z}^3)^*} \mathbb{1}\{\exists t>0 : vt\cap B_{zr,3r}\}\mathbb{1}\{\exists t>0 : wt\cap B_{zr,3r}\} e^{-2cr|z|} \\
&\le Cr^2 e^{-cr} \le Cr^2,
\end{aligned}
$$

since $v \neq w$ only finitely many $z \in \mathbb{Z}$ contribute to the sum, from which the second line follows.

□

Note that Lemma 4.3.1 is stated for G and H and not G^* and H^*. However similar bounds hold for the backwards excursions. Thus (omitting these details), we use Lemma 4.3.1 to insert Lemma 4.3.2 into (4.3.12) to get:

Proposition 4.3.3. *There exists a constant $C > 0$ such that for all $j \geq 1$*

$$
\mathbf{P}\left(\widehat{W}_j\right) \le Cr, \qquad \mathbf{P}\left(\widetilde{W}_j\right) \le Cr. \tag{4.3.15}
$$

4.3.6 Mismatches within one Excursion

Define the following indicator functions

$$
\begin{aligned}
\widehat{\eta}_j &= \widehat{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{\min_{0\le t\le \xi_{j-2}} (tu_{j-2} + y_{j-1} + \beta_{j-1}) \in \mathcal{Q}_r\right\} \\
\widetilde{\eta}_j &= \widetilde{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{\min_{0\le t\le \xi_{j}} (y_{j-1} + tu_j - \beta_{j-2}) \in \mathcal{Q}_r\right\} \\
\eta_j &:= \max\{\widetilde{\eta}_j, \widehat{\eta}_j\}
\end{aligned} \tag{4.3.16}
$$

In words, $\widehat{\eta}_j$ is the event that the $(j-1)$-labelled collision is shadowed by the immediately preceding path (i.e a *direct* shadowing event). And $\widetilde{\eta}_j$ is the event that during the j^{th} path segment there is a recollision with the immediately preceding obstacle (i.e a *direct* recollision) - see the left hand side of Figure 4.1.

Lemma 4.3.4. *For any $i, j < \gamma$ with $i \neq j$ there exists a $C < \infty$ such that*

$$\mathbf{E}\left(\eta_j\right) \leq Cr \tag{4.3.17}$$
$$\mathbf{E}\left(\eta_j \eta_i\right) \leq Cr^2 \tag{4.3.18}$$

(4.3.18) is not needed to prove the result for $T = o(r^{-1})$ however will be used to prove Theorem 4.2.2.

Proof of Lemma 4.3.4. Suppose $u_{j-2} = U$. Then throughout the two subsequent collisions we know for some $i = 1, 2, 3$ - $(u_{j-1})_i = (u_j)_i = U_i$ (i.e one coordinate of the velocity remains unchanged). Thus to (directly) recollide with $Y'_{j-2} + \mathcal{Q}_r$ we require $\xi_{j-1} < Cr$ which implies (4.3.17). The same is true for shadowing events, that is $\widehat{\eta}_j = 1$ implies $\xi_{j-1} > Cr$ for some constant.

(4.3.18) follows for the same reason. Suppose $i \neq j$, then for $\eta_j \eta_i = 1$, requires $\max\{\xi_{j-1}, \xi_{i-1}\} < Cr$ for some constant. As these are independent exponentials (4.3.18) is immediate.

□

Lemma 4.3.4 controls the probability of a *direct* mismatch. However we also need to control indirect mismatches. To that end define

$$\begin{aligned} \widehat{\eta}^o_j &:= \mathbb{1}\left\{\min_{0 \leq t \leq \tau_{j-3}} \left(Y(t) - Y'_{j-1}\right) \in \mathcal{Q}_r\right\} \\ \widetilde{\eta}^o_j &:= \mathbb{1}\left\{\min_{\tau_{j-1} \leq t \leq \tau_j} \left(\min_{0 \leq k \leq j-3} \left(Y(t) - Y'_k\right)\right) \in \mathcal{Q}_r\right\} \\ \eta^o_j &:= \max\{\widetilde{\eta}^o_j, \widehat{\eta}^o_j\}. \end{aligned} \tag{4.3.19}$$

In words $\widehat{\eta}^o_j$ is the indicator that an *indirect* (virtual) shadowing event occurs and $\widetilde{\eta}^o_j$ is the event an *indirect* (virtual) recollision occurs. That is a mismatch which involves more than the immediately preceding obstacle or path.

Lemma 4.3.5. *For any $3 < j \leq \gamma$ there exists a constant $C > 0$ such that*

$$\mathbf{E}\left(\eta^o_j\right) \leq C\gamma^2 r^2 \tag{4.3.20}$$

Proof of Lemma 4.3.5. Under time reversal Markovian flight processes remain Markovian flight process while recollisions become shadowed events. Hence recollisions and shadowing events happen with the same probability and thus we may restrict to proving the statement for recollisions.

By the union bound

$$\mathbf{E}\left(\widetilde{\eta}^o_j\right) \leq \sum_{k \leq j-3} \mathbf{P}\left(\left\{\min_{\tau_{j-1} \leq t \leq \tau_j} \left(Y(t) - Y'_k\right) \in \mathcal{Q}_r\right\}\right). \tag{4.3.21}$$

Write $\mathscr{A}_k = \{\min_{\tau_{j-1} \leq t \leq \tau_j} \left(Y(t) - Y'_k\right) \in \mathcal{Q}_r\}$ - the event there is a indirect recollision after $k-1$ fresh collisions. To have an indirect recollision, requires at least three distinct velocities along the path, thus

$$\mathbf{P}\left(\mathscr{A}_k\right) = \mathbf{P}\left(\mathscr{A}_k \cap \{\exists i \in [k+1, j-2] : u_i \neq u_j, u_{j-1}\}\right).$$

Moreover at each collision exactly one of the velocity coordinates changes sign. Hence we know u_j and u_{j-1} differ by a sign change in one coordinate therefore the event in the right hand side of (4.3.6) implies there is a third velocity which is linearly independent of u_j and u_{j-1}. Therefore

$$(4.3.6) = \mathbf{P}\left(\mathscr{A}_k \cap \{\exists i \in [k+1, j-2] \ : \ u_i, u_j, u_{j-1} \text{ lin. ind.}\}\right)$$

Moreover note that if we fix i

$$\mathcal{A}_k = \{\min_{0\le t\le \xi_j} (\xi_i u_i + \xi_{j-1}u_{j-1} + tu_j - s_i) \in \mathcal{Q}_r\}$$

where

$$s_i = \sum_{\substack{l=k+1\\ l\neq i}}^{j-2} u_l \xi_l.$$

Let B_i denote the event u_i, u_{j-1}, u_j are linearly independent. In this case

$$\begin{aligned}\mathbf{P}(\mathcal{A}_k) &\le \sum_{i=k+1}^{j-2} \mathbf{P}(B_i \cap \mathcal{A}_k)\\ &\le \sum_{i=k+1}^{j-2} \mathbf{E}\left(\mathbf{P}\left(B_i \cap \{\min_{0\le t\le \xi_j} (\xi_i u_i + \xi_{j-1}u_{j-1} + tu_j - s_i) \in \mathcal{Q}_r\} \;\middle|\; s_i\right)\right).\end{aligned}$$

Lemma 4.3.6 (below) implies that the probability inside the expectation is bounded by Cr^2. As $j-2-k \le \gamma$ this implies

$$\mathbf{P}(\mathcal{A}_k) \le C\gamma r^2.$$

Inserting this into (4.3.21) then implies (4.3.20).

□

Lemma 4.3.6. *Suppose $U_1, U_2, U_3 \in \Omega$ are linearly independent and $\xi_1, \xi_2, \xi_3 \sim EXP(1)$ are i.i.d exponentials. Then there exists a constant $C < \infty$ such that for any $s \in \mathbb{R}^3$*

$$\mathbf{P}\left(\min_{0\le t\le \xi_3} (U_1\xi_1 + U_2\xi_2 + U_3 t - s) \in \mathcal{Q}_r\right) \le Cr^2. \tag{4.3.22}$$

Proof. We can assume

$$U_1 = (\nu_1, \nu_2, \nu_3) \quad , \quad U_2 = (-\nu_1, \nu_2, \nu_3) \quad , \quad U_3 = (-\nu_1, -\nu_2, \nu_3)$$

in which case for any $t \le \xi_3$

$$U_1\xi_1 + U_2\xi_2 + U_3 t = ((\xi_1 - \xi_2 - t)\nu_1, (\xi_1 + \xi_2 - t)\nu_2, (\xi_1 + \xi_2 + t)\nu_3). \tag{4.3.23}$$

Therefore the event on the left hand side of (4.3.22) is the event that there exists a $t \le \xi_3$ satisfying the system of inequalities

$$\begin{array}{ccccc} s_1 - \frac{r}{2} & \le & (\xi_1 - \xi_2 - t)\nu_1 & \le & s_1 - \frac{r}{2}\\ s_2 - \frac{r}{2} & \le & (\xi_1 + \xi_2 - t)\nu_2 & \le & s_2 - \frac{r}{2}\\ s_3 - \frac{r}{2} & \le & (\xi_1 + \xi_2 + t)\nu_3 & \le & s_3 - \frac{r}{2}\end{array}$$

solving these equations, we find that regardless of t there exist c_1, c_2, C_1, C_2 such that

$$\xi_1 \in [c_1 - C_1 r, c_1 + C_1 r], \qquad \xi_2 \in [c_2 - C_2 r, c_2 + C_2 r]$$

since ξ_1 and ξ_2 are i.i.d exponentials (4.3.22) follows immediately. □

4.4 Beyond the Naïve Coupling

In the following sections we extend the results of Section 4.3 to times on the order $o(r^{-2})$. In order to reduce the amount of notation we will use the same notation for the *analogous* objects and will give the redefinitions explicitly. Recall the definition of the process $\{t \mapsto Z(t)\}$ given in Subsection 4.2.3. We will split the process $\{t \mapsto Z(t)\}$ into legs (similar to the excursions of the previous section).

4.4.1 Legs

Similar to Subsection 4.3.1 we split $t \mapsto Z(t)$ into legs. However to ensure that the different legs are independent we impose the restriction that each leg begins and ends with two path segments of length greater than 1. Let $\widetilde{\xi}_n = \widetilde{\tau}_n - \widetilde{\tau}_{n-1}$ for all $n \geq 1$. Let

$$\gamma := \min\{i > 1 : \widetilde{\xi}_{i-1}, \widetilde{\xi}_i, \widetilde{\xi}_{i+1}, \widetilde{\xi}_{i+1} > 1\ ,\ \ \widetilde{w}_{i+1} = \widetilde{w}_1 = v_0\}. \tag{4.4.1}$$

Note that the condition on $\widetilde{\xi}_i$ implies that $\gamma \in \{2\} \cup \{5, \dots\}$. If we define $\theta := \sum_{i=1}^{\gamma} \widetilde{\xi}_i$ then

$$\mathbf{P}(\gamma > s) \leq Ce^{-cs}, \qquad \mathbf{P}(\theta > s) \leq Ce^{-cs}. \tag{4.4.2}$$

The definition of a pack is then similar to Subsection 4.3.1: a *pack* is a collection

$$\varpi := \left(\gamma; \{\widetilde{\xi}_i\}_{i=1}^{\gamma}, \{\widetilde{\beta}_i\}_{i=1}^{\gamma}, \{\widetilde{w}_i\}_{i=1}^{\gamma}\right),$$

Given a pack we consider the process $t \mapsto Z(t)$ associated to it via the rules set forth in Subsection 4.2.3 and call such a segment a *leg*. Note that, in order to have a direct mismatch at step n requires that $\widetilde{\xi}_{n-1} < Cr$ for some constant $C < \infty$. Hence the beginning and end of a leg are Markovian steps.

Furthermore given a pack ϖ a *backwards leg* is defined to be

$$(\theta; Z^*(t); 0 \leq t \leq \theta)$$

where

$$Z^*(t) = Z(\theta - t, \varpi^*) - \overline{Z}(\varpi^*)$$

(we use the notation $Z(t, \varpi)$ to denote the forward forgetful process built from the pack ϖ) where

$$\varpi^* := (\gamma; \{\widetilde{\xi}_{\gamma-j}\}_{j=0}^{\gamma-1}, \{\widetilde{\beta}_{\gamma-j}\}_{j=0}^{\gamma-1}, \{\widetilde{w}_{\gamma-j}\}_{j=0}^{\gamma-1})$$

As before denote

$$Z_j^* := Z^*(\widetilde{\tau}_j), \quad 0 \leq j \leq \gamma, \qquad \overline{Z^*} = Z_\gamma^*.$$

Note the processes $t \mapsto Z(t)$ and $t \mapsto Z^*(t)$ do not have the same distribution.

4.4.2 Concatenation

Let $\varpi_n = \left(\gamma_n;\ \{\widetilde{\xi}_{n,j}\}_{j=1}^{\gamma_n}\{\widetilde{\beta}_{n,j}\}_{j=1}^{\gamma_n}, \{\widetilde{w}_{n,j}\}_{j=1}^{\gamma_n}\right)$, $n \geq 1$, be a sequence of i.i.d *packs* and consider the associated forwards legs $(Z_n(t) : 0 \leq t \leq \theta_n)$, $(Z_{n,j} : 1 \leq j \leq \gamma_n)$ and backwards legs $(Z_n^*(t) : 0 \leq t \leq \theta_n)$, $(Z_{n,j}^* : 1 \leq j \leq \gamma_n)$.

To construct the concatenated forward and backward processes $t \mapsto Z(t)$, $t \mapsto Z^*(t)$, $0 \leq t < \infty$, define for $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$

$$\begin{aligned} \Gamma_n &:= \sum_{k=1}^{n} \gamma_k, \quad \nu_n := \max\{m : \Gamma_m \leq n\}, \quad \{n\} := n - \Gamma_{\nu_n}, \\ \Theta_n &:= \sum_{k=1}^{n} \theta_k, \quad \nu_t := \max\{m : \Theta_m < t\}, \quad \{t\} := t - \Theta_{\nu_t}. \end{aligned} \tag{4.4.3}$$

The concatenated (multi-leg) forward and backward Z-processes are

$$\begin{aligned} \Xi_n &:= \sum_{k=1}^{n} \overline{Z}_k, & Z_n &:= \Xi_{\nu_n} + Z_{\nu_n+1,\{n\}}, & Z(t) &:= \Xi_{\nu_t} + Z_{\nu_t+1}(\{t\}), \\ \Xi_n^* &:= \sum_{k=1}^{n} \overline{Z}_k^*, & Z_n^* &:= \Xi_{\nu_n}^* + Z_{\nu_n+1,\{n\}}^*, & Z^*(t) &:= \Xi_{\nu_t}^* + Z_{\nu_t+1}^*(\{t\}). \end{aligned} \tag{4.4.4}$$

4.4.3 Mismatches in a Leg

Let $\varpi = (\gamma; \{\widetilde{\xi}_j\}_{j=1}^{\gamma}, \{\widetilde{\beta}_j\}_{j=1}^{\gamma}, \{\widetilde{w}_j\}_{j=1}^{\gamma})$ be a pack. Let $u \in \Omega_{v_0}$ a velocity and $\beta_0 \in B(u, v_0)$ an impact parameter.

Let $t \mapsto \mathscr{X}(t)$ be the wind-tree process coupled to the pack ϖ. That is, given the processes $t \mapsto Y(t)$ and $t \mapsto Z(t)$ follow the rules in Subsection 4.2.3 until time τ_γ.

Consider the jointly realized triple $((Y(t), \mathscr{X}(t), Z(t)) : 0^- < t < \theta^+)$ - a Markovian flight process, a wind-tree exploration process and a forgetful process all coupled to ϖ. The time interval $0^- < t < \theta^+$ indicates that the velocity immediately prior to the position at 0 is u, there is a collision with a scatterer at β_0, and at θ^+ the velocity of Y and Z is w.

Proposition 4.4.1. *There exists a $C < \infty$ such that for all $w \in \Omega$ and $u \in \Omega_w$ and $\beta_0 \in B(u, w)$*

$$\mathbf{P}\left(\mathscr{X}(t) \not\equiv Z(t) : 0^- < t < \theta^+\right) \leq r^2. \tag{4.4.5}$$

This proposition will be proved in Section 4.6.

4.4.4 Inter-Leg Mismatches

Consider a forgetful process $t \mapsto Z(t)$ built from legs. Define the following events

$$\begin{aligned} \widehat{W}_j &:= \big\{\{Z(t) - Z'_k : \quad 0 < t < \Theta_{j-1}, \quad \Gamma_{j-1} < k \leq \Gamma_j\} \cap \mathcal{Q}_r \neq \emptyset\big\}, \\ \widetilde{W}_j &:= \big\{\{Z'_k - Z(t) : \quad 0 \leq k < \Gamma_{j-1}, \quad \Theta_{j-1} < t < \Theta_j\} \cap \mathcal{Q}_r \neq \emptyset\big\}, \end{aligned} \tag{4.4.6}$$

i.e $\widehat{W}_j$ is the event a collision during the j^{th} leg is (virtually) shadowed by a path segment in a previous leg. $\widetilde{W}_j$ is the event that during the j^{th} leg the process (virtually) collides with an obstacle placed during a previous leg.

Proposition 4.4.2. *There exists a $C < \infty$ such that for all $j \geq 1$,*

$$\mathbf{P}\left(\widetilde{W}_j\right) \leq Cr^2, \qquad \mathbf{P}\left(\widehat{W}_j\right) \leq Cr^2. \tag{4.4.7}$$

The proof of this proposition is the content of Section 4.5.

4.5 Proof of Proposition 4.4.2

The proof of Proposition 4.4.2 follows the similar lines to that of Proposition 4.3.3. However as we have redefined legs we shall go through the full proof. In this section we redefine the Green's functions g, h, G, and H.

4.5.1 Occupation Measures

Let $t \mapsto Z(t)$ be a forward forgetful process with initial velocity v_0 and $t \mapsto Z^*(t)$ a backward process with initial velocity in $\Omega_{-\widetilde{w}_1}$ (distributed according to m_{-v_0}). Define the events

$$\begin{aligned}
\widehat{W}^*_j &:= \big\{\{Z^*(t) - Z'_k : && 0 < t < \Theta_{j-1}, && 0 < k \leq \gamma\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widetilde{W}^*_j &:= \big\{\{Z^{*\prime}_k - Z(t) : && 0 < k \leq \Gamma_{j-1}, && 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widehat{W}^*_\infty &:= \big\{\{Z^*(t) - Z'_k : && 0 < t < \infty, && 0 < k \leq \gamma\} \cap \mathcal{Q}_r \neq \emptyset\big\},\\
\widetilde{W}^*_\infty &:= \big\{\{Z^{*\prime}_k - Z(t) : && 0 < k < \infty, && 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset\big\}.
\end{aligned}$$

The same calculation as (4.3.10), (4.3.11), and (4.3.12) implies

$$\begin{aligned}
\mathbf{P}\left(\widetilde{W}_j\right) &\leq \mathbf{P}\left(\widetilde{W}^*_\infty\right) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} H^*(B_{zr,3r}) g(B_{zr,2r}),\\
\mathbf{P}\left(\widehat{W}_j\right) &\leq \mathbf{P}\left(\widehat{W}^*_\infty\right) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} G^*(B_{zr,3r}) h(B_{zr,2r}),
\end{aligned} \tag{4.5.1}$$

where the right hand side is in terms of the following Green's functions: for $A \subset \mathbb{R}^3$

$$\begin{aligned}
g(A) &:= \mathbf{E}\left(|\{1 \leq k \leq \gamma : Z_k \in A\}|\right), & g^*(A) &:= \mathbf{E}\left(|\{1 \leq k \leq \gamma : Z^*_k \in A\}|\right),\\
h(A) &:= \mathbf{E}\left(|\{0 < t \leq \theta : Z(t) \in A\}|\right), & h^*(A) &:= \mathbf{E}\left(|\{0 < t \leq \theta : Z^*(t) \in A\}|\right),\\
R^*(A) &:= \mathbf{E}\left(|\{1 \leq n < \infty : \Xi^*_n \in A\}|\right), & &\\
G^*(A) &:= \mathbf{E}\left(|\{1 \leq k < \infty : Z^*_k \in A\}|\right), & H^*(A) &:= \mathbf{E}\left(|\{0 < t < \infty : Z^*(t) \in A\}|\right).
\end{aligned}$$

Note that

$$\begin{aligned}
G^*(A) &= g^*(A) + \int_{\mathbb{R}^3} g^*(A - x) R^*(dx),\\
H^*(A) &= h^*(A) + \int_{\mathbb{R}^3} h^*(A - x) R^*(dx).
\end{aligned} \tag{4.5.2}$$

4.5.2 Bounds

Lemma 4.5.1. *The following bounds hold for any Borel set $A \subset \mathbb{R}^3$*

$$g(A) \leq M(A) + \widetilde{L}_{v_0}(A), \qquad g^*(A) \leq M(A) + \widetilde{L}^{\perp}_{v_0}(A), \tag{4.5.3}$$
$$h(A) \leq M(A) + L_{v_0}(A), \qquad h^*(A) \leq M(A) + L^{\perp}_{v_0}(A), \tag{4.5.4}$$
$$R^*(A) \leq K(A) + \widetilde{L}^{\perp}_{v_0}(A), \tag{4.5.5}$$
$$G^*(A) \leq K(A) + \widetilde{L}^{\perp}_{v_0}(A), \qquad H^*(A) \leq K(A) + L^{\perp}_{v_0}(A), \tag{4.5.6}$$

where K, L_{v_0}, and M are as defined in Lemma 4.3.1 and

$$L^{\perp}_{v_0}(A) := C \sum_{w \in \Omega_{-v_0}} \int_0^\infty \mathbb{1}\{tw \cap A\} e^{-ct} dt,$$
$$\widetilde{L}_{v_0}(A) := C \int_1^\infty \mathbb{1}\{tv_0 \cap A\} e^{-ct} dt, \qquad \widetilde{L}^{\perp}_{v_0}(A) := C \sum_{w \in \Omega_{-v_0}} \int_1^\infty \mathbb{1}\{tw \cap A\} e^{-ct} dt.$$

Proof. The proof of this Lemma follows the same lines as the proof of Lemma 4.3.1 however the legs in this section are conditioned to have the first step longer than 1. (4.5.5) follows from the fact that the steps of Ξ^*_n are i.i.d with exponentially decaying tails and the density of each step is bounded by $g^*(dx)$.

To bound $g(A)$ write:

$$\begin{aligned} g(A) &= \int_{\mathbb{R}^3} g_2(A - x) g_1(dx), \\ g_1(A) &:= \mathbf{P}\left(Z_1 \in A\right) = C \int_1^\infty \mathbb{1}\{tv_0 \in A\} e^{-t} dt, \\ g_2(A) &:= \mathbf{E}\left(|\{1 \leq k \leq \gamma_1 : Z_k - Z_1 \in A\}|\right). \end{aligned}$$

This follows since $Z_k - Z_1$ is independent of Z_1 for every $k \geq 2$. (4.5.3) then follows in the same way as did (4.3.3) in Lemma 4.3.1 from the bounds

$$g_2(\{x : |x| > s\}) \leq Ce^{-cs}, \qquad g_2(\mathbb{R}^3) = \mathbf{E}\left(\gamma\right) < \infty.$$

For $g^*(A)$ write

$$\begin{aligned} g^*(A) &= \mathbf{E}\left(|\{1 \leq k \leq \gamma_1 : Z^*_k \in A\}|\right) \\ &\leq \sum_{w \in \Omega_{-v_0}} \mathbf{E}\left(|\{1 \leq k \leq \gamma_1 : Z^*_k \in A\}| \;\middle|\; \widetilde{w}^*_1 = w\right) =: \sum_{w \in \Omega_{-v_0}} g^*_w(A), \end{aligned}$$

where $\widetilde{w}^*_1 := \dot{Z}^*(0^+)$. As for $g(A)$ we now split

$$\begin{aligned} g^*_w(A) &= \int_{\mathbb{R}^3} g^*_{2,w}(A - x) g^*_{1,w}(dx) \\ g^*_{1,w}(A) &:= \mathbf{P}\left(Z^*_1 \in A \;\middle|\; \widetilde{w}^*_1 = w\right) \\ g^*_{2,w}(A) &:= \mathbf{E}\left(|\{1 \leq k \leq \gamma_1 : Z_k - Z_1 \in A\}| \;\middle|\; \widetilde{w}^*_1 = w\right) \end{aligned}$$

Our bound for $g^*(A)$ now follows the same lines as for $g(A)$. $h^*(A)$ is very similar.

The bounds on G^* and H^* follow by inserting the bounds for g^*, h^*, R^* into (4.5.2).

□

4.5.3 Computations

Lemma 4.5.2. *The following bounds hold for some $C < \infty$ and r small enough*

$$\sum_{z\in\mathbb{Z}^3} \widetilde{L}^{\perp}_{v_0}(B_{zr,3r})L_{v_0}(B_{zr,2r}) = 0, \qquad \sum_{z\in\mathbb{Z}^3} L^{\perp}_{v_0}(B_{zr,3r})\widetilde{L}_{v_0}(B_{zr,2r}) = 0,$$
$$\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})\widetilde{L}_{v_0}(B_{zr,2r}) \leq Cr^3, \qquad \sum_{z\in\mathbb{Z}^3} \widetilde{L}^{\perp}_{v_0}(B_{zr,3r})M(B_{zr,2r}) \leq Cr^3,$$
$$\sum_{z\in\mathbb{Z}^3} L^{\perp}_{v_0}(B_{zr,3r})M(B_{zr,2r}) \leq Cr^3.$$

Proof. These bounds follow by observing

$$\begin{aligned}
\widetilde{L}_{v_0}(B_{zr,3r}) &\leq C\mathbb{1}\{\exists t \geq 1 : B_{zr,3r} \cap v_0 t\} r e^{-cr|z|},\\
\widetilde{L}^{\perp}_{v_0}(B_{zr,3r}) &\leq C \sum_{w\in\Omega_{-v_0}} \mathbb{1}\{\exists t \geq 1 : B_{zr,3r} \cap wt\} r e^{-cr|z|},\\
L^{\perp}_{v_0}(B_{zr,3r}) &\leq C\delta_{0,z} r^3 + C \sum_{w\in\Omega_{-v_0}} \mathbb{1}\{\exists t \geq 3r : B_{zr,3r} \cap wt\} r e^{-cr|z|},
\end{aligned} \tag{4.5.7}$$

and (4.3.13). With that the first two bounds are trivial. The third bound follows from:

$$\begin{aligned}
\sum_{z\in\mathbb{Z}^3} K(B_{zr,3r})\widetilde{L}_{v_0}(B_{zr,2r}) &\leq Cr^6 + Cr^4 \sum_{w\in\Omega_{-v_0}} \sum_{z\in\ (\mathbb{Z}^3)^*} \mathbb{1}\{\exists t \geq 3r : B_{zr,3r} \cap wt\} e^{-cr|z|}\\
&\leq Cr^6 + Cr^4 \sum_{w\in\Omega_{-v_0}} \sum_{z\in\mathbb{Z}^*} e^{-cr|vz|} \leq Cr^3,
\end{aligned}$$

where in the last line we approximate the sum by an integral in the same way as we did in (4.3.14).

Note that by (4.5.7)

$$\sum_{z\in\mathbb{Z}^3} \widetilde{L}^{\perp}_{v_0}(B_{zr,3r})M(B_{zr,2r}) \leq \sum_{z\in\mathbb{Z}^3} L^{\perp}_{v_0}(B_{zr,3r})M(B_{zr,2r}).$$

Moreover by (4.3.13) and (4.5.7)

$$\sum_{z\in\mathbb{Z}^3} L^{\perp}_{v_0}(B_{zr,3r})M(B_{zr,2r}) \leq Cr^6 + Cr^4 \sum_{w\in\Omega_{-v_0}} \mathbb{1}\{\exists t \geq 3r : B_{zr,3r} \cap wt\} e^{-2cr|z|} \leq Cr^3.$$

□

Proposition 4.4.2. The proof of Proposition 4.4.2 follows by inserting the bounds in Lemma 4.5.1 into (4.5.1) and then applying Lemma 4.5.2.

□

4.6 Proof of Proposition 4.4.1

In the setting of Section 4.4.3 the proof of Proposition 4.4.1 will follow from considering the following indicator functions

$$\begin{aligned}\widetilde{\eta}_j &:= \mathbb{1}\left\{\min_{\widetilde{\tau}_{j-1}<t<\widetilde{\tau}_j}\left(Z(t)-Z'_{j-2}\right)\in\mathcal{Q}_r\right\}\\ \widehat{\eta}_j &:= \mathbb{1}\left\{\min_{\widetilde{\tau}_{j-3}<t<\widetilde{\tau}_{j-2}}\left(Z(t)-Z(\widetilde{\tau}_{j-1})-\widetilde{\beta}_{j-1}\right)\in\mathcal{Q}_r\right\}\\ \eta_j &:= \max\{\widetilde{\eta}_j,\widehat{\eta}_j\}\end{aligned} \tag{4.6.1}$$

In particular, η_j is the probability of a mismatch for the Z-process in immediately before the j^{th} leg. It is important to note, the simple geometric fact (which follows simply from the fact that the collision angles are bounded) that $\eta^*_j = 1$ implies $\widetilde{\xi}_{j-1} < Cr$ for some constant $C < \infty$. This fact will make the geometric estimates vastly easier than for the Lorentz gas, where the equivalent statement is false.

The following statements will provide the proof of Proposition 4.4.1

$$\mathbf{P}\left(\{\mathcal{X}(t)\not\equiv Z(t):0^-\leq t\leq\theta^+\}\cap\{\sum_{j=1}^{\gamma}\eta_j>1\}\right)\leq Cr^2, \tag{4.6.2}$$

$$\mathbf{P}\left(\{\mathcal{X}(t)\not\equiv Z(t):0^-\leq t\leq\theta^+\}\cap\{\sum_{j=1}^{\gamma}=0\}\right)\leq Cr^2, \tag{4.6.3}$$

$$\mathbf{P}\left(\{\mathcal{X}(t)\not\equiv Z(t):0^-\leq t\leq\theta^+\}\cap\{\sum_{j=1}^{\gamma}=1\}\right)\leq Cr^2. \tag{4.6.4}$$

4.6.1 Proof of (4.6.2)

The simple geometric fact stated in the previous section implies

$$\mathbf{P}\left(\sum_{j=1}^{\gamma}\eta_j>1\right)\leq\frac{\gamma^2}{2}\max_{1\leq j<k\leq\gamma}\mathbf{P}\left(\eta_j\eta_k=1\right)\leq C\gamma^2r^2.$$

(4.6.2) now follows from the exponential tail bounds (4.4.2).

□

4.6.2 Proof of (4.6.3)

On $\left\{\sum_{j=1}^{\gamma}\eta_j=0\right\}$, the process $\{t\mapsto Z(t)\}$ is distributed like a Markovian flight process. Hence the event in (4.6.3) can be written

$$\{\mathcal{X}(t)\not\equiv Z(t):0^-\leq t\leq\theta^+\}\cap\left\{\sum_{j=1}^{\gamma}\eta_j=0\right\}=\{\exists\,3\leq j\leq\gamma:\eta^o_j=1\}\cap\{\sum_{j=1}^{\gamma}\eta_j=0\}$$

where η^o_j is the indicator of an indirect mismatch, as defined in (4.3.19). Therefore using Lemma 4.3.5

$$\mathbf{P}\left(\{\mathcal{X}(t)\not\equiv Z(t):0^-\le t\le\theta^+\}\cap\{\sum_{j=1}^{\gamma}\eta_j=1\}\right)\le\mathbf{P}\left(\{\exists\,3\le j\le\gamma:\eta_j^o=1\}\right)$$
$$\le\gamma\max_{3\le j\le\gamma}\mathbf{P}\left(\eta_j^o=1\right)$$
$$\le C\gamma^3r^2.$$

Thus (4.6.3) again follows from the exponential tail bounds (4.4.2).

□

4.6.3 Proof of (4.6.4)

Given a $\gamma\in\{2\}\cup\{5,\dots\}$, a signature $\underline{\epsilon}$ (recall the definition of a signature given at the end of Subsection 4.2.2) compatible with the definition of a pack, and a fixed label $3<k<\gamma$. Let $V_1,V_2\in\Omega$ and let ϖ be a pack with signature $\underline{\epsilon}$ and $\widetilde{w}_{k-2}=V_1$ and $\widetilde{w}_{k+1}=V_2$ (we assume V_1 and V_2 are compatible with this definition).

- On $0^-<t\le\widetilde{\tau}_{k-1}$ - $Z^{(k)}(t)=Y(t)$, conditioned such that $\widetilde{w}_{k-2}=V_1$.
- On $\widetilde{\tau}_{k-1}<t\le\widetilde{\tau}_k$ - $Z^{(k)}(t)$ is constructed like the Z-process, conditioned such that the final velocity is $\widetilde{w}_k\in\Omega_{V_2}$
- On $\widetilde{\tau}_k<t<\widetilde{\tau}_\gamma$ - $Z^{(k)}(t)=Y(t)$ a Markovian flight process starting at $Z^{(k)}(\widetilde{\tau}_k)$, conditioned such that $\widetilde{w}_{k+1}=V_2$.

On $\{\eta_j=\delta_{j,k}:1\le j\le\gamma\}$ - $Z^{(k)}$ is distributed like Z. The reason for conditioning on V_1 and V_2 is to ensure the following three parts are independent:

$$\begin{aligned}&(Z^{(k)}(t):0^-<t\le\widetilde{\tau}_{k-3})=(Y(t):0^-<t\le\widetilde{\tau}_{k-3}),\\&(Z^{(k)}(\widetilde{\tau}_{k-3}+t)-Z^{(k)}(\widetilde{\tau}_{k-3}):0\le t\le\widetilde{\tau}_k-\widetilde{\tau}_{k-3}),\\&(Z^{(k)}(\widetilde{\tau}_k+t)-Z(\widetilde{\tau}_k):0\le t<\theta^+-\widetilde{\tau}_k).\end{aligned}\qquad(4.6.5)$$

Let $A^{(k)}_{a,a}$, $1\le a\le3$ be the event that the a-th part of the trajectory is r-*inconsistent*. For $1\le a<b\le3$ we denote $A^{(k)}_{a,b}$ the event that the a and b-th parts are r-*incompatible*. Therefore to prove (4.6.4) we will bound

$$\left.\begin{aligned}&\max_{\underline{\epsilon},k,V_1,V_2}\mathbf{P}\left(\{\widehat{\eta}_k=1\}\cap A^{(k)}_{a,b}\mid\underline{\epsilon},V_1,V_2\right),\\&\max_{\underline{\epsilon},k,V_1,V_2}\mathbf{P}\left(\{\widetilde{\eta}_k=1\}\cap\{\widehat{\eta}_k=0\}\cap A^{(k)}_{a,b}\mid\underline{\epsilon},V_1,V_2\right),\end{aligned}\right.\qquad a,b=1,2,3.\qquad(4.6.6)$$

4.6.4 Bounds

First notice that $A^{(k)}_{1,1}, A^{(k)}_{3,3}$ and $A^{(k)}_{1,3}$ involve only Markovian segments hence the following estimates follow readily from Lemmas 4.3.1, 4.3.2, 4.3.4, and 4.3.5:

$$\left.\begin{aligned}&\max_{\underline{\epsilon},k,V_1,V_2}\mathbf{P}\left(\{\widehat{\eta}_k=1\}\cap A^{(k)}_{a,b}\mid\underline{\epsilon},V_1,V_2\right)\le C\gamma^3r^2,\\&\max_{\underline{\epsilon},k,V_1,V_2}\mathbf{P}\left(\{\widetilde{\eta}_k=1\}\cap\{\widehat{\eta}_k=0\}\cap A^{(k)}_{a,b}\mid\underline{\epsilon},V_1,V_2\right)\le C\gamma^3r^2,\end{aligned}\right.\qquad a,b=1,3.\qquad(4.6.7)$$

Therefore there remain 6 bounds.

Note that during middle segment in (4.6.5) the velocity of $Z^{(k)}(t)$ is restricted to only three possible velocities. Thus one component of the velocity remains unchanged throughout this segment. Therefore the middle segment can only be r-inconsistent if two of the path segments are shorter than Cr for some constant $C < \infty$. Thus

$$\begin{aligned}
&\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{2,2} \mid \underline{\epsilon}, V_1, V_2\right) \leq Cr^2, \\
&\mathbf{P}\left(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{2,2} \mid \underline{\epsilon}, V_1, V_2\right) \leq Cr^2.
\end{aligned} \tag{4.6.8}$$

It remains to prove

$$\begin{aligned}
&\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{b,2} \mid \underline{\epsilon}, V_1, V_2\right) \leq C\gamma r^2, \\
&\mathbf{P}\left(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{b,2} \mid \underline{\epsilon}, V_1, V_2\right) \leq C\gamma r^2,
\end{aligned} \qquad b = 1, 3. \tag{4.6.9}$$

We will only prove (4.6.9) for $b = 3$ as the proof for $b = 1$ is the same. Given a set $A \subset \mathbb{R}^3$ define the following occupation measures for the third part of (4.6.5)

$$\begin{aligned}
G^{(k)}_{\underline{\epsilon}}(A) :=& \mathbf{E}\left(\#\{1 \leq j \leq \gamma - k : Z^{(k)}(\widetilde{\tau}_{j+k}) - Z^{(k)}(\widetilde{\tau}_k) \in A\} \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2\right), \\
& \mathbf{E}\left(\#\{1 \leq j \leq \gamma - k : \widetilde{Y}(\widetilde{\tau}_j) \in A\} \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2\right), \\
H^{(k)}_{\underline{\epsilon}}(A) :=& \mathbf{E}\left(\left|\{\tau_j \leq \theta : Z^{(k)}(t) - Z^{(k)}(\widetilde{\tau}_k) \in A\}\right| \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2\right), \\
& \mathbf{E}\left(\left|\{0 \leq t \leq \tau_{\gamma-k} : \widetilde{Y}(t) \in A\}\right| \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2\right),
\end{aligned}$$

where $t \mapsto \widetilde{Y}(t)$ is a Markovian flight process with initial velocity in Ω_{V_2}. Similarly

$$\begin{aligned}
\widehat{G}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\#\{1 \leq j \leq 3 : Z^{(k)}(\widetilde{\tau}_{k-j}) - Z^{(k)}(\widetilde{\tau}_k) \in A\} \cdot \widehat{\eta}_k \mid \underline{\epsilon}, V_1, V_2\right), \\
\widehat{H}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\left|\{\widetilde{\tau}_{k-3} \leq t \leq \widetilde{\tau}_k : Z^{(k)}(t) - Z^{(k)}(\widetilde{\tau}_k) \in A\}\right| \cdot \widehat{\eta}_k \mid \underline{\epsilon}, V_1, V_2\right), \\
\widetilde{G}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\#\{1 \leq j \leq 3 : Z^{(k)}(\widetilde{\tau}_{k-j}) - Z^{(k)}(\widetilde{\tau}_k) \in A\} \cdot \widetilde{\eta}_k \cdot (1 - \widehat{\eta}_k) \mid \underline{\epsilon}, V_1, V_2\right), \\
\widetilde{H}^{(k)}_{\underline{\epsilon}}(A) &:= \mathbf{E}\left(\left|\{\widetilde{\tau}_{k-3} \leq t \leq \widetilde{\tau}_k : Z^{(k)}(t) - Z^{(k)}(\widetilde{\tau}_k) \in A\}\right| \cdot \widetilde{\eta}_k \cdot (1 - \widehat{\eta}_k) \mid \underline{\epsilon}, V_1, V_2\right).
\end{aligned}$$

As the middle and last parts in (4.6.5) are independent the following bounds apply

$$\begin{aligned}
&\mathbf{P}\left(\{\widehat{\eta}_k = 1\} \cap A^{(k)}_{3,2} \mid \underline{\epsilon}, V_1, V_2\right) \leq Cr^{-1}\left(\int_{\mathbb{R}^3} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widehat{H}^{(k)}_{\underline{\epsilon}}(dx) + \int_{\mathbb{R}^3} H^{(k)}_{\underline{\epsilon}}(B_{x,3r})\widehat{G}^{(k)}_{\underline{\epsilon}}(dx)\right), \\
&\mathbf{P}\left(\{\widetilde{\eta}_k = 1\} \cap \{\widehat{\eta}_k = 0\} \cap A^{(k)}_{3,2} \mid \underline{\epsilon}, V_1, V_2\right) \leq \\
&\qquad\qquad \leq Cr^{-1}\left(\int_{\mathbb{R}^3} G^{(k)}_{\underline{\epsilon}}(B_{x,2r})\widetilde{H}^{(k)}_{\underline{\epsilon}}(dx) + \int_{\mathbb{R}^3} H^{(k)}_{\underline{\epsilon}}(B_{x,3r})\widetilde{G}^{(k)}_{\underline{\epsilon}}(dx)\right).
\end{aligned} \tag{4.6.10}$$

By (4.3.4) there exists a constant $C < \infty$ such that

$$G_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \le CF(x), \qquad H_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \le CF(x) \tag{4.6.11}$$

where $F : \mathbb{R}^3 \to \mathbb{R}_+$

$$F(x) = r\{|x| \le r\} + \frac{r^3}{|x|^2}\{r < |x| \le 1\} + \frac{r^3}{|x|}\{|x| > 1\} + re^{-c|x|}\mathbb{1}\{\exists t > 0 : B_{x,2r} \cap tV_2\}\{|x| > r\}.$$

For simplicity we will only treat the first term on the right hand side in the second line of (4.6.10) (this is the most difficult), the other terms can be dealt with similarly.

Since during the middle section of (4.6.5) one component of the velocity does not change sign we can conclude

$$\widehat{G}_{\underline{\epsilon}}^{(k)}(B_{0,s}), \widetilde{G}_{\underline{\epsilon}}^{(k)}(B_{0,s}) \le Crs, \quad \widehat{H}_{\underline{\epsilon}}^{(k)}(B_{0,s}), \widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,s}) \le Crs, \tag{4.6.12}$$

and

$$\widehat{G}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3), \widetilde{G}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3) \le Cr, \quad \widehat{H}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3), \widetilde{H}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3) \le Cr. \tag{4.6.13}$$

First note that by (4.6.12)

$$\begin{aligned}\int_{|x|>r} re^{-c|x|}\mathbb{1}\{\exists t > 0 : B_{x,2r} \cap tV_2\}\widetilde{H}_{\underline{\epsilon}}^{(k)}(dx) &\le Cr^2 \int_{|x|>r} e^{-c|x|}\mathbb{1}\{\exists t > 0 : B_{x,2r} \cap tV_2\}dx \\ &\le Cr^4 \int_{t>r} e^{-c|tV_2|}dt \le Cr^4\end{aligned}$$

and

$$\int_{|x|>1} \frac{r^3}{|x|}\widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \le Cr^4.$$

Finally let $\widetilde{F}(u) = r\{u \le r\} + \frac{r^3}{u^2}\{r < u \le 1\}$, then by applying integration by parts

$$\begin{aligned}\int_{\{|x|<1\}} \widetilde{F}(|x|)\widetilde{H}_{\underline{\epsilon}}^{(k)}(dx) &\le C\int_0^1 \widetilde{F}(u)d\widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,u}) \\ &= Cr^3\widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,1}) - C\int_0^1 \widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,u})\widetilde{F}'(u)du \\ &\le Cr^4 + Cr^4\int_r^1 u^{-2}du \\ &\le Cr^4 + Cr^3.\end{aligned}$$

(4.6.9) follows by inserting these bounds into (4.6.10).

4.6.5 Proof of Theorem 4.2.2 - concluded

The proof of Theorem 4.2.2 now follows the same lines as Chapter 3 Section 3.7 repeated here for completeness.

Let $\{t \mapsto Y(t)\}$ be a Markovian flight process. Let $\{t \mapsto Z(t)\}$ be a coupled forgetful process. We split $\{t \mapsto Z(t)\}$ into i.i.d legs $(Z_n(t) : 0 \le t \le \theta_n)$, each associated to an i.i.d pack $\varpi_n = \left(\gamma_n; \{\widetilde{\xi}_{n,j}\}_{j=1}^{\gamma}, \{\widetilde{\beta}_{n,j}\}_{j=1}^{\gamma}, \{\widetilde{w}_{n,j}\}_{j=1}^{\gamma}\right)$. In addition, to each leg $(Z_n(t) : 0 \le t \le \theta_n)$ we associate a wind-tree process coupled to that leg $(\mathcal{X}_n(t) : 0 \le t \le \theta_n)$. From these components we construct the concatenated auxilliary process

$$\mathcal{X}(t) = \sum_{k=1}^{\nu_t} \mathcal{X}(\theta_n) + \mathcal{X}_{\nu_t+1}(\{t\}). \tag{4.6.14}$$

Note that $t \mapsto \mathcal{X}(t)$ is *not* a physical process. Each leg is independent of the others. Finally let $t \mapsto X(t)$ be the true wind-tree process, coupled to $t \mapsto Y(t)$ and $t \mapsto Z(t)$ as in Section 4.2.3.

We will use Propositions 4.4.1 and 4.4.2 to prove that until time $T = T(r) = o(r^{-2})$ the processes $t \mapsto X(t)$, $t \mapsto \mathcal{X}(t)$, and $t \mapsto Z(t)$ coincide with high probability.

For this define the (discrete) stopping times

$$\rho := \min\{n : \mathcal{X}_n(t) \not\equiv Z_n(t), 0 \le t \le \theta_n\}$$
$$\sigma := \min\{n : \max\{\mathbb{1}_{\widetilde{W}_n}, \mathbb{1}_{\widehat{W}_n} > 0\} = 1\},$$

and note that by construction

$$\inf\{t : Z(t) \ne X(t)\} \ge \Theta_{\min\{\rho,\sigma\}-1}.$$

Lemma 4.6.1. *Let $T = T(r)$ such that $\lim_{r\to\infty} T(r) = \infty$ and $\lim_{r\to\infty} r^2 T(r) = 0$. Then*

$$\lim_{r\to 0} \mathbf{P}\left(\Theta_{\min\{\rho,\sigma\}-1} < T\right) = 0. \tag{4.6.15}$$

Lemma 4.6.2. *Let $T = T(r)$ such that $\lim_{r\to\infty} T(r) = \infty$ and $\lim_{r\to\infty} r^2 T(r) = 0$. Then for any $\delta > 0$*

$$\lim_{r\to 0} \mathbf{P}\left(\max_{0\le t\le T} |Y(t) - Z(t)| > \delta\sqrt{T}\right) = 0. \tag{4.6.16}$$

Proof of Lemma 4.6.1.

$$\begin{aligned}\mathbf{P}\left(\Theta_{\min\{\rho,\sigma\}-1} < T\right) &\le \mathbf{P}\left(\rho \le 2\mathbf{E}\left(\theta\right)^{-1} T\right) + \mathbf{P}\left(\sigma \le 2\mathbf{E}\left(\theta\right)^{-1} T\right) + \mathbf{P}\left(\sum_{j=1}^{2\mathbf{E}(\theta)^{-1}T} \theta_j < T\right) \\ &\le Cr^2T + Cr^2T + Ce^{-cT}, \end{aligned} \tag{4.6.17}$$

where $C < \infty$ and $c > 0$. The first term on the right hand side of (4.6.17) is bounded by union bound and (4.4.5) from Proposition 4.4.1. Likewise the second term is bounded by union bound Proposition 4.4.2. In bounding the third term we use a large deviation upper bound for the sum of independent θ_j-s.

Finally (4.6.15) readily follows from (4.6.17). □

Proof of Lemma 4.6.2. Note first that

$$\max_{0\le t\le T} |Y(t)-Z(t)| \le \sum_{j=1}^{\nu_T+1} \eta_j \left(\sum_{i=j}^{\gamma_{\nu'_j}} \xi_i \right),$$

with ν_T and η_j defined in (4.2.5), respectively, (4.3.16) and ν'_j is ν_j from (4.4.3) (the label of the leg containing j). Hence,

$$\begin{aligned} \mathbf{P}\left(\max_{0\le t\le T} |Y(t)-Z(t)| > \delta\sqrt{T}\right) &\le \mathbf{P}\left(\sum_{j=1}^{2T} \eta_j \left(\sum_{i=j}^{\gamma_{\nu'_j}} \xi_i \right) > \delta\sqrt{T}\right) + \mathbf{P}\left(\nu_T > 2T\right) \\ &\le C\delta^{-1}\sqrt{T}r + e^{-cT}, \end{aligned} \tag{4.6.18}$$

with $C<\infty$ and $c>0$. The first term on the right hand side of (4.6.18) is bounded by Markov's inequality and the bound

$$\mathbf{E}\left(\eta_j \left(\sum_{i=j}^{\gamma_{\nu'_j}} \xi_i \right)\right) \le Cr.$$

To see this recall the exponential tail bound for γ (4.4.2). The bound on the second term follows from a straightforward large deviation estimate on $\nu_T \sim POI(T)$.

Finally (4.6.16) readily follows from (4.6.18).

□

(4.2.11) is a direct consequence of Lemmas 4.6.1 and 4.6.2 and this concludes the proof of Theorem 4.2.2. □

Part II

Statistics of Hyperbolic Orbits

Chapter 5

Hyperbolic Geometry

5.1 Hyperbolic Half-Plane

Herein we will give a broad overview of the background necessary to read Chapters 6 and 7. For an excellent reference we suggest the book [EW10, Chapter 9] or [BM00]. We do not prove all the statements in this section as most are classical and can be found in those and other texts.

5.1.1 Setup

Definition 5.1.1. Let

$$\mathbb{H} = \{z = x + iy : y > 0\} \tag{5.1.1}$$

denote the *hyperbolic half-plane*, with boundary $\partial\mathbb{H} \simeq \mathbb{R} \cup \{\infty\}$ endowed with the *hyperbolic metric*

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{5.1.2}$$

This hyperbolic metric induces an interesting non-Euclidean geometry. The real line at height $y = 0$ is infinitely far away from a point in the interior and distances are stretched as one moves towards this line. With this metric a pair of parallel lines will now always get infinitely close when approaching the point at infinity. Moreover horizontal parallel lines meet in both directions at infinity while all other pairs meet at infinity in one direction but diverge as they approach the real line.

At every point $z \in \mathbb{H}$ we consider the tangent space at z, $T_z = \{z\} \times \mathbb{C}$ - the set of velocity vectors associated to the point z. Then denote $T(\mathbb{H}) = \mathbb{H} \times \mathbb{C}$, the full *tangent space* and let $T^1(\mathbb{H}) = \mathbb{H} \times S^1_1$ denote the *unit tangent space*: all the points in $\mathbb{H}$ together with the unit length velocity vectors.

Recall that an *isometry* is a distance preserving map of a space. Let

$$G := \mathrm{SL}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}\ ,\ ad - bc = 1 \right\} \tag{5.1.3}$$

denote the *special linear group* and let

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\} \tag{5.1.4}$$

denote the *projective special linear group* (where I denotes the identity). Both these groups act on the half-plane via Möbius transformations: for $z \in \overline{\mathbb{H}}$ (the closure of $\mathbb{H}$) and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$

$$gz = \frac{az + b}{cz + d}. \tag{5.1.5}$$

Proposition 5.1.1. *The group* PSL$(2,\mathbb{R})$ *is the group of orientation-preserving isometries of* $\mathbb{H}$. *Moreover the volume measure* $d\mu := \frac{dxdy}{y^2}$ *is G-invariant.*

The tangent space can then be represented $T^1(\mathbb{H}) \simeq \mathrm{PSL}(2,\mathbb{R})$. That is for any two points $w, z \in T^1(\mathbb{H})$ there exists some $g \in \mathrm{PSL}(2,\mathbb{R})$ such that $gw = z$. In what follows we will sometimes write g for a point in $T^1(\mathbb{H})$ in that case we are referring to the point gX_i where X_i is the vector pointing towards ∞ based at i. Moreover for $u \in T^1(\mathbb{H})$ write $\pi(u)$ for the projection to $\mathbb{H}$.

5.1.2 Lattices

Definition 5.1.2. Given a discrete subgroup $\Gamma < \mathrm{SL}(2,\mathbb{R})$, a *fundamental domain* for the Γ-action on $\mathbb{H}$ is a subset $\mathcal{F} \subset \mathbb{H}$ such that $\bigcup_{\gamma\in\Gamma} \gamma\mathcal{F} = \mathbb{H}$ and for $\gamma_1 \neq \gamma_2 \in \Gamma$, $\mu(\gamma_1\mathcal{F} \cap \gamma_2\mathcal{F})$.

A discrete subgroup $\Gamma < G$ is a *lattice* if $\mu(\mathcal{F}) < \infty$ for any fundamental domain of the Γ-action.

In words the subgroup Γ is the symmetry group of a tiling of $\mathbb{H}$, each fundamental domain is a tile in one of these tilings. If the fundamental domains for this action have finite hyperbolic area then Γ is a lattice.

As an example PSL$(2,\mathbb{Z})$ is a lattice in PSL$(2,\mathbb{R})$. One fundamental domain for this group is

$$\{z \in \mathbb{H} : |z| > 1,\ -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}\},$$

see Figure 5.1. Note that this fundamental domain (and thus all fundamental domains for PSL$(2,\mathbb{Z})$) is not compact as it includes a cusp at infinity. However due to the hyperbolic metric this region does have finite area and thus PSL$(2,\mathbb{Z})$ is a lattice (although not a co-compact lattice).

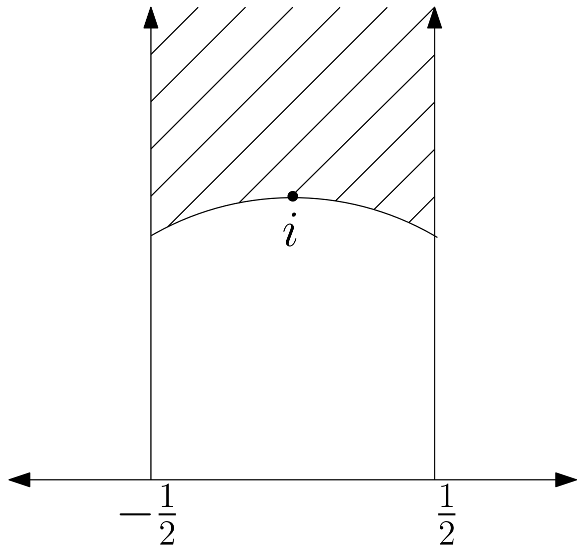

Figure 5.1: A fundamental domain (shaded region) for PSL$(2,\mathbb{Z})$. The left and right sides are glued together and the arc is glued to itself.

5.1.3 Geodesics and Horospheres

Using the hyperbolic metric the *geodesics* in $\mathbb{H}$ (i.e the shortest path between two points) are given by half-circles with centre on the boundary $\partial\mathbb{H}$, therefore to every point in $T^1(\mathbb{H})$ we associate a geodesic. For $u \in T^1(\mathbb{H})$ we denote the forward geodesic endpoint u^+ and the backwards geodesic endpoint u^-. In addition to its geodesic we can also associate to u, *contracting and expanding manifolds*:

$$\mathcal{H}^{\pm}(u) = \{v \in T^1(\mathbb{H}) : v^{\mp} = u^{\mp}\} \tag{5.1.6}$$

($\mathcal{H}^+$ denotes the expanding manifold). We say that the *contracting/expanding horospheres* are the subset of these manifolds that form a ball containing πu. These horospheres are then tangent to $\partial\mathbb{H}$ at $u^{\pm}$ (u^+ for the contracting and u^- for the expanding). See Figure 5.2. As such we can think of horospheres in $\mathbb{H}$ or $T^1(\mathbb{H})$.

There are several subgroups which will be useful later on. Denote

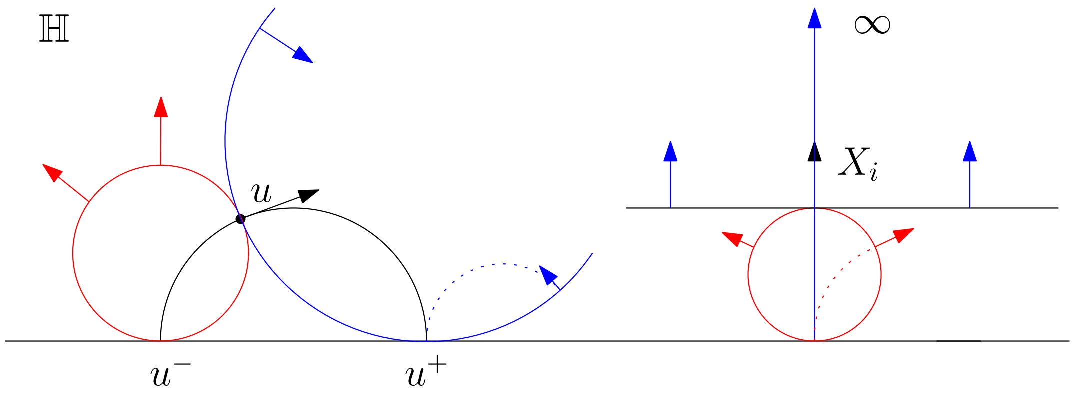

Figure 5.2: On the left, we show the point $u \in T^1(\mathbb{H})$. The black half-circle represents the geodesic. The blue circle with arrows pointing inwards is the contracting horosphere and the red circle the expanding horosphere. On the right we repeat this diagram for the point $X_i \in T^1(\mathbb{H})$. The dotted lines represent geodesics and show that the points on the stable/unstable horospheres share the forwards/backwards geodesic endpoints.

- $K = \mathrm{Stab}_G(i)$, hence $\mathbb{H} \cong G/K$.
- $A = \left\{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t > 0\right\}$, the geodesic flow.
- $N_- := \left\{n_-(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\right\}$, the contracting horosphere for a_t.
- $N_+ := \left\{n_+(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{R}\right\}$, the expanding horosphere for a_t.

We have identified points in G with points in $T^1(\mathbb{H})$ via the map $g \mapsto gX_i$, we can also identify points in G/K with points in $\mathbb{H}$ via the map $g \mapsto gi$. If we consider a point gX_i then multiplying ga_tX_i corresponds to a point a distance t further along the geodesic. $gn_-(x)X_i$ is a point a distance x along the contracting horosphere and $gn_+(x)X_i$ is a point on the expanding horosphere.

5.1.4 Classifying Isometries

There are three different ways in which elements of G act on $\mathbb{H}$. Namely given a matrix $M \neq I$ the group element can be classified as follows:

- **Elliptic:** If $\mathrm{Tr}(M) < 2$ then M corresponds to a rotation about a point, thus M has one fixed point in $\mathbb{H}$.
- **Parabolic:** If $Tr(M) = 2$ then M has one degenerate fixed point on $\partial\mathbb{H}$. For example $n_-(x)$ and $n_+(x)$ are parabolic for all x.
- **Hyperbolic:** If $Tr(M) > 2$ then M has two fixed points on $\partial\mathbb{H}$, one attracting and one repelling. For example a_t is hyperbolic for all t.

To see examples of each of these classifications see Figure 5.3.

Note that parabolic elements correspond to a *finite area cusp* (for example the region in Figure 5.1 has a parabolic element at ∞) while hyperbolic elements correspond to *infinite area funnels*.

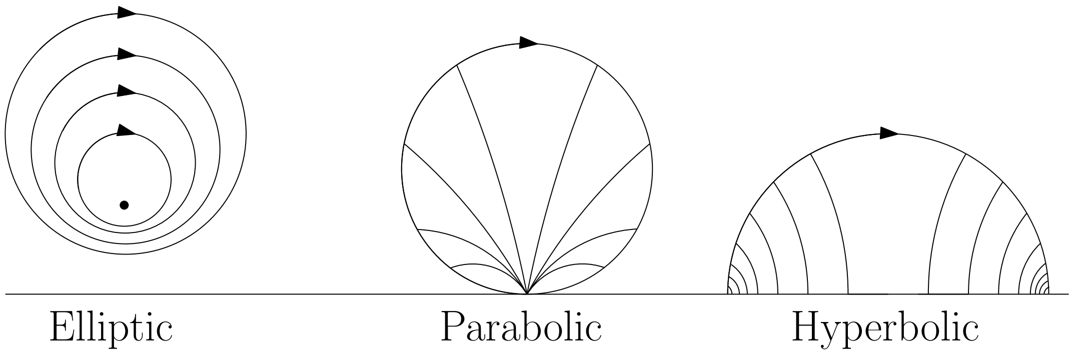

Figure 5.3: Above we show the three types of isometry of the half-plane. The elliptic element corresponds to a fixed rotation around a point in $\mathbb{H}$. For the parabolic each region inside the circle is taken to its right neighbour. Likewise a hyperbolic element shifts each region in the third diagram to the right.

5.1.5 Poincaré Disk

Definition 5.1.3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk with metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}. \tag{5.1.7}$$

We call this model the *Poincaré disk.*

Closely related to the upper half plane the Poincaré disk is another model of hyperbolic geometry (for excellent sources on the different hyperbolic models see [BV86] or [BKS91]). For example it can be shown that geodesics are again circular arcs perpendicular to the boundary.

Given a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{mat}(\mathbb{C})$ we define the same *Möbius map*

$$Mz = \frac{az + b}{cz + d}.$$

These Möbius maps can be shown to be conformal (angle preserving)

Proposition 5.1.2. *The set of automorphisms of the Poincaré disk,* $\text{Aut}(\mathbb{D})$ *is* $\text{SU}(1, 1)$ *where*

$$\text{SU}(1, 1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}) : d = \overline{a}, b = \overline{c}, |a|^2 - |b|^2 = 1 \right\},$$

where $\overline{a}$ *denotes the complex conjugate.*

To connect the Poincaré disk to the upper half-plane we note that the *Cayley map*

$$C : z \mapsto \frac{z - i}{z + i} \tag{5.1.8}$$

is a conformal automorphism of $\mathbb{H} \to \mathbb{D}$.

5.2 Homogeneous Dynamics

A *Euclidean lattice* in $\mathbb{R}^d$ is defined to be the $\mathbb{Z}$-span of d linearly independent vectors in $\mathbb{R}^d$. If one connects neighbouring points in this span the result is a tiling of $\mathbb{R}^d$ and the volume of each of these polyhedra is the *covolume* of the lattice. We denote the space of covolume 1, Euclidean lattices in $\mathbb{R}^d$, $\mathcal{L}_d$. With that definition in mind it is clear that

$$\mathcal{L}_d \cong \mathrm{SL}(d,\mathbb{R})/\,\mathrm{SL}(d,\mathbb{Z}). \tag{5.2.1}$$

Therefore dynamics on the space of lattices is equivalent to dynamics on the homogeneous space $\mathrm{SL}(d,\mathbb{R})/\,\mathrm{SL}(d,\mathbb{Z})$ - sometimes called the *modular group.*

The connection between the modular group and the space of lattices has many far-reaching applications, in particular to number theory. While it would be near impossible to give a full account of these applications a few of these connections are highlighted below.

5.2.1 Diophantine Approximation

One area to which homogeneous dynamics has been applied is Diophantine approximation. Diophantine approximation concerns the question of how 'well' an irrational number can be approximated by rationals. For example

Theorem 5.2.1 (Dirichlet's Theorem (see [Kle01, Theorem 3.1])**.** *For all $\alpha \in \mathbb{R}$ and all $R > 1$ there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $q < R$ and*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{Rq}. \tag{5.2.2}$$

In words this theorem states that for any irrational there are infinitely many good approximants. Starting from here the field of Diophantine approximation seeks to refine these approximation properties.

The approximation properties of irrationals can be linked to the properties of particular orbits in $\mathrm{PSL}(2,\mathbb{R})/\,\mathrm{PSL}(2,\mathbb{Z})$. In particular a number α is called *badly approximable* if Dirichlet's theorem is the best possible bound. Formally if there exists a $c > 0$ such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}. \tag{5.2.3}$$

On the other hand a number α is *singular* if for all $\epsilon > 0$ there exists an $R_0 > 0$ such that for all $R > R_0$ the inequality

$$\left|x - \frac{p}{q}\right| \leq \frac{\epsilon}{qR} \tag{5.2.4}$$

has infinitely many solutions with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $q \leq R$. If we let $\Lambda_x = n_+(x)\mathbb{Z}^2$ then the following theorem due to Dani is central to Diophantine Approximation (it can also be stated in higher dimensions):

Theorem 5.2.2 (Dani's Theorem [Dan95])**.** *Let $x \in \mathbb{R}$:*

1. *If $\{\Lambda_x a_t\}_{t\geq 0}$ is bounded then x is badly approximable.*
2. *If $\{\Lambda_x a_t\}_{t\geq 0}$ is divergent (eventually leaves every compact set forever) then x is singular.*

In addition to this connection there have been many advances in Diophantine approximation thanks to homogeneous dynamics. We list a few examples here and direct the interested reader to the surveys [Kle01, Mar02]:

- Given a non-increasing function $\psi : \mathbb{Z}^+ \to \mathbb{R}^+$ a number α is ψ-approximable if

$$\left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{|q|} \tag{5.2.5}$$

for infinitely many q and some p. *Khintchin's theorem* states that if $\sum_q \psi(q)$ diverges then almost every real α is ψ approximable. One proof (albeit not the only one and not the simplest) of this statement uses exponential mixing of the geodesic flow on the space of lattices. There are numerous refinements of this statement (see [Kle01]). There have been numerous refinements of Khintchin's theorem: recently Koukoulopoulos and Maynard [KM19] (using number theoretic methods) proved the celebrated Duffin-Schaeffer conjecture which allows one to remove the condition that ψ be non-increasing.

- Another problem to which homogeneous dynamics has been applied is *Littlewood's conjecture* which states that given $\alpha, \beta \in \mathbb{R}$

$$\liminf_{n\to\infty} \|n\alpha\|\|n\beta\| = 0, \tag{5.2.6}$$

where $\|\alpha\| = \min_{n\in\mathbb{Z}} |\alpha - n|$ is the distance to the nearest integer. While this conjecture remains open Einsiedler, Katok, and Lindenstrauss [EKL06], by classifying the invariant and ergodic measures on $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ (where $k \geq 3$) for a particular group were able to show that the set of exceptions to Littlewood's conjecture has 0 Hausdorff dimension.

5.2.2 Continued Fractions

As it will be useful in Chapter 7 we note that there is a fascinating relationship between flows on the modular surface and continued fraction expansions of real numbers. This relationship is described formally in great detail in [Ser85].

Let $\xi \in [0,1]$ with continued fraction expansion

$$\begin{aligned}\xi &= [0; a_1, a_2, \dots] \\ &= \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.\end{aligned}$$

Moreover define the Gauss map

$$\begin{aligned}T : &[0; a_1, a_2, \dots] \mapsto [0; a_2, \dots] \\ &\xi \mapsto \left\{\frac{1}{\xi}\right\},\end{aligned} \tag{5.2.7}$$

where $\{\cdot\}$ denotes the fractional part of a number. Given the modular group $\mathrm{SL}(2,\mathbb{Z})$ consider the image of the ideal triangle connecting $0, 1$ and $\infty \in \partial\mathbb{H}$. In Figure 5.4 we show some of the orbit of this ideal triangle by $\mathrm{SL}(2,\mathbb{Z})$. The resulting tessellation is called the *Farey tessellation*. Note that the cusps of the Farey tessellation are exactly the image of 0 by the modular group and generate the rationals $\mathbb{Q}$.

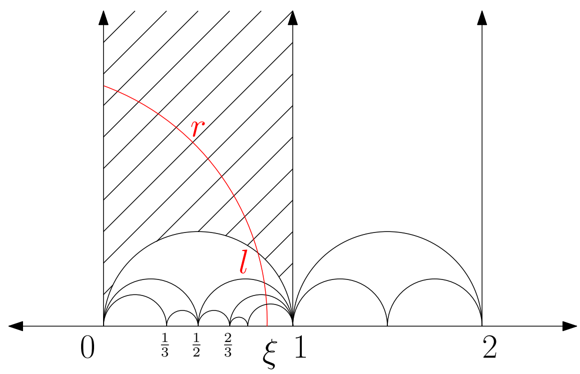

Figure 5.4: We show the image of the ideal triangle $(0, 1, \infty)$ (the shaded region) by the modular group. Moreover we show the first few terms in the cutting sequence for ξ which will be $rlll\ldots$

Now consider a geodesic, γ connecting $(-\infty, 0)$ to $\xi \in (0,1)$. This geodesic will cut each domain in the Farey tessellation as it passes through. In doing so it will separate one of the three cusps from the other two. We construct a sequence as follows: move along γ towards ξ and record whether the cusp which is separated is on the left or the right with an l or an r. This gives a sequence $r^{n_1} l^{n_2} r^{n_3} \ldots$. The cutting sequence for a number ξ is shown in Figure 5.4.

If the cutting sequence ends then $\xi \in \mathbb{Q}$ (i.e ξ lies at the end of a cusp) and we have the relation

$$\xi = [0; n_1, n_2, \ldots, n_k, 1]$$

and if the cutting sequence is infinite then we have the relation

$$\xi = [0; n_1, n_2, \ldots].$$

Since the cutting sequence is unique to the number ξ, as is the continued fraction expansion, there is a one-to-one correspondence between cutting sequences and continued fraction expansions.

The Gauss map represents a shift operator on the continued fraction expansion of ξ. Therefore it can be translated into a sort of shift on the cutting sequence. Formally, Series [Ser85] showed that the Gauss map is equivalent to the return time map for the geodesic flow to a particular cross-section of $\mathbb{T}^1(\mathbb{H})$.

Now if we consider the Haar measure on $T^1(\mathbb{H}) \cong \mathbb{H} \times S^1_1$ parameterised by

$$d\mu(u) = \frac{dxdy}{y^2} d\theta.$$

We can identify a point $u \in T^1(\mathbb{H})$ by its geodesic end points u^-, u^+ and the arc length along the geodesic, t. Then the measure μ can be reparamaeterised

$$d\mu(u) = \frac{du^+ du^-}{|u^+ - u^-|^2} dt$$

If we project this measure onto $\partial\mathbb{H} \times \partial\mathbb{H}$ we are left with $\frac{du^+ du^-}{|u^+ - u^-|^2}$. Now integrate the left end point of $[0, 1]$ and we are left with $\frac{du^+}{u^+(1+u^+)}$ a measure on $(1, \infty)$. Therefore, after changing variables and normalising we end up with the measure on $[0, 1]$:

$$\frac{1}{\log 2} \frac{d\beta}{1 + \beta} \tag{5.2.8}$$

which is the *Gauss measure*, - i.e invariant and ergodic for the Gauss map. Therefore what this train of reasoning tells us is that the Gauss measure is a projection of the Haar measure onto one of its geodesic endpoints. Moreover, the fact that this measure is ergodic and invariant for the Gauss map is a consequence of the fact that the Haar measure is invariant and ergodic for the geodesic flow.

Thus there is a deep connection between dynamics on the modular surface and continued fractions and the symbolic dynamics resulting from the Gauss map.

5.2.3 Local-Statistics of Point Processes

The last area to which homogeneous dynamics has applications which we discuss in this introduction is the study of local statistics of point processes.

Since the time of Mark Kac [Kac59] a fundamental question in modern probability theory is how to characterise independence or 'randomness'? Indeed when considering point processes there are a number of ways to characterise independence. Firstly, we can ask if the sequence is uniformly distributed with respect to a given measure. That is, does the proportion of points in a small set converge to the measure of that set, as would be the case for independently distributed points?

While this is a very interesting question, it is sometimes too coarse a measure of independence. One of the next questions typically asked is: what can be said about local statistics? That is, what does the presence of points tell one about the likelihood of finding another point nearby? Concretely one example of local statistics is the gap distribution which measures the distribution of the distances between neighbouring points. Thus one can ask if the local statistics (e.g gap distribution) of a sequence converge to those of an independently distributed sequence.

Homogeneous dynamics is well equipped to tackle many examples of these questions, in particular when the points are generated using some sort of periodic procedure. One example which we study in Chapter 6 is the local-statistics of hyperbolic groups. Roughly speaking, the idea is to place an observer in hyperbolic space or it's boundary and consider the orbit of another point by a discrete subgroup. Then we can generate a point set by considering the direction of the orbit points as viewed by the observer ordered by the distance from the observer. This is a fundamental way to study the orbit of a group. Moreover if the group can be connected to another object in mathematics then it may be possible to move from the local statistics of the group orbit to those of the object. We will return to the relevant literature in Chapters 6 and 7 but for now suffice it to say that the problem has been extensively studied for lattices [BPZ14, KK15, RS17, MV18], for some thin groups [Zha17, Zha19] and even for surfaces of variable curvature [Pol17]. In Chapter 6 we will study this problem for a wide class of (possibly thin) groups.

Another example of such a system which is still very relevant to modern mathematics (although not so much this thesis) is the system $\{\alpha\sqrt{n} \mod 1\}_{n\in\mathbb{N}}$. Elkies and McMullen [EM04] showed using homogeneous dynamics methods (i.e equidistirbution of expanding horospherical subgroups) that if $\alpha^2 \in \mathbb{Q}$ then these points obey a particular explicit limiting distribution, as do their gaps. It is conjectured that for α^2 irrational that these points are Poisson distributed.

As a last aside, characterising the limiting local-statistics is an important and interesting question which is asked in numerous contexts. For example the famous Berry-Tabor conjecture [BT77] states that for typical Riemannian surfaces, if the dynamics on the surface are integrable (i.e not chaotic), this implies that the eigenvalues of the Laplacian have Poisson distributed gaps (and it is conjectured that if the dynamics are chaotic then the gap distribution (typically) will be related to a random matrix ensemble).

5.3 Thin Groups

Thin groups have become a hot topic recently owing in part to two major developments which have made them significantly more accessible. For some detailed references we suggest the survey articles [Kon16, KLLR19], or the conference proceedings [BO14]. For what follows we will need to introduce a small amount of algebraic geometry.

Definition 5.3.1. An *algebraic variety*, over a field k is the common 0-set of a finite collection of polynomials over k. That is,

$$X := \{\mathbf{x} \in k^n : F_i(\mathbf{x}) = 0, \forall 1 \leq i \leq n\} \tag{5.3.1}$$

where $F_i \in k[x_1, \ldots x_n]$. An *algebraic group* is a group which is also an algebraic variety.

For example $\mathrm{SL}(n, \cdot)$ defined over a field k together with matrix multiplication is an algebraic group where the polynomial preserved is the determinant minus 1. Note that given two polynomials f_1 and f_2 defined over the same field, if we denote the varieties associated to each by $\mathbb{V}(f_1)$ and $\mathbb{V}(f_2)$ note that

$$\mathbb{V}(f_1) \cap \mathbb{V}(f_2) = \mathbb{V}(\{f_1, f_2\})$$
$$\mathbb{V}(f_1) \cup \mathbb{V}(f_2) = \mathbb{V}(f_1 \cdot f_2).$$

Therefore varieties induce a topology on k^n, where varieties form the closed sets. We call this topology the *Zariski Topology.* Therefore we say a subgroup is *Zariski dense* if it does *not* belong to the 0-set of any additional polynomials.

Given an algebraic group defined over $\mathbb{Q}$, $\mathbb{G}$, we say that a subgroup of $\mathbb{G}(\mathbb{Z})$ is an *arithmetic group* if it has finite index and we say a subgroup $\mathbb{G}(\mathbb{Z})$ is a *thin group* if it has infinite index. This part of the thesis is concerned with infinite volume hyperbolic subgroups, a subset of which are thin. However, since thin groups have been the source of a great deal of modern mathematical research lately we highlight this application here by giving a brief explanation as to why thin groups have been promoted from the side-lines of mathematical research.

In essence there are two reasons for which thin groups have become a hot topic recently: super-strong approximation and Patterson-Sullivan theory. We discuss Patterson-Sullivan theory in Section 5.5 as it will be crucial to our results and proofs. However to illustrate the importance of thin groups we will also briefly discuss some of the applications of strong and super-strong approximation.

5.3.1 Strong and Super-Strong Approximation

For a detailed exposition on super-strong approximation we recommend the notes by Emmanuel Breuillard [Bre14]. Strong approximation proved independently by [Nor87] and [Wei84] is the following theorem, for simplicity we state it for $\mathrm{SL}(n, \mathbb{Z})$, however the result holds for all simply connected, semi-simple algebraic groups defined over $\mathbb{Q}$.

Theorem 5.3.1. *[Strong-approximation for $\mathrm{SL}(n, \mathbb{Z})$ [Bre14]] Let $\Gamma \leq \mathrm{SL}(n, \mathbb{Z})$ be a Zariski dense subgroup. Then Γ_p (the reduction of Γ modulo p) is equal $\Gamma_p = \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ for all p large enough.*

In 2008 Bourgain and Gamburd [BG08] established super-strong approximation for thin subgroups $\Gamma < \mathrm{SL}(2, \mathbb{Z})$. In words their statement is the following, given a generating set for Γ, we consider the reduction of these generators $\mod p$. Super strong approximation is the statement that these generators 'fill out' $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ rapidly (specifically the family of Cayley graphs associated to $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ is an expander family). Strong approximation has also been generalised (see [SGV12]).

The power of Strong approximation is that rather than study the properties of a thin group, one can instead consider finitely many reductions modulo primes. Super strong approximation is a statement about the spectral gap of the Cayley graphs which can be translated into a statement about the mixing properties of a random walk on the graph. Without entering into the details, it will suffice to say that this statement has been tremendously powerful in allowing mathematicians to approach thin groups. Two frequently cited applications of strong and super-strong approximation are the affine sieve and local-global principles.

While sieving techniques have been around for many years, the affine sieve is a new variation developed by Bourgain, Gamburd and Sarnak [BGS11] and Salehi Golsefidy and Sarnak [SGS13]. The idea is the following, given a suitable thin group Γ, the orbit of a point $\Gamma\mathbf{v}$, and a suitable function f. The affine sieve is the statement that there is a constant R such that there exist infinitely many

R-almost primes (i.e integers with fewer than R prime factors) in f evaluated on the group orbit. As an example the affine sieve has been applied to Apollonian circle packings [Oh14] and Pythagorean triples [KO12].

A local-global principle states that given a sequence of numbers, every integer outside of finitely many congruence conditions appears in the sequence. Strong approximation has important applications for proving local-global principles. As one example we note that strong approximation played a role in Bourgain and Kontorovich's proof that Zaremba's conjecture holds, outside possibly a set of density 0 [BK14]. The conjecture states the following: For $A \in \mathbb{N}$ let

$$\mathcal{C}_A := \left\{ \frac{p}{q} = [0; a_1, \dots a_n] : (p,q) = 1 \ \& \ a_i < A \text{ for all } 1 \le i \le n \right\} \tag{5.3.2}$$

Then let $\mathcal{D}_A := \{q : p/q \in \mathcal{C}_A\}$. Zaremba's conjecture states that all sufficiently large natural numbers belong to $\mathcal{D}_A$ for some $A > 1$. Bourgain and Kontorovich showed that a set of asymptotic density 1 in the natural numbers appears in $\mathcal{D}_{50}$ (this result was subsequently improved to $\mathcal{D}_5$ by Huang [Hua15]).

5.4 Higher Dimensional Hyperbolic Space

In the next sections we will discuss Patterson-Sullivan theory, however in Chapter 6 we will require this theory in higher dimensions. Therefore to avoid repetition we will present the setup now for the higher dimensional case.

Let

$$\mathbb{H}^n := \{(x_1, \dots, x_n, y) : y > 0\}$$

with the hyperbolic metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2 + dy^2}{y^2}.$$

As in the two dimensional setting we consider the unit tangent space $T^1(\mathbb{H}^n)$.

For convenience we introduce the notion of Clifford numbers. This notation will be useful in describing the isometry group G using an extension of complex numbers and quaternions to higher dimensions and will help with some of the calculations. What follows is a condensed introduction to the concept. For a more in-depth introduction we suggest the paper by Waterman [Wat93].

Define the *Clifford Algebra*, C_m to be the real associative algebra generated by $\mathbf{i}_1, ..., \mathbf{i}_m$ such that $\mathbf{i}_j^2 = -1$ and $\mathbf{i}_j\mathbf{i}_k = -\mathbf{i}_k\mathbf{i}_j$ for all $k \neq j$. Thus for all $\mathbf{a} \in C_m$

$$\mathbf{a} = \sum_{\mathbf{I}} a_{\mathbf{I}} \mathbf{I} \tag{5.4.1}$$

where $\mathbf{I}$ ranges over the products of the $\mathbf{i}_j$ and $a_{\mathbf{I}} \in \mathbb{R}$. C_m forms a 2^m-dimensional vector space over $\mathbb{R}$, which we endow with the norm $|\mathbf{a}|^2 = \sum_{\mathbf{I}} a_{\mathbf{I}}^2$.

Consider the following three involutions on C_m

- $\mathbf{a} \mapsto \mathbf{a}'$ - replaces all $\mathbf{i}_l$ with $-\mathbf{i}_l$ for all l
- $\mathbf{a} \mapsto \mathbf{a}^*$ - replaces all $\mathbf{I} = \mathbf{i}_{\nu_1}, ..., \mathbf{i}_{\nu_l}$ with $\mathbf{i}_{\nu_l}, ..., \mathbf{i}_{\nu_1}$

- $\mathbf{a} \mapsto \bar{\mathbf{a}} := \mathbf{a}'^*$

Define *Clifford vectors* to be vectors $\mathbf{x} = x_0 + x_1\mathbf{i}_1 + ... + x_m\mathbf{i}_m$ with the corresponding vector space denoted V_m (which we identify with $\mathbb{R}^m$ in the natural way). We write Δ_m for the *Clifford group*, i.e the group generated by non-zero Clifford vectors.

Furthermore we define several matrix groups

$$\begin{aligned}
\mathrm{GL}(2, C_m) &:= \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} : \begin{array}{l} \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \Delta_m \cup \{0\} \\ \mathbf{ab}^*, \mathbf{cd}^*, \mathbf{c}^*\mathbf{a}, \mathbf{d}^*\mathbf{b} \in V_m \\ \mathbf{ad}^* - \mathbf{bc}^* \in \mathbb{R} \setminus \{0\} \end{array} \right\}, \\
\mathrm{SL}(2, C_m) &:= \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \mathrm{GL}(2, C_m) : \mathbf{ad}^* - \mathbf{bc}^* = 1 \right\}, \\
\mathrm{SU}(2, C_m) &:= \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ -\mathbf{b}' & \mathbf{a}' \end{pmatrix} \in \mathrm{SL}(2, C_m) \right\}.
\end{aligned} \tag{5.4.2}$$

We can then represent hyperbolic half-space by

$$\mathbb{H}^n = \{\mathbf{x} + \mathbf{i}y : \mathbf{x} \in V_{n-1},\ y \in \mathbb{R}_{>0}\} \tag{5.4.3}$$

with $\mathbf{i} := \mathbf{i}_{n-1}$ (and with the usual hyperbolic metric on $\mathbb{H}^n$). Moreover the action of $\mathrm{SL}(2, C_m)$ on $\mathbb{H}^n$ defined via Möbius transformations

$$\mathbf{z} \mapsto \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \mathbf{z} = (\mathbf{az} + \mathbf{b})(\mathbf{cz} + \mathbf{d})^{-1} \tag{5.4.4}$$

is isometric and orientation-preserving. Therefore

$$G \cong \mathrm{PSL}(2, C_{n-1}) = \mathrm{SL}(2, C_{n-1})/\{\pm 1\} \tag{5.4.5}$$

is isomorphic to the group of orientation-preserving isometries of $\mathbb{H}^n$. The boundary of $\mathbb{H}^n$ can be identified

$$\partial\mathbb{H}^n := V_{n-1} \cup \{\infty\}. \tag{5.4.6}$$

Now, as was done for the two dimensional case, consider a point $\mathbf{i} \in \mathbb{H}^n$, a vector based at that point $\mathbf{X}_\mathbf{i} \in T^1(\mathbb{H}^n)$ and the following relevant subgroups:

- The stabiliser of $\mathbf{i}$ is given by

$$K \cong \mathrm{PSU}(2, C_{n-1}) = \mathrm{SU}(2, C_{n-2})/\{\pm 1\}. \tag{5.4.7}$$

  Hence we identify $\mathbb{H}^n \cong G/K$.

- $M := \mathrm{Stab}_G(\mathbf{X}_\mathbf{i})$, hence $T^1(\mathbb{H}^n) \cong G/M$. Thus $M = \left\{ \begin{pmatrix} \mathbf{a} & 0 \\ 0 & (\mathbf{a}^{-1})^* \end{pmatrix} : |\mathbf{a}| = 1 \right\}$.

- $A := \{a_r : r \in \mathbb{R}\}$ - one-parameter subgroup in the centraliser of M such that $r \mapsto a_r\mathbf{X}$ is the unit speed geodesic flow for any $\mathbf{X} \in T^1(\mathbb{H}^n)$. For $\mathbf{X}$ pointed in the vertical direction this subgroup is given by the matrices $\begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$. For other vectors A is conjugate to this group.

- $N_+ := \{n_+ \in G : \lim_{t\to\infty} a_{-t}n_+a_t = I\}$ - the expanding horocycle subgroup, thus N^+ is conjugate to upper triangular matrices.

- $N_- := \{n_- \in G : \lim_{t\to\infty} a_t n_- a_{-t} = I\}$ - contracting horocycle subgroup (conjugate to lower triangular matrices).

Note that N_+ and N_- are defined for the left a_r action. Alternatively, given $g \in G$ one can define the right a_r action by right multiplication $g \mapsto ga_r$. Thus a point $\mathbf{u} = g_{\mathbf{u}}\mathbf{X_i} \in T^1(\mathbb{H}^n)$ is sent to $g_{\mathbf{u}}a_r\mathbf{X_i}$. In this case N_+ and N_- are contracting and expanding respectively (i.e their roles are reversed).

Notation: In Chapter 6 we will work with $\mathbb{H}^n$ for general $n \geq 2$. Therefore, in that chapter, we will use the bold-face notation for points in $\mathbb{H}^n$ and $T^1(\mathbb{H}^n)$ which we established here. In Chapter 7 we only work in $\mathbb{H}$ therefore, in keeping with standard practice, we will not use this bold-face notation.

5.5 Measure Theory of Infinite Volume Manifolds

The homogeneous dynamics described in Section 5.2 is restricted to the finite co-volume setting (and thus does not apply, for example, to thin groups). This is because the Haar measure, which is the invariant measure under the action of $\mathrm{SL}(d,\mathbb{R})$ has infinite volume in the infinite co-volume setting. As a result many of the ergodic properties exploited in 'classical' homogeneous dynamics do not apply. Fortunately in the 1970s, the measure theory needed to construct nice ergodic measures for infinite covolume subgroups was formulated.

In preparation for what follows we introduce the notion of Hausdorff dimension (however we remain brief as we will not need many details). For an extensive treatment of this subject we suggest Falconer's book [Fal05]. Given a set X, let $\mathcal{U} = \{U_i\}_i \in \mathcal{I}$ be an open cover of X (i.e $X \in \bigcup_{i\in\mathcal{I}} U_i$). For $\epsilon, \delta > 0$ we define the ϵ-Hausdorff measure of X to be

$$H_\epsilon^\delta := \inf_{\mathcal{U}} \left\{ \sum_{i\in\mathcal{I}} \mathrm{diam}(U_i)^\delta \right\}$$

where the infimum is taken over all open covers $\mathcal{U}$ such that for all i, the diameter $\mathrm{diam}(U_i) \leq \epsilon$. Now we define the Hausdorff measure to be the limit $H^\delta(X) := \lim_{\epsilon\downarrow 0} H_\epsilon^\delta(X)$. With that the *Hausdorff dimension* is

$$\dim_H(X) := \inf\{\delta > 0 : H^\delta(X) = 0\}. \tag{5.5.1}$$

For our purposes it suffices to know that the Hausdorff is a measure of how large a set is. It coincides with the standard definition of integer dimension. But also gives a measure to the size of fractal sets.

5.5.1 Patterson-Sullivan Theory

We now give an introduction to measure theory on infinite volume hyperbolic manifolds. For a more in-depth introduction in 2 dimensions we recommend the opening sections of the book by Borthwick [Bor07] or the book [BKS91]. To begin with, let Γ be a discrete subgroup of $G = \mathrm{Isom}^+(\mathbb{H}^n)$.

For $\mathbf{u} \in T^1(\mathbb{H}^n)$ define the geodesic endpoints in terms of the right a_t action for $\mathbf{u} = g_{\mathbf{u}}\mathbf{X_i}$

$$\mathbf{u}^\pm = \lim_{t\to\pm\infty} g_{\mathbf{u}} a_t \mathbf{X_i}. \tag{5.5.2}$$

Let δ_Γ denote the *critical (or Poincaré) exponent of the subgroup* Γ. That is, for arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{H}^n$

$$\delta_\Gamma := \inf\left\{ s > 0 : \sum_{\gamma\in\Gamma} e^{-sd(\gamma\mathbf{x},\mathbf{y})} < \infty \right\}. \tag{5.5.3}$$

Let $\mathcal{L}(\Gamma)$ denote the *limit set of* Γ (i.e the set of accumulation points of the orbit of any point in $\mathbb{H}^n$, say $\mathbf{i}$). For the Γ we are considering $\mathcal{L}(\Gamma) \subset \partial\mathbb{H}^n$. Moreover it is well-known ([Sul79]) that δ_Γ is the Hausdorff dimension of $\mathcal{L}(\Gamma)$.

In this thesis rather than work with general discrete subgroups we will work with *geometrically finite, Zariski dense, non-elementary* subgroups (which includes a large class of relevant thin groups, as well as lattices). A group Γ is non-elementary if the limit set $\mathcal{L}(\Gamma)$ contains more than 2 points (and thus is uncountable - see [BKS91]). Consider the set of geodesics connecting any two points in $\mathcal{L}(\Gamma)$ together, the convex core of Γ is the projection to $\Gamma\backslash\mathbb{H}^n$ of the minimal convex set containing all these geodesics. A group Γ is *geometrically finite* if the unit neighbourhood of the convex core has finite Riemannian volume. As noted in [OS13] any group admitting a finite sided polyhedron as its fundamental domain is geometrically finite.

For $\boldsymbol{\xi} \in \partial\mathbb{H}^n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{H}^n$ denote the *Busemann function*, $\beta : \partial\mathbb{H}^n \times \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$

$$\beta_{\boldsymbol{\xi}}(\mathbf{x},\mathbf{y}) = \lim_{t\to\infty} d(\mathbf{x},\boldsymbol{\xi}_t) - d(\mathbf{y},\boldsymbol{\xi}_t) \tag{5.5.4}$$

where $\boldsymbol{\xi}_t$ lie on any geodesic ray such that as $\lim_{t\to\infty}\boldsymbol{\xi}_t = \boldsymbol{\xi}$ (the limiting value is independent of the choice of ray). In words $\beta_{\boldsymbol{\xi}}(\mathbf{x},\mathbf{y})$ is the signed geodesic distance between two horospheres each based at $\boldsymbol{\xi}$ containing $\mathbf{x}$ and $\mathbf{y}$ respectively.

With that, let $\{\mu_{\mathbf{x}} : \mathbf{x} \in \mathbb{H}^n\}$ denote a family of measures on $\partial\mathbb{H}^n$. We call such a family a Γ-*invariant conformal density of dimension* $\delta_\mu > 0$ if: for each $\mathbf{x} \in \mathbb{H}^n$, $\mu_{\mathbf{x}}$ is a finite Borel measure such that

$$\begin{aligned} &\gamma_*\mu_{\mathbf{x}}(\cdot) := \mu_{\mathbf{x}}(\gamma^{-1}(\cdot)) = \mu_{\gamma\mathbf{x}}(\cdot) \\ &\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}}(\boldsymbol{\xi}) = e^{\delta_\mu\beta_{\boldsymbol{\xi}}(\mathbf{y},\mathbf{x})}, \end{aligned} \tag{5.5.5}$$

for all $\mathbf{y} \in \mathbb{H}^n$, $\boldsymbol{\xi} \in \partial\mathbb{H}^n$, and $\gamma \in \Gamma$.

Patterson in dimension 2 [Pat76] and Sullivan [Sul79] for general dimension, proved the existence of a Γ-invariant conformal density of dimension δ_Γ, the critical exponent, supported on $\Lambda(\Gamma)$ which we will denote $\{\nu_{\mathbf{x}} : \mathbf{x} \in \mathbb{H}^n\}$ - the *Patterson-Sullivan density*. In particular, for $s > \delta_\Gamma$ define the probability measures

$$\mu_{\mathbf{x}}^{(s)} := \left(\sum_{\gamma\in\Gamma} e^{-sd(\mathbf{i},\gamma\mathbf{i})}\right)^{-1} \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{x},\gamma\mathbf{i})}\delta_{\gamma\mathbf{i}}$$

where $\delta_{\mathbf{w}}$ is the point measure supported at $\mathbf{w} \in \mathbb{H}^n$. In which case we define $\nu_{\mathbf{x}}$ to be the weak star limit as $s \to \delta_\Gamma$ from above. Moreover let the *Lebesgue density*, $\{\mathrm{m}_{\mathbf{x}} : \mathbf{x} \in \mathbb{H}^n\}$ denote the G-invariant conformal density of dimension $(n-1)$, unique up to homothety.

From here we can define several measures on $T^1(\mathbb{H}^n)$ which will be essential to what follows. For $\mathbf{u} \in T^1(\mathbb{H}^n)$, let $\pi(\mathbf{u})$ be the projection to $\mathbb{H}^n$, $s := \beta_{\mathbf{u}^-}(i, \pi(\mathbf{u}))$ and define

- The *Bowen-Margulis-Sullivan* measure, given by

$$d\mathrm{m}^{BMS}(\mathbf{u}) = e^{\delta_\Gamma\beta_{\mathbf{u}^+}(\mathbf{i},\pi(\mathbf{u}))} e^{\delta_\Gamma\beta_{\mathbf{u}^-}(\mathbf{i},\pi(\mathbf{u}))} d\nu_{\mathbf{i}}(\mathbf{u}^+) d\nu_i(\mathbf{u}^-) ds. \tag{5.5.6}$$

This measure is supported on $\{\mathbf{u} \in T^1(\mathbb{H}^n) : \mathbf{u}^+, \mathbf{u}^- \in \Lambda(\Gamma)\}$ and is finite on $T^1(\Gamma\backslash\mathbb{H}^n)$ for geometrically finite Γ [Sul79].

- The *Burger-Roblin* measure

$$d\mathrm{m}^{BR}(\mathbf{u}) = e^{\delta_\Gamma \beta_{\mathbf{u}^-}(\mathbf{i},\pi(\mathbf{u}))} e^{(n-1)\beta_{\mathbf{u}^+}(\mathbf{i},\pi(\mathbf{u}))} d\nu_{\mathbf{i}}(\mathbf{u}^-) d\mathrm{m}_{\mathbf{i}}(\mathbf{u}^+) ds. \tag{5.5.7}$$

This measure is supported on $\{\mathbf{u} \in T^1(\mathbb{H}^n) : \mathbf{u}^- \in \Lambda(\Gamma)\}$ and is, in general, not finite on $T^1(\Gamma\backslash\mathbb{H}^n)$.

These are both measures on $T^1(\mathbb{H}^n) \cong G/M$. We extend them to measures on G. That is, let μ be either m^{BR} or m^{BMS} defined on $T^1(\mathbb{H}^n)$, for $\phi \in \mathcal{C}_c(G)$

$$\int_G \phi(g) d\mu(g) = \int_{T^1(\mathbb{H}^n)} \int_M \phi(\mathbf{u}m) d\mu_M^{Haar}(m) d\mu(\mathbf{u}) \tag{5.5.8}$$

where $\mu_M^{Haar}(m)$ is the normalised probability Haar measure on M. Thus we simply average out the extra dependence. To avoid too much notation we denote the BR-measures on G and $T^1(\mathbb{H}^n)$ both by m^{BR} and likewise for the BMS-measure.

Furthermore, let $H < G$ be an expanding horosperical subgroup for the *right* a_r-action (i.e a subgroup of N_-). Let $\overline{H} := H/(M \cap H)$ be the projection to $T^1(\mathbb{H}^n)$. For a fixed $g \in G$, define

$$d\mu_{g\overline{H}}^{PS}(gh) := e^{\delta_\Gamma \beta_{gh\mathbf{X}_{\mathbf{i}}^+}(\mathbf{i},gh\mathbf{i})} d\nu_{\mathbf{i}}(gh\mathbf{X}_{\mathbf{i}}^+). \tag{5.5.9}$$

In what follows in the next two chapters, it will be useful to consider the push-forward of these measures via parameterisations. Given a horospherical subgroup H, $\overline{H}$ is isomorphic with a horosphere in $T^1(\mathbb{H}^n)$. Hence there exists a group isomorphism

$$\mathrm{hor} : \mathbb{R}^{n-1} \to \overline{H} \tag{5.5.10}$$

such that the push-forward of the Haar measure is equal to the Lebesgue measure

$$d\mu_{\overline{H}}^{Haar}(\mathrm{hor}^{-1}(\mathbf{x})) = d\mathbf{x}. \tag{5.5.11}$$

Define the measure on $\mathbb{R}^{n-1}$

$$d\omega_{\Gamma,g,\overline{H}}^{PS}(\mathbf{x}) := d\mu_{\Gamma g\overline{H}}^{PS}(g\,\mathrm{hor}^{-1}(\mathbf{x})). \tag{5.5.12}$$

Lastly for what follows we would also like to define spherical Patterson-Sullivan measures. That is, a measure supported on the rotation group K/M. Since the Patterson-Sullivan measure is supported on the limit set which lives on the boundary $\partial\mathbb{H}^n$ and since the boundary is isomorphic to S_1^{n-1} (the unit circle) this can be done. However the parameterisation is more delicate than for horospheres since there is not a single natural parameterisation of the rotation group.

Let $\overline{K} = K/M$ and define the *spherical Patterson-Sullivan* measure to be

$$d\mu_{\Gamma g\overline{K}}^{PS}(gk) := e^{\delta_\Gamma \beta_{gk\mathbf{X}_{\mathbf{i}}}(\mathbf{i},gke^{-1}\mathbf{i})} d\nu_{\mathbf{i}}(gk\mathbf{X}_{\mathbf{i}}). \tag{5.5.13}$$

For a fixed $g \in G$, the prefactor $e^{\delta_\Gamma \beta_{gk\mathbf{X}_{\mathbf{i}^+}}(\mathbf{i},gke^{-1}\mathbf{i})}$ is constant.

As mentioned, unlike for horospheres there is not a single natural way to parameterise spheres. Therefore we add a Jacobian to ensure the parameterised Patterson-Sullivan measure is invariant for different parameterisations. Specifically we use the following polar coordinate change of variables.

Lemma 5.5.1. *For $k \in \overline{K}$ let $u = k^{-1}\mathbf{0}$. Writing $k = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ -\mathbf{b}' & \mathbf{a}' \end{smallmatrix}\right)$ we have the following change of variables*

$$du = |\mathbf{a}|^{n-1}dk. \tag{5.5.14}$$

Proof. While this is classical we present a proof using conformal densities for completeness. First note

$$du = e^{(n-1)\beta_{n_+(-u)\mathbf{X}_\mathbf{i}^-}(\mathbf{i}, n_+(-u)\mathbf{i})}dm_\mathbf{i}(n_+(-u)\mathbf{X_i}). \tag{5.5.15}$$

Since $u = -\mathbf{ba}^{-1}$ we can write

$$({}^tk)^{-1} = -n(-u)\begin{pmatrix} |\mathbf{a}^{-1}| & 0 \\ 0 & |\mathbf{a}| \end{pmatrix}\begin{pmatrix} \frac{\mathbf{a}'+\mathbf{b}'\mathbf{a}^{-1}\mathbf{b}}{|\mathbf{a}|^{-1}} & 0 \\ 0 & \frac{-\mathbf{a}}{|\mathbf{a}|} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\mathbf{a}^{-1}\mathbf{b} & 1 \end{pmatrix} \tag{5.5.16}$$

where ${}^tk$ denotes the transpose. Note that the rightmost matrix is in N_-, the second from the right is in M and the third is in A. Therefore

$$n_+(-u)\mathbf{X}_\mathbf{i}^- = {}^tk^{-1}\mathbf{X}_\mathbf{i}^-. \tag{5.5.17}$$

Moreover

$$\beta_{n_+(-u)\mathbf{X}_\mathbf{i}^-}(\mathbf{i}, n_+(-u)\mathbf{i}) = \ln|\mathbf{a}| + \beta_{({}^tk)^{-1}\mathbf{X}_\mathbf{i}^-}(\mathbf{i}, ({}^tk)^{-1}\mathbf{i}) = \ln|\mathbf{a}| + \beta_{({}^tk)^{-1}\mathbf{X}_\mathbf{i}^-}(\mathbf{i},\mathbf{i}) = \ln|\mathbf{a}|. \tag{5.5.18}$$

Thus

$$du = |\mathbf{a}|^{n-1}dm_\mathbf{i}(({}^tk)^{-1}\mathbf{X_i}). \tag{5.5.19}$$

The measure $dm_\mathbf{i}(({}^tk)^{-1}\mathbf{X_i}) = d({}^tk)^{-1} = dk$. Proving Lemma (5.5.14).

□

Now fix $g \in G$ and a parameterisation $\mathbf{x} \mapsto R(\mathbf{x}) \in \overline{K}$ with $\mathbf{x}$ ranging in a non-empty open set $\mathcal{U} \subset \mathbb{R}^{n-1}$. Let $\tilde{\mathbf{x}} = R(\mathbf{x})\mathbf{0}$ and $\left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right|$ the standard Jacobian on $\mathbb{R}^{n-1}$. Define the *parameterised spherical Patterson-Sullivan measure* for $\mathcal{U}$ to be

$$d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x}) = \left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right|^{-1}|\mathbf{a}|^{n-1}d\mu^{PS}_{\Gamma g\overline{K}}(gR(\mathbf{x})). \tag{5.5.20}$$

5.5.2 Some Properties

We mention here a few of the properties of the BMS and BR measures. There are numerous results (e.g [Rob03] or [FS90]) therefore we will only present the theorems which are necessary in what follows. With regards ergodictiy we have that

Theorem 5.5.2 ([Win15])**.** *Let m^{BR} and m^{BMS} be as above, normalised to be a probability measures.*

1. *The Burger-Roblin measure is ergodic with respect to the flow N_+ (the expanding horosphere flow).*
2. *The Bowen-Margulis-Sullivan measure is mixing on $\Gamma\backslash G$ with respect to the frame flow $\{a_t\}$: for any $\psi_1, \psi_2 \in \mathcal{C}_c(\Gamma\backslash G)$*

$$\lim_{t\to\infty}\int_{\Gamma\backslash G}\psi_1(ga_t)\psi_2(g)d\mathrm{m}^{BMS} = \frac{\mathrm{m}^{BMS}(\psi_1)\mathrm{m}^{BMS}(\psi_2)}{|\mathrm{m}^{BMS}|}. \tag{5.5.21}$$

We note that there are effective versions of the mixing theorem in various contexts which we will not need.

The other property of these measure we will make use of is the following decomposition (due to [OS13, Proposition 7.3]) which generalises the so-called Iwasawa decomposition [Iwa49] for the Burger-Roblin measure:

Proposition 5.5.3 ([OS13, Proposition 7.3])**.** *For any $\phi \in C_c(T^1(\mathbb{H}^n))$*

$$\mathrm{m}^{BR}(\phi) = \int_{k\in K}\int_{r\in\mathbb{R}}\int_{n_+\in N_+} \phi(ka_r n_+)e^{-\delta_\Gamma r} dn_+ dr d\nu_{\mathbf{i}}(kX_0^-). \tag{5.5.22}$$

For a more in-depth account of properties of the BMS and BR measures see either [Moh13] or [MO11].

5.6 Horospherical Equidistribution

Classically horospherical equidistribution is one of the powerhouse tools of homogeneous dynamics. To the author's knowledge, the idea goes back to Margulis' thesis [Mar04]. More recently there have been effective versions of this equidistribution result by Strömbergsson [Str04] and Sarnak [Sar81] which have countless important implications. These theorems (and similar ones) have proved tremendously useful, as a few examples we note that the main theorem of [Str04] plays a role in Venkatesh's proof of an important step towards a conjecture in sparse equidsitribution [Ven10]. Moreover horospherical equidsitribution theorems were used in [MS11] wherein Marklof and Strömbergsson studied the periodic Lorentz gas in the Boltzmann-Grad limit (see Chapter 2 Section 2.2.2). Complimenting these results for the equidistribution of expanding horospheres, Dani and Smillie [DS84] showed that for any finite volume hyperbolic surface, all horocyclic orbits are either periodic or equidistribute.

The above mentioned results are concerned with how horospheres equidistribute when acted on by the geodesic flow in *finite* volume manifolds (e.g the modular surface). For our purposes we will make use of the analogous results for infinite volume manifolds. What follows are several equidistribution results converging to the result we will need, starting from a theorem of Oh and Shah.

Our goal is to start with an equidistribution theorem of Oh and Shah [OS13, Theorem 3.6]. However their theorem applies only to M-invariant functions whereas we need an equidistribution theorem for functions on G. A similar equidistribution theorem for functions of G was proved by Mohammadi and Oh [MO15, Theorem 5.3] - however they use spectral methods and hence assume a lower bound on the critical exponent (thus giving them an exponential rate), which does not suffice for our purposes.

Fortunately the exact proof of [OS13, Theorem 3.6] can be used to prove the necessary theorem (without the exponential rate). Let H be an unstable horospherical subgroup for *right* multiplication by a_t, therefore $H < N_-$. Again, let Γ be a geometrically finite, non-elementary, Zariski dense subgroup.

Theorem 5.6.1. *For any $g \in G$, any $\Psi \in \mathcal{C}_c(\Gamma\backslash G)$ and $\phi \in \mathcal{C}_c(gH)$*

$$\begin{aligned}\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\overline{H}}\int_{H\cap M} \Psi(\Gamma g h m a_t)\phi(ghm) d\mu_{\overline{H}}^{Haar}(h) d\mu_{H\cap M}^{Haar}(m)\\ = \frac{1}{|\mathrm{m}^{BMS}|}\int_{H\times\Gamma\backslash G}\Psi(\alpha)\phi(gh)d\mathrm{m}^{BR}(\alpha)d\mu_{\Gamma gH}^{PS}(gh).\end{aligned} \tag{5.6.1}$$

The proof of this theorem is omitted as it is identical to the proof of [OS13, Theorem 3.6] with one exception: rather than use the mixing theorem of Rudolph, Roblin and Babillot on $T^1(\Gamma\backslash\mathbb{H}^n)$, (which appears as [OS13, Theorem 3.2]) use the mixing theorem for the BMS measure under the frame flow on G proved by Winter - Theorem 5.5.2. Namely, write $g \in G$ as $g = \mathbf{u}m$ for $\mathbf{u} \in T^1(\mathbb{H})$ and $m \in M$. From there, using Winter's mixing theorem and the fact that the frame flow is in the centraliser of M, the same proof will give the above theorem.

Theorem 5.6.1 then leads to the following corollary:

Corollary 5.6.2. *Under the assumptions of Theorem 5.6.1, let λ be a Borel probability measure on $\mathbb{R}^{n-1}$ with density $\lambda' \in \mathcal{C}_c(\mathbb{R}^{n-1})$. Then for any $g \in G$*

$$\begin{aligned}\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{R}^{n-1}} \int_{M\cap H} \Psi(\Gamma g \operatorname{hor}(\mathbf{x}) m a_t) d\lambda(\mathbf{x}) d\mu^{Haar}_{H\cap M}(m) \\ = \frac{1}{|\mathrm{m}^{BMS}|} \int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \lambda'(\mathbf{x}) \Psi(\alpha) d\mathrm{m}^{BR}(\alpha) d\mathrm{m}^{BR}(\alpha) d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x}).\end{aligned} \tag{5.6.2}$$

Proof. Inserting the definition of λ' and then applying Theorem 5.6.1 with $\phi(\cdot) = \lambda' \circ \operatorname{hor}^{-1}(g^{-1}(\cdot)M)$ gives

$$\begin{aligned}&\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{R}^{n-1}} \int_{M\cap H} \Psi(\Gamma g \operatorname{hor}(\mathbf{x}) m a_t) d\mu^{Haar}_{H\cap M}(m) d\lambda(\mathbf{x}) \\ &= \lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\overline{H}} \int_{H\cap M} \Psi(\Gamma g h m a_t) \lambda'(\operatorname{hor}^{-1}(g^{-1}(ghm)M)) d\mu^{Haar}_{H\cap M}(m) d\mu^{Haar}_{\overline{H}}(h) \\ &= \frac{1}{|m^{BMS}|} \int_{\overline{H}\times\Gamma\backslash G} \Psi(\alpha) \lambda'(\operatorname{hor}^{-1}(h)) d\mathrm{m}^{BR}(\alpha) d\mu^{PS}_{\Gamma g \overline{H}}(h)\end{aligned}$$

Now inserting the parameterisation $\operatorname{hor}^{-1} : \overline{H} \to \mathbb{R}^{n-1}$ gives (5.6.2).

□

From here, the proof of [MS10, Theorem 5.3] allows us to extend to functions of $\mathbb{R}^{d-1} \times \Gamma\backslash G$ and to sequences of functions

Theorem 5.6.3. *Let λ be as in Corollary 5.6.2. Let $f : \mathbb{R}^{n-1} \times \Gamma\backslash G \to \mathbb{R}$ be compactly supported and continuous. Let $f_t : \mathbb{R}^{n-1} \times \Gamma\backslash G \to \mathbb{R}$ be a family of continuous functions all supported on a compact set such that $f_t \to f$ uniformly. Then for any $g \in G$*

$$\begin{aligned}\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{R}^{n-1}\times H\cap M} f_t(\mathbf{x}, \Gamma g \operatorname{hor}(\mathbf{x}) m a_t) d\mu^{Haar}_{H\cap M}(m) d\lambda(\mathbf{x}) \\ = \frac{1}{|\mathrm{m}^{\mathrm{BMS}}|} \int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \lambda'(\mathbf{x}) f(\mathbf{x}, \alpha) d\mathrm{m}^{BR}(\alpha) d\omega^{PS}_{\Gamma g \overline{H}}(\mathbf{x})\end{aligned} \tag{5.6.3}$$

Proof. Let $\mathcal{S} \subset \Gamma\backslash G := \{\alpha \in \Gamma\backslash G : \exists t > 0, \mathbf{x} \in \mathbb{R}^{n-1} \text{ s.t } f_t(\mathbf{x}, \alpha) \neq 0\}$ (which is compact as the support of the entire family f_t is compact) and let $\zeta(\alpha)$ be a smooth compactly supported bump function equal to 1 on $\mathcal{S}$. As f_t converges to f uniformly and all functions are uniformly continuous, for all $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $t_0 > 0$ such that for all $\mathbf{x}_0 \in \mathbb{R}^{n-1}$

$$\begin{aligned}f(\mathbf{x}_0, g) - \delta\zeta(g) \leq f(\mathbf{x}, g) \leq f(\mathbf{x}_0, g) + \delta\zeta(g) \\ f(\mathbf{x}_0, g) - \delta\zeta(g) \leq f_t(\mathbf{x}, g) \leq f(\mathbf{x}_0, g) + \delta\zeta(g)\end{aligned} \tag{5.6.4}$$

for all $\mathbf{x} \in \mathbf{x}_0 + [0, \epsilon)^{n-1}$ and $t > t_0$. We fix $\delta > 0$ and let $\epsilon = \epsilon(\delta)$ to be adjusted later in the proof, and decompose $\mathbb{R}^{n-1}$ as follows

$$\int_{H\cap M}\int_{\mathbb{R}^{n-1}} f_t(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m)$$
$$= \sum_{\mathbf{k}\in\mathbb{Z}^{n-1}}\int_{H\cap M}\int_{\epsilon\mathbf{k}+[0,\epsilon)^{n-1}} f_t(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m) \tag{5.6.5}$$
$$\leq \sum_{\mathbf{k}\in\mathbb{Z}^{n-1}}\int_{H\cap M}\int_{\epsilon\mathbf{k}+[0,\epsilon)^{n-1}} f(\epsilon\mathbf{k},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)+\delta\zeta(\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m)$$

For each $\mathbf{k}$ and $\mathcal{E}_{\mathbf{k}} = \epsilon\mathbf{k}+[0,\epsilon)^{n-1}$ we can apply Corollary 5.6.2 to the r.h.s of (5.6.5), and then use that the ζ has compact support to conclude:

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{H\cap M}\int_{\mathcal{E}_{\mathbf{k}}} f_t(\epsilon\mathbf{k},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m)$$
$$\leq \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathcal{E}_{\mathbf{k}}\times\Gamma\backslash G}\lambda'(\mathbf{x})(f(\mathbf{x},\alpha)+\delta\zeta(\alpha))d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x}) \tag{5.6.6}$$
$$= \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathcal{E}_{\mathbf{k}}\times\Gamma\backslash G}\lambda'(\mathbf{x})f(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})+C_{\mathbf{k}}\delta.$$

Since $\delta\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\zeta(\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x}) < \infty$ we know that $\sum_{\mathbf{k}\in\mathbb{Z}^{n-1}} C_{\mathbf{k}} < \infty$. Putting this all together we get, that there exists a $C<\infty$ such that for any $\delta>0$,

$$\limsup_{t\to\infty}\; e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}\times H\cap M} f_t(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\mu^{Haar}_{H\cap M}(m)d\lambda(\mathbf{x})$$
$$\leq \frac{1}{|\mathrm{m}^{\mathrm{BMS}}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})f(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})+C\delta$$
$$+\delta\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{H\cap M}\int_{\mathbb{R}^{n-1}}\zeta(\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m). \tag{5.6.7}$$

Since ζ does not depend on t we may replace the $\limsup$ on the right hand side by a $\lim$. Then, since ζ is bounded and compactly supported we may apply Corollary 5.6.2 to bound the last term

$$\delta\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{H\cap M}\int_{\mathbb{R}^{n-1}}\zeta(\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m)$$
$$\leq \frac{\delta}{|m^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\zeta(\alpha)dm^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x}) \leq C'\delta, \tag{5.6.8}$$

for some $C'<\infty$. Therefore there exists a $C''<\infty$ such that

$$\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}\times H\cap M} f_t(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\mu^{Haar}_{H\cap M}(m)d\lambda(\mathbf{x})$$
$$\leq \frac{1}{|\mathrm{m}^{\mathrm{BMS}}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})f(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})+C''\delta. \tag{5.6.9}$$

A similar lower bound can be achieved for the $\liminf$ from which the Theorem follows.

□

For a given $t_0>0$, let $\{\mathcal{E}_t\}_{t\geq t_0}$ be bounded subsets of $\mathbb{R}^{n-1}\times\Gamma\backslash G$ all with boundary of $\omega^{PS}_{\Gamma,g,\overline{H}}\times\mathrm{m}^{BR}$-measure 0, and define

$$\lim\,(\inf \mathcal{E}_t)^o \;:=\; \bigcup_{t\geq t_0}\left(\bigcap_{s\geq t}\mathcal{E}_s\right)^o \tag{5.6.10}$$

$$\lim \overline{\sup \mathcal{E}_t} \;:=\; \bigcap_{t\geq t_0}\overline{\bigcup_{s\geq t}\mathcal{E}_s} \tag{5.6.11}$$

$$\lim\sup \mathcal{E}_t \;:=\; \bigcap_{t\geq t_0}\bigcup_{s\geq t}\mathcal{E}_s \tag{5.6.12}$$

In which case it is possible to prove a similar corollary to [MS10, Theorem 5.6] (with the exception that, as the m^{BR} is not finite on $\Gamma\backslash G$ we require our sets to be uniformly bounded):

Corollary 5.6.4. *Let λ be a Borel probability measure on $\mathbb{R}^{n-1}$ as in Corollary 5.6.2. Then for any bounded family of subsets $\mathcal{E}_t \subset \mathbb{R}^{n-1}\times\Gamma\backslash G$ all with boundary of $\omega^{PS}_{\Gamma,g,\overline{H}}\times\mathrm{m}^{BR}$-measure 0, for any $g\in\Gamma\backslash G$*

$$\begin{aligned}\liminf_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}}\int_{M\cap H}\chi_{\mathcal{E}_t}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{M\cap H}(m)\\ \geq \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\chi_{\lim(\inf\mathcal{E}_t)^o}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})\end{aligned} \tag{5.6.13}$$

$$\begin{aligned}\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}}\int_{M\cap H}\chi_{\mathcal{E}_t}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{M\cap H}(m)\\ \leq \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\chi_{\lim\overline{\sup\mathcal{E}_t}}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})\end{aligned} \tag{5.6.14}$$

Moreover, if $\lim\overline{\sup\mathcal{E}_t}\setminus\lim(\inf\mathcal{E}_t)^o$ has $\omega^{PS}_{\Gamma,g,\overline{H}}\times\mathrm{m}^{BR}$-measure 0 then

$$\begin{aligned}\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}}\int_{M\cap H}\chi_{\mathcal{E}_t}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{M\cap H}(m)\\ = \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\chi_{\lim\sup\mathcal{E}_t}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})\end{aligned} \tag{5.6.15}$$

Proof. This Corollary follows from Theorem 5.6.3 in exactly the same way as [MS10, Theorem 5.6], with one exception. Addressing only (5.6.14) (as the other results follow similarly). Let

$$\tilde{\mathcal{E}}_t := \overline{\bigcup_{s\geq t}\mathcal{E}_s}, \tag{5.6.16}$$

thus $\mathcal{E}_t\subset\tilde{\mathcal{E}}_t\subset\tilde{\mathcal{E}}_{t_1}$ for $t\geq t_1$. Hence

$$\begin{aligned}\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}}\int_{M\cap H}\chi_{\mathcal{E}_t}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m)\\ \leq \limsup_{t_1\to\infty}\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}}\int_{M\cap H}\chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda(\mathbf{x})d\mu^{Haar}_{H\cap M}(m).\end{aligned} \tag{5.6.17}$$

From here we apply Theorem 5.6.3 for a fixed $f=f_t=\chi_{\mathcal{E}_{t_1}}$ by approximating compactly supported characteristic functions with bounded, compactly supported, continuous functions. That is, consider

$$\left|\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{H\cap M}\int_{\mathbb{R}^{n-1}} \chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda d\mu^{Haar}_{M\cap H} - \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \lambda'(\mathbf{x})\chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})\right|. \quad (5.6.18)$$

Fix $\epsilon > 0$ and let ϕ be a bounded, compactly supported function such that $\phi = \chi_{\tilde{\mathcal{E}}_{t_1}}$ outside of a δ-neighborhood of the boundary of $\tilde{\mathcal{E}}_{t_1}$. $\delta = \delta(\epsilon) > 0$ will be fixed later in the proof. Write

$$(5.6.18) = \left|\limsup_{t\to\infty}\Big(e^{(n-1-\delta_\Gamma)t}\int_{H\cap M}\int_{\mathbb{R}^{n-1}} \chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t) + \phi(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t) - \phi(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)\Big)\,d\lambda d\mu^{Haar}_{M\cap H} - \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x})\right|. \quad (5.6.19)$$

Applying Theorem 5.6.3 to the second term in the second line then gives that (5.6.18) is less than or equal

$$(5.6.18) \le \limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{H\cap M}\int_{\mathbb{R}^{n-1}}\left|\chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t) - \phi(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)\right| d\lambda d\mu^{Haar}_{M\cap H}(m) + \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\left|\chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\alpha)-\phi(\mathbf{x},\alpha)\right| d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x}). \quad (5.6.20)$$

Now let $\tilde{\phi}$ be a continuous, bounded, function supported on the δ-neighbourhood of $\tilde{\mathcal{E}}_{t_1}$ such that $\tilde{\phi} \ge \left|\chi_{\tilde{\mathcal{E}}_{t_1}} - \phi\right|$ everywhere. Hence

$$(5.6.18) \le \limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{H\cap M}\int_{\mathbb{R}^{n-1}}\tilde{\phi}(\mathbf{x},\Gamma g\,\mathrm{hor}(\mathbf{x})ma_t)d\lambda d\mu^{Haar}_{M\cap H}(m) + \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\left|\chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\alpha)-\phi(\mathbf{x},\alpha)\right| d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x}). \quad (5.6.21)$$

Now we may apply Theorem 5.6.3 once again to $\tilde{\phi}$ to conclude

$$(5.6.18) \le \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G}\lambda'(\mathbf{x})\left(\tilde{\phi}(\mathbf{x},\alpha) + \left|\chi_{\tilde{\mathcal{E}}_{t_1}}(\mathbf{x},\alpha)-\phi(\mathbf{x},\alpha)\right|\right) d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\mathbf{x}). \quad (5.6.22)$$

Now note that by assumption the Patterson-Sullivan measure is finite and the Burger-Roblin measure is finite on bounded subsets. Since both terms in the integrand are bounded and supported on the δ-neighbourhood of $\tilde{\mathcal{E}}_{t_1}$, we may choose δ small enough such that the right hand side of (5.6.22) is less than ϵ. (5.6.14) then follows from (5.6.17) from which it follows that (5.6.20) is less than $C\epsilon$ for some $C < \infty$.

The rest of the Theorem follows similarly.

□

Chapter 6

Directions in Thin Orbits

6.1 Introduction

Patterson-Sullivan theory describes the asymptotic density of points near the boundary of hyperbolic space. Hence a very natural question one can ask is 'what about higher order spatial statistics?' For example what can one say about the gap (or nearest neighbour) distribution? Herein we will answer these questions and give a full characterisation of the spatial statistics of such a point set as viewed from a fixed observer in hyperbolic space or its boundary. These questions have been addressed previously for lattices [BPZ14, KK15, RS17, MV18], and for certain thin groups [Zha17, Zha19]. However we will treat a much more general class of subgroups in arbitrary dimension.

Our main results are in general dimension $n \geq 2$. For the purpose of this introduction we restrict our attention to dimension 2 and gap statistics. The main theorem in all dimensions will follow in Section 6.2. Let $G := \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ and consider the left action on an element $\mathbf{z} \in \mathbb{H}^2$ via Möbius transformations. Let $\Gamma < G$ be a *Zariski dense, non-elementary, geometrically finite* subgroup (see Chapter 5, Section 5.5.1) and consider the orbit of a point $\mathbf{w} \in \mathbb{H}^2$, $\overline{\mathbf{w}} = \Gamma\mathbf{w}$.

For a given $t \in \mathbb{R}_{\geq 0}$ consider the radial projection to the boundary of all the points in $\overline{\mathbf{w}}$ a distance less than t from $\mathbf{i}$. As we can identify $\partial\mathbb{H}^2 \cong S_1^1$ this generates a point set on S_1^1. Formally, let $\xi(\mathbf{z}) \subset \mathbb{H}^2$ be the geodesic connecting $\mathbf{i}$ to $\mathbf{z}$ and let $\xi_s(\mathbf{z}) \subset \mathbb{H}^2$ be the point along said geodesic a distance s from $\mathbf{i}$ in the direction of $\mathbf{z}$. Define

$$\mathcal{Q}_t(\overline{\mathbf{w}}) := \left\{ \lim_{s\to\infty} \xi_s(\gamma\mathbf{w}) : \gamma \in \Gamma/\Gamma_{\mathbf{w}},\ d(\gamma\mathbf{w}, \mathbf{i}) < t \right\} \subset S_1^1, \tag{6.1.1}$$

where $d(\cdot,\cdot)$ denotes the hyperbolic distance and $\Gamma_{\mathbf{w}} := \mathrm{Stab}_\Gamma(\mathbf{w})$. Let $N_t = \#\mathcal{Q}_t(\overline{\mathbf{w}})$ and label the points in $\mathcal{Q}_t(\overline{\mathbf{w}})$ sequentially as $\{x_i\}_{i=1}^{N_t} \subset S_1^1$. Asymptotically the points x_i will be distributed according to the Patterson-Sullivan density (see Chapter 5 Section 5.5). That is, a consequence of [OS13, Theorem 1.2] is that for a subset $F \subset S_1^1$

$$\#\mathcal{Q}_t(\overline{\mathbf{w}}) \cap F \sim C\nu_{\mathbf{i}}(F)e^{\delta_\Gamma t} \tag{6.1.2}$$

where $\nu_{\mathbf{i}}$ is the conformal density of dimension δ_Γ (the critical exponent of Γ). (6.1.2) is a consequence of Theorem 6.2.1 below.

Denote the j^{th} scaled gap

$$s_j := \{x_{j+1} - x_j\}e^t, \tag{6.1.3}$$

where $\{\cdot\}$ denotes the distance to the nearest integer and let $S(t)$ denote all the scaled gaps coming from $\mathcal{Q}_t$. Define the cumulative gap distribution to be

$$F_t(L) := \frac{1}{N_t}\#\{j \leq N_t : s_j \geq L\}. \tag{6.1.4}$$

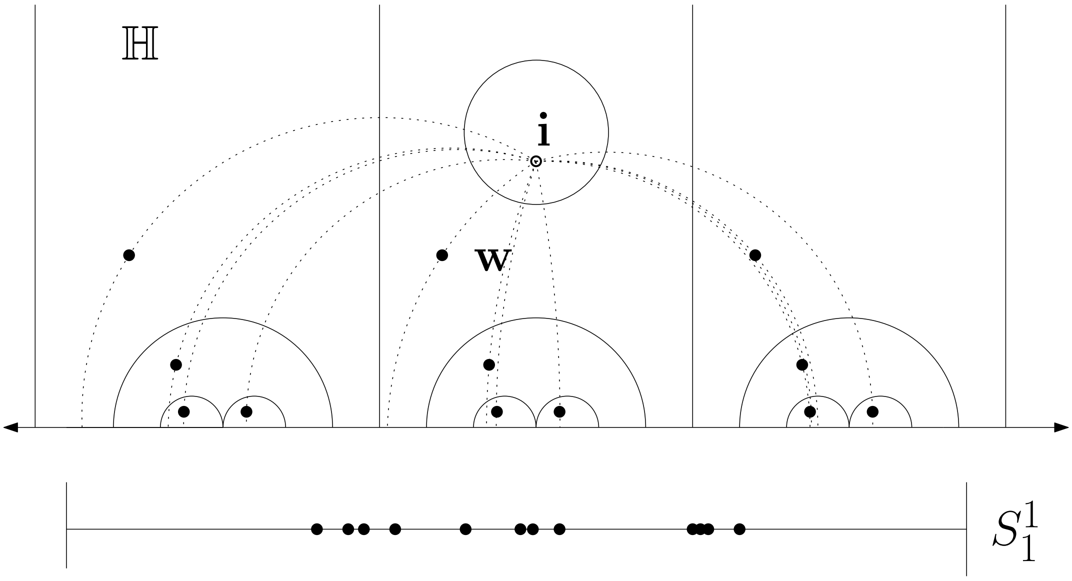

Figure 6.1: On top we show a schematic diagram of the setting in 2 dimensions. The bold lines cut the half-plane $\mathbb{H}$ into fundamental domains. Then we consider a point $\mathbf{w} \in \mathbb{H}$ and the orbit $\overline{\mathbf{w}} = \Gamma\mathbf{w}$ - the black dots. The dotted lines represent the geodesics connecting the points of $\overline{\mathbf{w}}$ to $\mathbf{i}$. We consider the intersection of the geodesics with the unit hyperbolic sphere centred at $\mathbf{i}$ (this is equivalent to projection to the boundary $\partial\mathbb{H}$). Giving a projected point set on S^1_1 (illustrated below the upper half-plane). If we include all points in $\overline{\mathbf{w}}$ such that $d(\gamma\mathbf{w}, \mathbf{i}) < t$ then this point set corresponds to $\mathcal{Q}_t(\overline{\mathbf{w}})$.

Theorem 6.1.1. *The limiting function $F : [0, \infty) \to \mathbb{R}$ defined $F(L) := \lim_{t\to\infty} F_t(L)$ exists, is monotone decreasing and continuous. Moreover if the fundamental domain for Γ is made of a finite number of non-intersecting half circles then there exists some $L_0 > 0$ such that*

$$F(L) = 1 \tag{6.1.5}$$

for all $L < L_0$.

Remark. The proof of this Theorem will come in Section 6.7. This theorem generalises a theorem by Zhang [Zha17] in the case of certain Schottky groups to the general geometrically finite case. In fact, we will (in Subsection 6.7.3) express explicitly and prove convergence of the nearest neighbour distribution in all dimensions.

Moreover the gap distribution satisfies the following formula

$$F(L) = C_{\mathbf{w}} \int_0^\infty e^{\delta_\Gamma r} \int_0^\pi \prod_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \left(1 - \chi_{\mathcal{E}(\gamma)}(r, \theta)\right) d\nu_{\mathbf{i}}(\theta) dr, \tag{6.1.6}$$

where $C_{\mathbf{w}}$ is an explicit constant, $\mathcal{E}(\gamma)$ is an explicit set depending on the choice of γ, and here and throughout $\chi_{\mathcal{A}}$ is the characteristic function of the set $\mathcal{A}$. In the lattice case $\delta_\Gamma = 1$ and $\nu_{\mathbf{i}}(\theta) = d\theta$. To the best of the author's knowledge this formula was not known previously. The proof of this formula is the content of Subsection 6.7.5 (where we will also take a derivative to arrive at the density). More explicit formula than this for the gap distribution are known only in the Euclidean case due to Marklof and Strömbergsson [MS14] and in the hyperbolic lattice case for certain circle packing examples due to Rudnick and Zhang [RZ17].

In this Chapter we will extend Theorem 6.1.1 to more general statistics and arbitrary dimension $n \geq 2$. Similar results are known only for more restricted contexts. Using number theoretic methods Boca, Popa and Zaharescu [BPZ14] proved a theorem about the pair correlations of angles between directions in the *modular group*. They posed a conjecture later proved by Kelmer and Kontorovich [KK15] who proved a limiting distribution for the *pair correlation* of angles between directions in more general *hyperbolic lattices*. More recently Risager and Södergren [RS17] extended these results to arbitrary dimension in the lattice case, giving effective results with explicit rates.

Marklof and Vinogradov [MV18] then characterised the full limiting behaviour of such projected point sets for hyperbolic lattices. Their result is a special case of Theorem 6.2.2, our main theorem, restricted to the lattice case. Zhang then proved a limiting theorem for the gap distribution of directions for *certain Schottky groups* [Zha17] (hence this was the first treatment of the infinite volume case, in 2 dimensions). Following that, Zhang proved a limiting distribution for the directions of centres of Apollonian circle packings [Zha19] (another non-lattice example, this time in 3 dimensions). As an application of one of our main theorems (Theorem 6.3.2), in Subsection 6.2.2 we will discuss how our methods apply to a general class of sphere packings. That is, any sphere packing (possibly overlapping) invariant under the action of a suitable subgroup. Theorem 6.3.2 allows us to characterise the statistical regularity of the centers of the spheres in such a packing.

The general strategy to prove the results in this Chapter is the same as that used in [MV18]. They use an argument of Margulis' [Mar04] to prove equidistribution of large horospheres and spheres. Then they use those equidistribution theorems to establish the limiting distribution. Our work will follow the same plan but will instead use the equidistribution theorems stated in Chapter 5 Section 5.6. As the limiting measure is no longer the invariant Haar measure there are a number of added complications.

Plan of the Chapter: In Section 6.2 we setup and state our main result in general dimensions. Then we explain how our result applies to a general class of sphere packings.

In Sections 6.3 and 6.4 we prove a theorem analogous to the main theorem with the observer on the boundary, $\partial\mathbb{H}^n$, rather than the interior, $\mathbb{H}^n$. Moreover we show how this limiting theorem can be used to prove convergence of the moment generating function.

In Sections 6.5 and 6.6 we prove our main theorem, Theorem 6.2.2 for an observer in $\mathbb{H}^n$.

In Section 6.7 we present several applications: we prove the convergence of higher moments in both the boundary and interior cases, prove existence and express the limiting two-point correlation function, prove existence and express the limiting nearest neighbour distribution. Then, in dimension $n = 2$, we explain how to prove Theorem 6.1.1 for gap statistics as a consequence of Theorem 6.2.2 and arrive at the explicit formula described.

6.2 Statement of Main Result

Our main result is in general dimension $n \geq 2$.

6.2.1 Main Theorem

Given two points $\mathbf{w}, \mathbf{z} \in \mathbb{H}^n$ define the *direction function*, $\varphi_{\mathbf{z}}(\mathbf{w})$, to be the intersection of the geodesic connecting $\mathbf{z}$ to $\mathbf{w}$ with the hyperbolic unit sphere centered at $\mathbf{z}$ (i.e $\overline{K}e^{-1}\mathbf{i} + \mathbf{z}$). Thus $\varphi : \mathbb{H}^n \times \mathbb{H}^n \to S_1^{n-1}$.

Fix $\Gamma < G$ a *Zariski dense, non-elementary, geometrically finite* subgroup. Given the orbit $\overline{\mathbf{w}} = \Gamma\mathbf{w}$ and $s < t \in \mathbb{R}_{\geq 0}$ define

$$\mathcal{P}^{\mathbf{z}}_{t,s}(\overline{\mathbf{w}}) := \{\varphi_{\mathbf{z}}(\gamma\mathbf{w}) : \gamma \in \Gamma/\Gamma_{\mathbf{w}},\ s < d(\gamma\mathbf{w}, \mathbf{z}) < t\}, \tag{6.2.1}$$

Thus $\mathcal{P}^{\mathbf{z}}_{t,s}(\overline{\mathbf{w}})$ represents the set of directions of orbit points of $\mathbf{w}$ within an annulus (of inner radius s and outer radius t) around the observer at $\mathbf{z}$.

Without loss of generality we can use the left-invariance of the metric d to move $\mathbf{w}$ and set $\mathbf{z}$ to be $\mathbf{i}$ (keeping Γ the same). Set

$$\mathcal{P}_{t,s}(\overline{\mathbf{w}}) := \mathcal{P}^{\mathbf{i}}_{t,s}(\overline{\mathbf{w}}). \tag{6.2.2}$$

The first order statistics of this projected point set are characterised by a result of Oh and Shah [OS13]

Theorem 6.2.1. *Let $F \subset \overline{K} \cong S_1^{n-1}$ with $\nu_{\mathbf{i}}(\partial F) = 0$. Then the following asymptotic formula holds as $t \to \infty$*

$$\#(\mathcal{P}_{t,0}(\overline{\mathbf{w}}) \cap F) \sim \frac{|\mu^{PS}_{\Gamma\overline{K}}|}{\delta_\Gamma |m^{BMS}|} \nu_{\mathbf{i}}(F) e^{\delta_\Gamma t}. \tag{6.2.3}$$

This theorem follows from [OS13, Theorem 7.16].

Turning now to our main object of study: the higher order spatial statistics. Let ω denote the solid angle measure on S_1^{n-1} normalised to be a probability measure. Hence, for a subset $\mathcal{A} \subset S_1^{n-1}$,

$$\omega(\mathcal{A}) = \frac{\mathrm{vol}_{S_1^{n-1}}(\mathcal{A})}{\mathrm{vol}_{S_1^{n-1}}(S_1^{n-1})}. \tag{6.2.4}$$

For $\sigma > 0$ let $\mathcal{D}_{t,s}(\sigma, \mathbf{v}, g\overline{\mathbf{w}}) \subset S_1^{n-1}$ be the (shrinking with t) open disk centred at $\mathbf{v} \in S_1^{n-1}$ of volume

$$\omega(\mathcal{D}_{t,s}(\sigma, \mathbf{v}, g\overline{\mathbf{w}})) = \frac{\sigma}{\#\mathcal{P}_{t,s}(g\overline{\mathbf{w}})^{\frac{n-1}{\delta_\Gamma}}}, \tag{6.2.5}$$

the scaling in the exponent is chosen in such a way that $\mathcal{D}$ scales like in the lattice-case (we will discuss this scaling after the statement of Theorem 6.2.2). Let

$$\mathcal{N}_{t,s}(\sigma, \mathbf{v}, g\overline{\mathbf{w}}) := \#(\mathcal{P}_{t,s}(g\overline{\mathbf{w}}) \cap \mathcal{D}_{t,s}(\sigma, \mathbf{v}, g\overline{\mathbf{w}})). \tag{6.2.6}$$

Finally define the cuspidal cone:

$$\mathcal{Z}_0(s, \sigma) := \{\mathbf{z} \in \mathbb{H}^n : \mathrm{Re}(\mathbf{z}) \in \vartheta^{-1/\delta_\Gamma} \mathcal{B}_\sigma,\ 1 \le \mathrm{Im}(z) \le e^s\}, \tag{6.2.7}$$

where $\vartheta = \frac{|\nu_{\mathbf{i}}|}{\delta_\Gamma |\mathrm{m}^{BMS}|}$ and $\mathcal{B}_\sigma$ is a ball (in $\mathbb{R}^{n-1}$) of volume σ centred at the origin. With that, the main theorem is:

Theorem 6.2.2. *Let λ be a Borel probability measure on S_1^{n-1} absolutely continuous with respect to Lebesgue with continuous density. Then for every $g \in G$, $r \in \mathbb{Z}_{>0}$, $s \in [0, \infty]$ and $\sigma \in (0, \infty)$*

$$E_s(r, \sigma; g\overline{\mathbf{w}}) := \lim_{t \to \infty} e^{(n-1-\delta_\Gamma)t} \lambda(\{\mathbf{v} \in S_1^{n-1} : \mathcal{N}_{t,s}(\sigma, \mathbf{v}; g\overline{\mathbf{w}}) = r\}) \tag{6.2.8}$$

exists and is given by:

$$E_s(r, \sigma; g\overline{\mathbf{w}}) = \frac{C_\lambda}{|\mathrm{m}^{BMS}|} \mathrm{m}^{BR}(\{\alpha \in \Gamma\backslash G : \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}_0(s, \sigma)) = r\}) \tag{6.2.9}$$

where $C_\lambda = C_\lambda(g, \Gamma) = \int_{\overline{K}} \lambda'(k) d\mu^{PS}_{\Gamma g \overline{K}}$. Moreover the limit distribution $E_s(\cdot, \sigma; g\overline{\mathbf{w}})$ is continuous in $s \in (0, \infty]$ and $\sigma \in (0, \infty)$ and satisfies:

$$\lim_{\sigma\to 0} E_s(r,\sigma,\overline{\mathbf{w}}) = 0 \tag{6.2.10}$$

Remark. In Section 6.7 we will show several consequences of the above theorem. Namely we show how to prove convergence of moments and prove existence and write explicitly the two-point correlation and gap statistics.

Remark. The above theorem is not true in general for $r = 0$, unlike the case for lattices. When considering lattices, Marklof and Vinogradov also have a theorem of the same form with $r \geq 0$. The reason for this discrepancy is that the scaling of the set $\mathcal{D}_{t,s}(\sigma, \mathbf{v}, g\overline{\mathbf{w}})$, (6.2.5) is the same scaling as one would expect for lattices. Hence, when we consider orbit-point-free sets the scaling factor $e^{(n-1-\delta_\Gamma)t}$ is too large and causes the integral to blow up. In other words, there are two scales to this problem. For the two dimensional problem this translates to the fact that most gaps between neighbouring directions are of size e^{-t} but there are very big gaps of size $e^{-\delta_\Gamma t}$. This dichotomy was pointed out by Zhang [Zha17].

6.2.2 Sphere Packings

In Section 6.3 we will replicate Theorem 6.2.2, with the observer moved to ∞ and rather than consider a ball centred at the observer, we will consider an expanding horosphere based at the point ∞. This will induce a similar point set to (6.2.1) which we will denote $\mathcal{P}^{\infty}_{t,s}(\overline{\mathbf{w}})$. In which case Theorem 6.3.2 below, implies the analogous result as Theorem 6.2.2 for this point set. Using that, we can describe the spatial regularity of general sphere packings. For a general discussion of such packings see [Oh14, Section 7]. We include here a brief discussion of this application as a motivating example.

For $n \geq 3$, by a sphere packing, we mean the union of a collection of (possibly intersecting) $(n-2)$-spheres. Let $\mathcal{P}$ be a sphere packing in $\mathbb{R}^{n-1}$ invariant under the right action of a Zariski dense, non-elementary, geometrically finite subgroup. When $n = 3$ the canonical example of such a sphere packing is the Apollonian circle packing, however many other examples exist. Another nice example is considered in [Kon17], wherein Kontorovich considers so-called Soddy packings which generalise the Apollonian case to dimension $n = 4$ (our discussion here holds for more general packings as well).

A natural problem is to understand the asymptotic characteristics of such a collection as one restricts the set of spheres to those of radius larger than a certain cut off. Asymptotic counting formula for these packings are given in [Oh14, Theorem 7.5]. And, in the Apollonian case for $n = 3$, [Zha19] studied the spatial statistics of the centres of these packings. In fact, a special case of Theorem 6.3.2 (below) characterises the spatial statistics of these packings. To see this, we simply point out a well known relationship.

Let $\mathcal{P}$ be a Γ-invariant sphere packing in $\mathbb{R}^{n-1} \cong \partial\mathbb{H}^n$. Now let $\tilde{\mathcal{P}}$ be the collection of hemispheres supported on $\mathcal{P}$ (i.e whose intersection with $\partial\mathbb{H}^n$ is $\mathcal{P}$). In this case $\tilde{\mathcal{P}}$ is also Γ invariant.

Let $\mathbf{w} \in \mathbb{H}^n$ denote the apex of one of the spheres in $\tilde{\mathcal{P}}$. Then $\overline{\mathbf{w}} = \Gamma\mathbf{w}$ denotes the collection of apices of the spheres in $\tilde{\mathcal{P}}$. Hence, using the notation of Section 4, the set

$$\mathcal{P}^{\infty}_{t,s}(\overline{\mathbf{w}}) := \{\mathrm{Re}(\gamma\mathbf{w}) : \gamma \in \Gamma_\infty\backslash\Gamma/\Gamma_w,\ e^{-t} \leq \mathrm{Im}(\gamma\mathbf{w}) < e^{s-t}\}, \tag{6.2.11}$$

is equivalent to

$$\mathcal{P}^{\infty}_{t,s}(\overline{\mathbf{w}}) := \{c(S) : S \in \mathcal{P},\ e^{-t} \leq r(S) < e^{s-t}\}, \tag{6.2.12}$$

where $c(S)$ is the location of the centre of the sphere $S \in \mathcal{P}$ and $r(S)$ is the radius of S. In particular $\mathcal{P}^{\infty}_{t,\infty}(\overline{\mathbf{w}})$ denotes the centres of all of the spheres with radius larger than e^{-t}. Hence Theorem 6.3.2

describes the asymptotic spatial characteristics of this point set for any sphere packing (invariant under the action of non-elementary, Zariski dense subgroups).

6.3 Observer at Infinity

Our goal is to consider observers inside hyperbolic half-space but it will be more convenient to first consider an observer on the boundary (w.l.o.g at ∞) as this will allow us to use the horospherical equidistribution theorem stated above directly. Consider the projection of $\Gamma\mathbf{w}$ onto a horosphere centered at ∞. Hence there are two situations, either ∞ is the location of a cusp in a fundamental domain of Γ, or it is in a funnel. We will treat these two situations together.

Consider the cusp with rank $0 \leq l \leq n-1$ at ∞ (a rank 0 cusp is trivial and hence describes the situation with no cusp). Γ contains the (possibly trivial) subgroup Γ_∞. We may furthermore write

$$\Gamma_\infty = \{n_+(\mathbf{m}) : \mathbf{m} \in \mathcal{L}\}, \tag{6.3.1}$$

where $\mathcal{L}$ is a (possibly trivial) discrete subgroup of $\mathbb{R}^{n-1}$ of rank l.

Define

$$\mathcal{P}^\infty_{t,s}(\overline{\mathbf{w}}) := \{\mathrm{Re}(\gamma\mathbf{w}) \bmod \mathcal{L} : \gamma \in \Gamma_\infty\backslash\Gamma/\Gamma_{\mathbf{w}},\ e^{-t} \leq \mathrm{Im}(\gamma\mathbf{w}) < e^{s-t}\}. \tag{6.3.2}$$

$\mathcal{P}^\infty_{t,s}(\overline{\mathbf{w}})$ can be identified with a subset of the horospherical subgroup $\overline{H}$ by identifying $\overline{H}$ with $\mathbb{R}^{n-1}$ via group isomorphism hor.

The first order statistics for a boundary observer are given by:

Theorem 6.3.1. *In the present context. Let $F \subset \overline{H}$ be a Borel subset of the horospherical subgroup, $\overline{H}$, with $\mu^{PS}_{\overline{H}}(F) < \infty$ and $\mu^{PS}_{\overline{H}}(\partial F) = 0$. Then the following asymptotic formula holds as $t \to \infty$*

$$\#(\mathcal{P}^\infty_{t,\infty}(\overline{\mathbf{w}}) \cap F) \sim \vartheta\mu^{PS}_{\overline{H}}(F)e^{\delta_\Gamma t} \tag{6.3.3}$$

for ϑ defined below (6.2.7) depending only on Γ.

Remark. Asymptotic formulas for the number of lattice points in balls and sectors have been studied previous, for example by Good [Goo83]. Bourgain-Kontorovich-Sarnak [BKS10] described the asymptotics of orbit points in growing balls when the critical exponent is less than $1/2$ in dimension $n = 2$. Oh and Shah [OS13] then extended these results to full generality, including the sector case. Theorem 6.3.1 concerns horospherical sectors which is also covered by Oh and Shah [OS13, Theorem 7.16].

Consider the following rescaled test sets in $\mathbb{T}^l \times \mathbb{R}^{n-1-l}$ (scaled to match the scaling in (6.2.5))

$$\mathcal{B}_{t,s}(\mathcal{A}, \mathbf{x}) = N_{t,s}(\overline{\mathbf{w}})^{-1/\delta_\Gamma}\mathcal{A} - \mathbf{x} + \mathcal{L} \subset \mathbb{T}^l \times \mathbb{R}^{n-1-l}, \tag{6.3.4}$$

where $N_{t,s}(\overline{\mathbf{w}}) := \#\mathcal{P}^\infty_{t,s}(\overline{\mathbf{w}})$ and $\mathcal{A} \subset \mathbb{R}^{n-1}$ is bounded. The base point $\mathbf{x}$ will be chosen with law λ. Let

$$\mathcal{N}^\infty_{t,s}(\mathcal{A}, \mathbf{x}; \overline{\mathbf{w}}) := \#(\mathcal{P}^\infty_{t,s}(\overline{\mathbf{w}}) \cap \mathcal{B}_{t,s}(\mathcal{A}, \mathbf{x})). \tag{6.3.5}$$

Let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be bounded test sets with boundary of Lebesgue measure 0. Given a compactly supported density λ' on $\mathbb{T}^l \times \mathbb{R}^{n-1-l}$ write

$$A_\lambda = \int_{\mathbb{T}^l \times \mathbb{R}^{n-1-l}} \lambda'(\mathbf{x}) d\omega^{PS}_{\Gamma,\overline{H}}(\mathbf{x}) \tag{6.3.6}$$

($\omega^{PS}_{\Gamma,\overline{H}} := \omega^{PS}_{\Gamma,g,\overline{H}}$ with $g = Id$ the identity).

Theorem 6.3.2. *Let λ be a compactly supported Borel probability measure on $\mathbb{T}^l \times \mathbb{R}^{n-1-l}$ absolutely continuous with respect to Lebesgue measure, with continuous density. Then for any $r = (r_1, ..., r_m) \in \mathbb{Z}^m_{>0}$, $s \in (0, \infty]$ and $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_m$*

$$E_s(r, \mathcal{A}; \overline{\mathbf{w}}) := \lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\lambda(\{\mathbf{x} \in \mathbb{T}^l \times \mathbb{R}^{n-1-l} : \mathcal{N}^{\infty}_{t,s}(\mathcal{A}_j, \mathbf{x}; \overline{\mathbf{w}}) = r_j, \forall j\}) \tag{6.3.7}$$

exists and is given by

$$E_s(r, \mathcal{A}; \overline{\mathbf{w}}) = \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}(\{\alpha \in \Gamma\backslash G : \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(s, \mathcal{A}_j)) = r_j \forall j\}), \tag{6.3.8}$$

with

$$\mathcal{Z}(s, \mathcal{A}_j) := \{\mathbf{z} \in \mathbb{H}^n : \operatorname{Re} \mathbf{z} \in \vartheta^{-1/\delta_\Gamma}\mathcal{A}_j, 1 \le \operatorname{Im} \mathbf{z} < e^s\}. \tag{6.3.9}$$

Moreover, $E_s(r, \mathcal{A}; \overline{\mathbf{w}})$ is continuous in s and $\mathcal{A}$.

Borrowing notation from [MV18], by continuous in the set $\mathcal{A}$ we mean that there exists a constant $C < \infty$ such that

$$|E_s(r, \mathcal{A}; \overline{\mathbf{w}}) - E_s(r, \mathcal{B}; \overline{\mathbf{w}})| \le C \operatorname{vol}_{\mathbb{R}^{m(n-1)}}(\mathcal{B} \setminus \mathcal{A}) \tag{6.3.10}$$

for any two sets $\mathcal{A} \subset \mathcal{B} \subset \mathbb{R}^{m(n-1)}$ as in Theorem 6.3.2.

With the exception of the proof of Proposition 6.3.3 and some other details, the proof of Theorem 6.3.2 follows similar lines as proof of [MV18, Theorem 4].

For a set $\mathcal{A} \subset \mathbb{H}^n$ with boundary of BR-measure 0 (i.e $\mathrm{m}^{BR}(\pi^{-1}(\mathcal{A})) = 0$) and $r \in \mathbb{Z}_{>0}$ define the following sets

$$[\mathcal{A}]_{\le r} := \{\alpha \in \Gamma\backslash G : 0 < \#(\mathcal{A} \cap \alpha^{-1}\overline{\mathbf{w}}) \le r\} \tag{6.3.11}$$

$$[\mathcal{A}]_{\ge r} := \{\alpha \in \Gamma\backslash G : \#(\mathcal{A} \cap \alpha^{-1}\overline{\mathbf{w}}) \ge r\} \tag{6.3.12}$$

$$[\mathcal{A}]_{= r} := \{\alpha \in \Gamma\backslash G : \#(\mathcal{A} \cap \alpha^{-1}\overline{\mathbf{w}}) = r\}. \tag{6.3.13}$$

Finally let $\mathbf{w} = g_{\mathbf{w}}\mathbf{i}$, then

Proposition 6.3.3. *Consider a measurable set with finite volume and boundary of BR-measure 0, $\mathcal{B} \subset \mathbb{H}^n$ such that $\inf\{t : n_+a_{-t}\mathbf{i} \in g^{-1}_{\mathbf{w}}\mathcal{B}\} =: t_0 > -\infty$ and $\mathcal{A} \subset \mathcal{B}$ (also with boundary of BR-measure 0). In that case, with $r \in \mathbb{N}_{>0}$*

$$\mathrm{m}^{BR}([\mathcal{A}]_{\ge 1}) \le \frac{C_{t_0}}{\#\Gamma_{\mathbf{w}}} \operatorname{vol}_{\mathbb{H}^n}(\mathcal{A}), \tag{6.3.14}$$

$$|\mathrm{m}^{BR}([\mathcal{A}]_{=r}) - \mathrm{m}^{BR}([\mathcal{B}]_{=r})| \le \frac{C_{t_0}}{\#\Gamma_{\mathbf{w}}} \operatorname{vol}_{\mathbb{H}^n}(\mathcal{B} \setminus \mathcal{A}), \tag{6.3.15}$$

and

$$0 \le \mathrm{m}^{BR}([\mathcal{A}]_{\le r}) - \mathrm{m}^{BR}([\mathcal{B}]_{\le r}) \le \frac{C_{t_0}}{\#\Gamma_{\mathbf{w}}} \operatorname{vol}_{\mathbb{H}^n}(\mathcal{B} \setminus \mathcal{A}), \tag{6.3.16}$$

with $C_{t_0} < \infty$ depending on t_0 and $\mathbf{w}$.

Proof. The proof of this Lemma will follow from a Siegel type estimate. Consider

$$\int_G \chi_{\mathcal{A}}(\alpha^{-1}\mathbf{w})d\mathrm{m}^{BR}(\alpha) \tag{6.3.17}$$

Now write $\mathbf{w} = g_{\mathbf{w}}\mathbf{i}$. By making the change of variables $\alpha \mapsto g_{\mathbf{w}}^{-1}\alpha g_{\mathbf{w}}$ we can then consider the Burger-Roblin measure associated to the group $\Gamma^{\mathbf{w}} := g_{\mathbf{w}}^{-1}\Gamma g_{\mathbf{w}}$. Thus

$$\int_G \chi_{\mathcal{A}}(\alpha^{-1}\mathbf{w})d\mathrm{m}^{BR}(\alpha) = \int_G \chi_{\mathcal{A}}(g_{\mathbf{w}}\alpha^{-1}\mathbf{i})dm_{\Gamma^{\mathbf{w}}}^{BR}(\alpha). \tag{6.3.18}$$

The decomposition of the Burger-Roblin measure from Chapter 5, Proposition 5.5.3 together with the fact $\chi_{\mathcal{A}} \in \mathcal{C}(T^1(\mathbb{H}^n)$ give

$$\int_G \chi_{\mathcal{A}}(\alpha^{-1}\mathbf{w})d\mathrm{m}^{BR}(\alpha) = \int_{KAN_+} \chi_{g_{\mathbf{w}}^{-1}\mathcal{A}}((ka_t n_+)^{-1}\mathbf{i})e^{-\delta_\Gamma t}d\mu_{N_+}^{Haar}(n_+)dtd\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-), \tag{6.3.19}$$

$\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-)$ is the conformal density of dimension $\delta_\Gamma = \delta_{\Gamma^{\mathbf{w}}}$ supported on $\Lambda(\Gamma^{\mathbf{w}})$. Applying the inverse inside the bracket and recalling that K is the stabiliser of $\mathbf{i}$ gives

$$\int_G \chi_{\mathcal{A}}(\alpha^{-1}\mathbf{w})d\mathrm{m}^{BR}(\alpha) = |\nu_{\mathbf{i}}^{\mathbf{w}}| \int_{AN_+} \chi_{g_{\mathbf{w}}^{-1}\mathcal{A}}(n_+^{-1}a_{-t}\mathbf{i})e^{-\delta_\Gamma t}d\mu_{N_+}^{Haar}(n_+)dt. \tag{6.3.20}$$

As the integral on N_+ is with respect to Haar measure we can change variables giving

$$\int_G \chi_{\mathcal{A}}(\alpha^{-1}\mathbf{w})d\mathrm{m}^{BR}(\alpha) \leq \tilde{C}_{t_0} \int_{AN_+} \chi_{g_{\mathbf{w}}^{-1}\mathcal{A}}(n_+a_{-t}\mathbf{i})d\mu_{N_+}^{Haar}(n_+)dt, \tag{6.3.21}$$

with $\tilde{C}_{t_0} = |\nu_{\mathbf{i}}^{\mathbf{w}}|\, e^{-\delta_\Gamma t_0}$. Now, changing variables gives

$$\begin{aligned}\int_G \chi_{\mathcal{A}}(\alpha^{-1}\mathbf{w})d\mathrm{m}^{BR}(\alpha) &\leq \tilde{C}_{t_0} \int_{\mathbb{R}^{n-1}\times\mathbb{R}} \chi_{g_{\mathbf{w}}^{-1}\mathcal{A}}(a_{-t}n_+(e^t\mathbf{x})\mathbf{i})d\mathbf{x}dt \\ &\leq \tilde{C}_{t_0} \int_{\mathbb{R}^{n-1}\times\mathbb{R}} e^{(n-1)t}\chi_{g_{\mathbf{w}}^{-1}\mathcal{A}}(a_t n_+(\mathbf{x})\mathbf{i})d\mathbf{x}dt \\ &\leq C_{t_0}\,\mathrm{vol}_{\mathbb{H}^n}(g_{\mathbf{w}}^{-1}\mathcal{A}) = C_{t_0}\,\mathrm{vol}_{\mathbb{H}^n}(\mathcal{A}),\end{aligned} \tag{6.3.22}$$

with $C_{t_0} = |\nu_{\mathbf{i}}^{\mathbf{w}}|\, e^{(n-1-\delta_\Gamma)t_0}$.

The proof of Proposition 6.3.3 now follows from (6.3.22), Chebyshev's inequality and some simple set manipulations (see [MV18, Lemma 5]) and is simply a consequence of the following

$$\int_{\Gamma\backslash G} \#(\mathcal{A}\cap\alpha^{-1}\overline{\mathbf{w}})d\mathrm{m}^{BR}(\alpha) = \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma/\Gamma_{\mathbf{w}}} \chi_{\mathcal{A}}(\alpha^{-1}\gamma\mathbf{w})d\mathrm{m}^{BR}(\alpha) \tag{6.3.23}$$

$$= \frac{1}{\#\Gamma_{\mathbf{w}}}\int_G \chi_{\mathcal{A}}(\alpha^{-1}\mathbf{w})d\mathrm{m}^{BR}(\alpha). \tag{6.3.24}$$

□

Lemma 6.3.4. *Under the hypothesis of Theorem 6.3.2, given an $\epsilon > 0$ there exists a $t_0 \in \mathbb{R}$ and bounded sets $\mathcal{A}_j^-, \mathcal{A}_j^+ \subset \mathbb{R}^{n-1}$ with boundary of Lebesgue measure 0 such that*

$$\mathcal{A}_j^- \subset \mathcal{A}_j \subset \mathcal{A}_j^+, \tag{6.3.25}$$

$$\mathrm{vol}_{\mathbb{R}^{n-1}}(\mathcal{A}_j^+ \setminus \mathcal{A}_j^-) < \epsilon \tag{6.3.26}$$

and for all $t \geq t_0$

$$\#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(s, \mathcal{A}_j^-)) \leq \mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j, \mathbf{x}; \overline{\mathbf{w}}) \leq \#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(s, \mathcal{A}_j^+)) \tag{6.3.27}$$

Proof. Write

$$\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j, \mathbf{x}; \overline{\mathbf{w}}) = \#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(s, e^t \vartheta^{1/\delta_\Gamma} N_{t,s}(\overline{\mathbf{w}})^{-1/\delta_\Gamma} \mathcal{A}_j)) \tag{6.3.28}$$

and note that $e^t \vartheta^{1/\delta_\Gamma} N_{t,s}(\overline{\mathbf{w}})^{-1/\delta_\Gamma} \to 1$ from which the lemma follows (see [MV18, Lemma 6] for more details).

□

Furthermore the analogue of [MV18, Lemma 7] applies in this context as well.

Lemma 6.3.5. *Under the hypothesis of Theorem 6.3.2, for all $s \geq 0$ we have*

$$\begin{aligned}\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \big|\lambda(\{\mathbf{x} \in \mathbb{T}^l \times \mathbb{R}^{n-1-l} : 0 < \#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(\infty, \mathcal{A}_j)) \leq r_j, \ \forall j\}) \\ - \lambda(\{\mathbf{x} \in \mathbb{T}^l \times \mathbb{R}^{n-1-l} : 0 < \#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(s, \mathcal{A}_j)) \leq r_j \forall j\})\big| \leq C e^{-\delta_\Gamma s/2} (\operatorname{vol}_{\mathbb{R}^{n-1}} \tilde{\mathcal{A}})^{1/2},\end{aligned} \tag{6.3.29}$$

where $\tilde{\mathcal{A}} = \bigcup_j \mathcal{A}_j$ and $C > 0$ is some constant.

Proof. Suppose $-\infty < a < b \leq \infty$ and $\mathcal{A} \subset \mathbb{R}^{n-1}$, define

$$\mathcal{Z}(a, b, \mathcal{A}) := \{\mathbf{z} \in \mathbb{H}^n : \operatorname{Re} \mathbf{z} \in \vartheta^{-1/\delta_\Gamma} \mathcal{A}, e^a \leq \operatorname{Im} \mathbf{z} \leq e^b\}. \tag{6.3.30}$$

The left hand side of (6.3.29) without the lim sup is less than or equal to

$$e^{(n-1-\delta_\Gamma)t} \lambda(\{\mathbf{x} \in \mathbb{T}^l \times \mathbb{R}^{n-1-l} : \#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(s, \infty, \tilde{\mathcal{A}})) \geq 1\}) \tag{6.3.31}$$

and $\#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(s, \infty, \tilde{\mathcal{A}}) = \mathcal{N}_{t-s,\infty}^{\infty}(\eta_{t-s} e^{t-s} \tilde{\mathcal{A}}, \mathbf{x}; \overline{\mathbf{w}})$, where $\eta_{t-s} = \frac{N_{t-s,\infty}^{1/\delta_\Gamma}}{\vartheta^{1/\delta_\Gamma} e^{t-s}} \to 1$ as $t \to \infty$

Chebyshev's inequality then implies

$$(6.3.31) \leq e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{T}^l \times \mathbb{R}^{n-1-l}} \mathcal{N}_{t-s,\infty}^{\infty}(\eta_{t-s} e^{t-s} \tilde{\mathcal{A}}, \mathbf{x}; \overline{\mathbf{w}}) d\lambda(\mathbf{x}) \tag{6.3.32}$$

Note further, by Theorem 6.3.1

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{T}^l \times \mathbb{R}^{n-1-l}} \mathcal{N}_{t,s}^{\infty}(\mathcal{A}, \mathbf{x}; \overline{\mathbf{w}}) d \operatorname{vol}(\mathbf{x}) = \operatorname{vol}_{\mathbb{R}^{n-1}} \mathcal{A}. \tag{6.3.33}$$

Hence, for any $R \geq c$ for some constant $c > 0$

$$\begin{aligned}e^{(n-1-\delta_\Gamma)t} &\int_{\mathbb{T}^l \times \mathbb{R}^{n-1-l}} \mathcal{N}_{t-s,\infty}^{\infty}(\eta_{t-s} e^{t-s} \tilde{\mathcal{A}}, \mathbf{x}; \overline{\mathbf{w}}) d\lambda(\mathbf{x}) \\ &\leq R e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{T}^l \times \mathbb{R}^{n-1-l}} \mathcal{N}_{t-s,\infty}^{\infty}(\eta_{t-s} e^{t-s} \tilde{\mathcal{A}}, \mathbf{x}; \overline{\mathbf{w}}) \chi_{\operatorname{supp}(\lambda)}(\mathbf{x}) d \operatorname{vol}(\mathbf{x}) \\ &\to R e^{-(n-1)s} \operatorname{vol}_{\mathbb{R}^{n-1}} \tilde{\mathcal{A}},\end{aligned} \tag{6.3.34}$$

as $t \to \infty$. Choosing

$$R := C \frac{e^{(n-1)s} (\operatorname{vol}_{\mathbb{R}^{n-1}} \tilde{\mathcal{A}})^{-1/2}}{e^{\delta_\Gamma s/2}} \tag{6.3.35}$$

proves the theorem (the constant C is there to ensure $R > c$).

□

Proof of Theorem 6.3.2. This proof is similar to [MV18, Proof of Theorem 4]. It suffices to show that for all $r = (r_1, ..., r_m) \in \mathbb{Z}^m_{>0}$ and all sets $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_m$ with $\mathcal{A}_j \subset \mathbb{R}^{n-1}$ bounded with boundary of Lebesgue measure 0 the following limit holds as $t \to \infty$

$$e^{(n-1-\delta_\Gamma)t}\lambda(\{\mathbf{x} \in \mathbb{T}^l \times \mathbb{R}^{n-1-l} : 0 < \mathcal{N}^{\infty}_{t,s}(\mathcal{A}_j, \mathbf{x}; \overline{\mathbf{w}}) \leq r_j,\ \forall j\})$$
$$\to \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}(\{\alpha \in \Gamma\backslash G : 0 < \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(s, \mathcal{A}_j) \leq r_j,\ \forall j\}). \quad (6.3.36)$$

The left hand side is equal

$$e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,s}}((a_t n_+(\mathbf{x}))^{-1})d\lambda(\mathbf{x}) \quad (6.3.37)$$

with

$$\mathcal{E}_{t,s} := \{\alpha \in \Gamma\backslash G : 0 < \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(s, e^t \vartheta^{1/\delta_\Gamma} N_{t,s}(\overline{\mathbf{w}})^{-1/\delta_\Gamma}\mathcal{A}_j)) \leq r_j,\ \forall j\}. \quad (6.3.38)$$

<u>Assume $s < \infty$</u>: Fix $\epsilon > 0$. By Lemma 6.3.4 there exist sets $\mathcal{A}^\pm$ with $\mathrm{vol}_{\mathbb{R}^{n-1}}(\mathcal{A}^+ \setminus \mathcal{A}^-) < \epsilon$. Such that if we write

$$\mathcal{E}^\pm_s := \{\alpha \in \Gamma\backslash G : 0 < \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(s, \mathcal{A}^\pm_j) \leq r_j,\ \forall j\}, \quad (6.3.39)$$

then $\mathcal{E}^+_s \subset \mathcal{E}_{t,s} \subset \mathcal{E}^-_s$ for all $t \geq t_0$. Since $\mathcal{Z}(s, e^t\vartheta^{1/\delta_\Gamma} N_{t,s}(\overline{\mathbf{w}})^{-1/\delta_\Gamma}\mathcal{A}_j)$ is bounded, we know that $\overline{\mathcal{E}_{t,s}}$ is compact as are $\overline{\mathcal{E}^\pm_s}$. Hence (because λ is compactly supported, and is absolutely continuous with respect to Lebesgue measure) we can apply Chapter 5, Corollary 5.6.4. Giving

$$\begin{aligned}\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,s}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x}) &\leq \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}(\overline{\mathcal{E}^-_s}),\\ \liminf_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,s}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x}) &\geq \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}((\mathcal{E}^+_s)^o).\end{aligned} \quad (6.3.40)$$

Finally Proposition 6.3.3, Lemma 6.3.4 and the fact $\mathcal{Z}(s, \mathcal{A}^\pm_j)$ is bounded for $s < \infty$ imply that

$$\lim_{\epsilon\to 0} \mathrm{m}^{BR}(\overline{\mathcal{E}^-_s} \setminus (\mathcal{E}^+_s)^o) = 0 \quad (6.3.41)$$

which proves Theorem 6.3.2 for $s < \infty$.

<u>Assume $s = \infty$:</u> The equidistribution theorems stated in Chapter 5, Section 5.6 hold only for compactly supported functions χ. Hence an approximation argument is needed to get around this.

Consider

$$\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,\infty}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x}). \quad (6.3.42)$$

Fix $\epsilon > 0$, by Lemma 6.3.5, there exists an $s_\epsilon < \infty$ such that

$$\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,\infty}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x})$$
$$\leq \limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,s_\epsilon}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x})+\epsilon. \quad (6.3.43)$$

By Lemma 6.3.4 for any $\rho=\rho(\epsilon)>0$ there exist sets $\mathcal{A}^{\pm}_{s_\epsilon,\rho}$, with $\mathrm{vol}(\mathcal{A}^{+}_{s_\epsilon,\rho}\setminus\mathcal{A}^{-}_{s_\epsilon,\rho})\leq\rho$ and associated

$$\mathcal{E}^{\pm}_{s_\epsilon,\rho}=\{\alpha\in\Gamma\backslash G: 0<\#(\alpha^{-1}\overline{\mathbf{w}}\cap\mathcal{Z}(s_\epsilon,\mathcal{A}^{\pm}_{s_\epsilon,\rho}))<r_j\ \forall j\}, \quad (6.3.44)$$

such that the right hand side of (6.3.43) is less than

$$(6.3.43)\leq \limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}^{+}_{s_\epsilon,\rho}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x})+\epsilon. \quad (6.3.45)$$

Therefore, applying Chapter 5, Corollary 5.6.4 to (6.3.45) we can bound

$$\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,\infty}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x})\leq\frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}(\overline{\mathcal{E}^{+}_{s_\epsilon,\rho}})+\epsilon \quad (6.3.46)$$

and similarly

$$\liminf_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \chi_{\mathcal{E}_{t,\infty}}(n_+(-\mathbf{x})a_{-t})d\lambda(\mathbf{x})\leq\frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}((\mathcal{E}^{-}_{s_\epsilon,\rho})^o)-\epsilon. \quad (6.3.47)$$

Therefore it remains to use $\rho=\rho(\epsilon)$ to control

$$\lim_{\epsilon\to 0}\mathrm{m}^{BR}(\overline{\mathcal{E}^{+}_{s_\epsilon,\rho}}\setminus(\mathcal{E}^{-}_{s_\epsilon,\rho})^o) \quad (6.3.48)$$

by Proposition 6.3.3 we have

$$\lim_{\epsilon\to 0}\mathrm{m}^{BR}(\overline{\mathcal{E}^{+}_{s_\epsilon,\rho}}\setminus(\mathcal{E}^{-}_{s_\epsilon,\rho})^o)\leq\lim_{\epsilon\to 0}c_{s_\epsilon,\rho}\,\mathrm{vol}(\overline{\mathcal{A}^{+}_{s_\epsilon,\rho}}\setminus(\mathcal{A}^{-}_{s_\epsilon,\rho})^o) \quad (6.3.49)$$

where $c_{s_{\epsilon,\rho}}$ is the constant C_{t_0} defined below (6.3.22), here t_0 depends on the set $\overline{\mathcal{E}^{+}_{s_\epsilon,\rho}}$.

$$t_0=\inf(\tilde{t}: 0<\#((n_+a_{-\tilde{t}})^{-1}\overline{\mathbf{w}}\cap\mathcal{Z}(\infty,\mathcal{A}^{\pm}_{s_\epsilon,\rho}))<r_j,\ \forall j) \quad (6.3.50)$$

For fixed ϵ, $\mathcal{Z}(s_\epsilon,\mathcal{A}^{+}_{s_\epsilon,\rho})$ is a cuspidal cone of fixed height. Therefore t_0 is bounded below, independent of $\rho>0$. Thus there exists a constant C'_{s_ϵ} depending only on s_ϵ such that

$$\lim_{\epsilon\to 0}\mathrm{m}^{BR}(\overline{\mathcal{E}^{+}_{s_\epsilon,\rho}}\setminus(\mathcal{E}^{-}_{s_\epsilon,\rho})^o)\leq\lim_{\epsilon\to 0}C'_{s_\epsilon}\rho(\epsilon)=0 \quad (6.3.51)$$

for $\rho(\epsilon)$ suitably chosen. Hence

$$\lim_{\epsilon\to 0}\mathrm{m}^{BR}(\overline{\mathcal{E}^{+}_{s_\epsilon,\rho(\epsilon)}})=\lim_{\epsilon\to 0}\mathrm{m}^{BR}((\mathcal{E}^{-}_{s_\epsilon,\rho(\epsilon)})^o)=\mathrm{m}^{BR}(\{\alpha\in\Gamma\backslash G: 0<\#(\alpha^{-1}\overline{\mathbf{w}}\cap\mathcal{Z}(\infty,\mathcal{A}))\leq r_j,\ \forall j\}), \quad (6.3.52)$$

proving the Theorem 6.3.2.

□

6.4 Moment Generating Function for Cuspidal Observer

Continuing to follow the example set by [MV18], for test sets $\mathcal{A}_1, ..., \mathcal{A}_m \subset \mathbb{R}^{n-1}$ with boundary of Lesbesgue measure 0 and for complex $\tau_i \in \mathbb{C}$, define the moment generating function

$$\mathbb{G}^{\infty}_{t,s}(\tau_1, ..., \tau_m; \mathcal{A}) := \int_{\mathbb{T}^l \times \mathbb{R}^{n-1-l}} \mathbb{1}(\mathcal{N}^{\infty}_{t,s}(\mathcal{A}_j, \mathbf{x}, \overline{\mathbf{w}}) \neq 0, \forall j) \exp\left(\sum_{j=1}^{m} \tau_j \mathcal{N}^{\infty}_{t,s}(\mathcal{A}_j, \mathbf{x}, \overline{\mathbf{w}})\right) d\lambda(\mathbf{x}) \quad (6.4.1)$$

and similarly for the limit distribution let

$$\mathbb{G}_s(\tau_1, ..., \tau_m; \mathcal{A}) := \sum_{r_1, ..., r_m = 1}^{\infty} \exp\left(\sum_{j=1}^{m} \tau_j r_j\right) E_s(r, \mathcal{A}, \overline{\mathbf{w}}). \quad (6.4.2)$$

Where E_s is defined as in Theorem 6.3.2 and $r = (r_1, ..., r_m)$. Let $\mathrm{Re}_+ \tau := \max(\mathrm{Re}(\tau), 0)$.

Theorem 6.4.1. *Let λ be a Borel probability measure on $\mathbb{T}^l \times \mathbb{R}^{n-1-l}$ as in Theorem 6.3.2, and $\{\mathcal{A}\}_{j=1}^m \subset \mathbb{R}^{(n-1)}$ bounded with boundary of Lesbegue measure 0. Then there exists a constant $c_0 > 0$ such that for $\mathrm{Re}_+ \tau_1 + ... + \mathrm{Re}_+ \tau_m < c_0$, $s \in (0, \infty]$*

1. *$\mathbb{G}_s(\tau_1, ..., \tau_m; \mathcal{A})$ is analytic*
2. *$\lim_{t \to \infty} e^{(n-1-\delta_\Gamma)t} \mathbb{G}^{\infty}_{t,s}(\tau_1, ..., \tau_m; \mathcal{A}) = \frac{A_\lambda}{|\mathrm{m}^{BMS}|} \mathbb{G}_s(\tau_1, ..., \tau_m; \mathcal{A})$.*

Suppose $-\infty < a < b \leq \infty$ and $\mathcal{A} \subset \mathbb{R}^{n-1}$. For $b < \infty$, $\mathcal{Z}(a, b, \mathcal{A})$ (see (6.3.30)) is bounded. Now note that there exists a lattice $\tilde{\Gamma}$ such that $\Gamma < \tilde{\Gamma}$, hence

$$\begin{aligned} \#(\alpha\overline{\mathbf{w}} \cap \mathcal{Z}(a, b, \mathcal{A})) &\leq \#(\alpha\tilde{\Gamma}\mathbf{w} \cap \mathcal{Z}(a, b, \mathcal{A})) \\ &\leq C \operatorname{vol}_{\mathbb{H}^n}(\alpha\mathcal{Z}(a, b, \mathcal{A})) \end{aligned} \quad (6.4.3)$$

which, by the left invariance of the volume is uniformly bounded from above in α. Thus $\#(\alpha\overline{\mathbf{w}} \cap \mathcal{Z}(a, b, \mathcal{A}))$ is bounded from above uniformly in $\alpha \in G$. This implies that all moments converge. Therefore we are concerned with the case $b = \infty$.

For that, let

$$\delta(\alpha\overline{\mathbf{w}}) := \min_{\substack{\gamma_1, \gamma_2 \in \Gamma \\ \gamma_1 \notin \gamma_2 \Gamma_{\mathbf{w}}}} d(\alpha\gamma_1\mathbf{w}, \alpha\gamma_2\mathbf{w}). \quad (6.4.4)$$

Note, because α is an isometry and because G acts properly discontinuously

$$\delta(\alpha\overline{\mathbf{w}}) = \min_{\gamma \in \Gamma/\Gamma_{\mathbf{w}}} d(\mathbf{w}, \gamma\mathbf{w}) = \delta(\overline{\mathbf{w}}) > 0. \quad (6.4.5)$$

In order to prove Theorem 6.4.1 we first require three lemmas.

Lemma 6.4.2. *Fix $a \in \mathbb{R}$ and a bounded subset $\mathcal{A} \subset \mathbb{R}^{n-1}$. There exist positive constant ζ, η such that for all $\alpha \in G$, $r \in \mathbb{N}$*

$$[\#(\alpha\overline{\mathbf{w}} \cap \mathcal{Z}(a, \infty, \mathcal{A})) \geq r] \Rightarrow [\#(\alpha\overline{\mathbf{w}} \cap \mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A})) \geq 1] \quad (6.4.6)$$

Lemma 6.4.2 is a statement about the definition of $\mathcal{Z}$. As the definition of $\mathcal{Z}$ is the same as in [MV18] we do not include the proof (see [MV18, Lemma 10]).

Lemma 6.4.3. *Fix a bounded subset $\mathcal{A} \subset \mathbb{R}^{n-1}$ and ζ, η as in Lemma 6.4.2. Then*

$$\int_{\Gamma\backslash G} \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A}))\mathrm{dm}^{BR}(\alpha) \leq \frac{e^{(\eta-\zeta r)}\vartheta^{(n-1)/\delta_\Gamma} \mathrm{vol}_{\mathbb{H}^n}(\mathcal{A})}{\#\Gamma_{\mathbf{w}}(n-1)} \tag{6.4.7}$$

Proof. This statement follows quite straightfowardly from Proposition 6.3.3 and specifically (6.3.22). To see this note

$$\int_{\Gamma\backslash G} \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A}))\mathrm{dm}^{BR}(\alpha) = \frac{1}{\#\Gamma_{\mathbf{w}}} \int_G \chi_{\mathcal{Z}(\zeta r-\eta,\infty,\mathcal{A})}(\alpha^{-1}\mathbf{w})\mathrm{dm}^{BR}(\alpha) \tag{6.4.8}$$

Now if we apply (6.3.22) and then insert the volume of $\mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A})$:

$$\int_{\Gamma\backslash G} \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A}))\mathrm{dm}^{BR}(\alpha) \leq \frac{e^{(-n+1-\delta_\Gamma)(\zeta r-\eta)}}{\#\Gamma_{\mathbf{w}}} \mathrm{vol}_{\mathbb{H}^n}(g_{\mathbf{w}}^{-1}\mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A})) \tag{6.4.9}$$

$$= \frac{e^{\delta_\Gamma(\eta-\zeta r)}\vartheta^{(n-1)/\delta_\Gamma} \mathrm{vol}_{\mathbb{H}^n}(\mathcal{A})}{\#\Gamma_{\mathbf{w}}(n-1)}. \tag{6.4.10}$$

□

Lemma 6.4.4. *Fix a bounded subset $\mathcal{A} \subset \mathbb{R}^{n-1}$ and ζ, η as in Lemma 6.4.2. Let λ be a probability measure on $\mathbb{T}^l \times \mathbb{R}^{n-1-l}$ as in Theorem 6.3.2. Then, there exists a constant C such that*

$$\sup_{t\geq 0} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A}))d\lambda(\mathbf{x}) \leq Ce^{-\zeta r\delta_\Gamma}. \tag{6.4.11}$$

Proof. The proof is the same as the proof of [MV18, Lemma 12]. Firstly by taking $C > 0$ large we may assume λ is the Lesbegue measure on the support of λ. Then

$$\begin{aligned}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} &\#(a_t n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(\zeta r - \eta, \infty, \mathcal{A}))\chi_{\mathrm{supp}(\lambda)}(\mathbf{x})d\mathbf{x} \\ &= \int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}} \#(n_+(\mathbf{x})\overline{\mathbf{w}} \cap \mathcal{Z}(\zeta r - \eta - t, \infty, e^{-t}\mathcal{A}))\chi_{\mathrm{supp}(\lambda)}(\mathbf{x})d\mathbf{x} \\ &\leq C\,\mathrm{vol}_{\mathbb{R}^{n-1}}(e^{-t}\mathcal{A})\#\{\gamma \in \Gamma_\infty\backslash\Gamma/\Gamma_{\mathbf{w}}, \mathrm{Im}(\gamma\mathbf{w}) \geq e^{-t+\zeta r-\eta}\}.\end{aligned} \tag{6.4.12}$$

By (6.3.3) there exists a constant such that

$$\#\{\gamma \in \Gamma_\infty\backslash\Gamma/\Gamma_{\mathbf{w}}, \mathrm{Im}(\gamma\mathbf{w}) \geq e^{-t+\zeta r-\eta}\} \leq C' \max\{1, e^{-\delta_\Gamma(t-\zeta r)}\}, \tag{6.4.13}$$

from which (6.4.11) follows.

□

Proof of Theorem 6.4.1. To begin with we once more note that for $s < \infty$, $\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j, \mathbf{x}; \overline{\mathbf{w}})$ is uniformly bounded and thus $E_s(r, \mathcal{A}; \overline{\mathbf{w}}) = 0$ for $|r| := \max_j r_j$ large enough. From here Theorem 6.4.1 follows from Theorem 6.3.2. Thus we set $s = \infty$ for the remainder of the proof.

Set $\tilde{\mathcal{A}} = \bigcup_j \mathcal{A}_j$

$$
\begin{aligned}
\sum_{|r|\geq R} E_s(r,\mathcal{A};\overline{\mathbf{w}}) &\leq \sum_{r'=R}^{\infty} E_s(r',\tilde{\mathcal{A}};\overline{\mathbf{w}}) \\
&\leq \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}(\{\alpha\in\Gamma\backslash G : \#(\alpha^{-1}\overline{\mathbf{w}}\cap\mathcal{Z}(0,\infty,\tilde{\mathcal{A}})\geq R\}) \\
&\leq \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}(\{\alpha\in\Gamma\backslash G : \#(\alpha^{-1}\overline{\mathbf{w}}\cap\mathcal{Z}(\zeta R-\eta,\infty,\tilde{\mathcal{A}})\geq 1\})
\end{aligned}
\tag{6.4.14}
$$

where we have used Lemma 6.4.2. Now by Chebyshev's inequality,

$$
\sum_{|r|\geq R} E_s(r,\mathcal{A};\overline{\mathbf{w}}) \leq \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\int_{\Gamma\backslash G}\#(\alpha^{-1}\overline{\mathbf{w}}\cap\mathcal{Z}(\zeta R-\eta,\infty,\tilde{\mathcal{A}}))d\mathrm{m}^{BR}(\alpha). \tag{6.4.15}
$$

We can then use Lemma 6.4.3 to say

$$
\sum_{|r|\geq R} E_s(r,\mathcal{A};\overline{\mathbf{w}}) \leq C_1 e^{-\delta_\Gamma\zeta R} \tag{6.4.16}
$$

from which analyticity follows.

Theorem 6.3.2 implies

$$
\begin{aligned}
\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}}\prod_{j=1}^{m}\mathbb{1}(0<\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}})<R)\exp(\tau_j\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}}))d\lambda(\mathbf{x}) \\
= \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\sum_{r_1,\ldots,r_m=1}^{R-1}\exp\left(\sum_{j=1}^{m}\tau_j r_j\right)E_s(r,\mathcal{A};\overline{\mathbf{w}}).
\end{aligned}
\tag{6.4.17}
$$

Therefore it remains to show

$$
\begin{aligned}
\lim_{R\to\infty}\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\left|\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}}\prod_{j=1}^{m}\mathbb{1}(\max_j\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}})\geq R,\ \min_j\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}})>0)\cdot\right. \\
\left.\exp(\tau_j\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}}))d\lambda(\mathbf{x})\right| = 0.
\end{aligned}
\tag{6.4.18}
$$

Note that

$$
\begin{aligned}
e^{(n-1-\delta_\Gamma)t}\left|\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}}\prod_{j=1}^{m}\mathbb{1}(\max_j\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}})\geq R)\exp(\tau_j\mathcal{N}_{t,s}^{\infty}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}}))d\lambda(\mathbf{x})\right| \\
\leq e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}}\mathbb{1}(\mathcal{N}_{t,s}^{\infty}(\mathcal{A},\mathbf{x};\overline{\mathbf{w}})\geq R)\exp(\tilde{\tau}\mathcal{N}_{t,s}^{\infty}(\tilde{\mathcal{A}},\mathbf{x};\overline{\mathbf{w}}))d\lambda(\mathbf{x}),
\end{aligned}
\tag{6.4.19}
$$

where $\tilde{\mathcal{A}} = \bigcup_j \mathcal{A}_j$ and $\tilde{\tau} = \sum_j \mathrm{Re}_+\,\tau_j$. From there, performing the same decomposition as [MV18, proof of Theorem 8] we get that the right hand side of (6.4.19) is less than or equal

$$
(6.4.19) \leq e^{(n-1-\delta_\Gamma)t}\sum_{r=R}^{\infty}e^{\tilde{\tau}r}\int_{\mathbb{T}^l\times\mathbb{R}^{n-1-l}}\mathbb{1}(\mathcal{N}_{t,s}^{\infty}(\tilde{\mathcal{A}},\mathbf{x};\overline{\mathbf{w}})\geq r)d\lambda(\mathbf{x}). \tag{6.4.20}
$$

Now using Lemma 6.4.2 and Lemma 6.4.4 we can bound (6.4.20) (uniformly in $t \geq 0$) by

$$(6.4.20) \leq \sum_{r=R}^{\infty} C e^{\tilde{\tau} r} e^{-\delta_\Gamma \zeta r}. \tag{6.4.21}$$

Thus, for $\tilde{\tau} < \delta_\Gamma \zeta$

$$\lim_{R\to\infty} \sum_{r=R}^{\infty} e^{\tilde{\tau} r} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{T}^l \times \mathbb{R}^{n-1-l}} \mathbb{1}(\mathcal{N}_{t,s}^{\infty}(\tilde{\mathcal{A}}, \mathbf{x}; \overline{\mathbf{w}}) \geq r) d\lambda(\mathbf{x}) = 0 \tag{6.4.22}$$

uniformly in t. Taking $c_0 = \delta_\Gamma \zeta$ proves Theorem 6.4.1.

□

6.5 Spherical Averages

We now present a theorem analogous to Chapter 5, Theorem 5.6.3 however we will replace the horospherical average with a spherical average. This will allow us to move the observer to the interior and replace the shrinking horospherical subset with a shrinking subset of the sphere centred on the observer. Fix $g \in G$ and recall the definition of the spherical Patterson-Sullivan measure, $\mu^{PS}_{\Gamma g \overline{K}}$ – Chapter 5, (5.5.13). Moreover, given a subset $\mathcal{U} \subset \mathbb{R}^{n-1}$ and parameterisation $R : \mathbf{x} \to \overline{K}$ from $\mathcal{U}$, as in Chapter 5, Section 5.5, recall the definition of $\omega^{PS}_{\Gamma,g,\overline{K}}$.

Theorem 6.5.1. *Let $\mathcal{U}$ be a nonempty open subset and let $R : \mathcal{U} \to \overline{K}$ such that the map $\mathcal{U} \ni \mathbf{x} \mapsto \mathbf{0}R^{-1}(\mathbf{x}) \in \partial\mathbb{H}^n$ has nonsingular differential at almost all $\mathbf{x} \in \mathcal{U}$. Let λ be a compactly supported Borel probability measure on $\mathcal{U}$ with continuous density. Then for any compactly supported, right M-invariant, continuous $f : \mathcal{U} \times \Gamma\backslash G \to \mathbb{R}$, and any family of right M-invariant, continuous $f_t : \mathcal{U} \times \Gamma\backslash G \to \mathbb{R}$ all supported on a single compact set, with $f_t \to f$ as $t \to \infty$ uniformly, for any $g \in G$*

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\mathcal{U}} f_t(\mathbf{x}, \Gamma g R(\mathbf{x}) a_t) d\lambda(\mathbf{x}) = \frac{1}{|\mathrm{m}^{BMS}|} \int_{\mathcal{U}\times\Gamma\backslash G} \lambda'(\mathbf{x}) f(\mathbf{x}, \alpha) d\mathrm{m}^{BR}(\alpha) d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x}). \tag{6.5.1}$$

Proof. The proof is similar to [MS10, Corollary 5.4] but requires some significant additions since we are no longer working with the Haar measure, but rather fractal measures and the invariance properties are not so nice.

Let $\mathbf{x}_0$ be a point where the map $\mathbf{x} \mapsto R^{-1}(\mathbf{x})\mathbf{0}$ has non-singular differential. We first show that (6.5.1) holds for any Borel subset of an open set $\mathcal{U}_0 \subset \mathcal{U}$ containing $\mathbf{x}_0$. As $R(\mathbf{x}) \in \overline{K}$ we can write

$$R(\mathbf{x}) = \begin{pmatrix} \mathbf{a}(\mathbf{x}) & \mathbf{b}(\mathbf{x}) \\ -\mathbf{b}'(\mathbf{x}) & \mathbf{a}'(\mathbf{x}) \end{pmatrix} \tag{6.5.2}$$

where $\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in \Delta_{n-2}$.

Case 1: Assume $\mathbf{a}(\mathbf{x}_0) \neq 0$. In that case we write

$$\begin{aligned} R(\mathbf{x}) &= \begin{pmatrix} \mathbf{a}(\mathbf{x}) & \mathbf{b}(\mathbf{x}) \\ -\mathbf{b}'(\mathbf{x}) & \mathbf{a}'(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\mathbf{b}'(\mathbf{x})\mathbf{a}(\mathbf{x})^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}(\mathbf{x}) & \mathbf{b}(\mathbf{x}) \\ 0 & \mathbf{b}'(\mathbf{x})\mathbf{a}(\mathbf{x})^{-1}\mathbf{b}(\mathbf{x}) + \mathbf{a}'(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \tilde{\mathbf{x}} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}(\mathbf{x}) & \mathbf{b}(\mathbf{x}) \\ 0 & -\tilde{\mathbf{x}}\mathbf{b}(\mathbf{x}) + \mathbf{a}'(\mathbf{x}) \end{pmatrix}, \end{aligned}$$

with $\tilde{\mathbf{x}} := -\mathbf{b}'(\mathbf{x})\mathbf{a}^{-1}(\mathbf{x}) = R(\mathbf{x})^{-1}\mathbf{0}$. Note further that

$$R(\mathbf{x})a_t = \begin{pmatrix} 1 & 0 \\ \tilde{\mathbf{x}} & 1 \end{pmatrix} a_t \begin{pmatrix} \mathbf{a}(\mathbf{x}) & e^{-t}\mathbf{b}(\mathbf{x}) \\ 0 & -\tilde{\mathbf{x}}\mathbf{b}(\mathbf{x}) + \mathbf{a}'(\mathbf{x}) \end{pmatrix} \tag{6.5.3}$$

$$= n_-(\tilde{\mathbf{x}})a_t \begin{pmatrix} \mathbf{a}(\mathbf{x}) & e^{-t}\mathbf{b}(\mathbf{x}) \\ 0 & -\tilde{\mathbf{x}}\mathbf{b}(\mathbf{x}) + \mathbf{a}'(\mathbf{x}) \end{pmatrix}. \tag{6.5.4}$$

As the map $\mathbf{x} \mapsto \mathbf{x}_0$ has nonsingular differential at $\mathbf{x}_0$ there exists an open set $\mathcal{V} \ni \mathbf{x}_0$ such that $\overline{\mathcal{V}} \subset \mathcal{U}$ and $\mathbf{x} \mapsto \mathbf{x}_0$ is a diffeomorphism on $\mathcal{V}$. We call the image under this map $\tilde{\mathcal{V}}$ (and adopt this notation for all subsets of $\mathcal{V}$).

Let $\mathcal{U}_0$ be an open neighbourhood of $\mathbf{x}_0$ such that $\overline{\mathcal{U}}_0 \subset \mathcal{V}$. For any Borel subset $B \subset \mathcal{U}_0$ we have

$$\tilde{B} \subset \tilde{\mathcal{U}_0} \subset \tilde{\mathcal{V}}. \tag{6.5.5}$$

Assume $\lambda(B) > 0$ and let $\tilde{\lambda}$ be the push-forward measure on $\mathbb{R}^{n-1}$ of $\frac{1}{\lambda(B)}\lambda\mid_B$ by the map $\mathbf{x} \mapsto \tilde{\mathbf{x}}$. Note $\tilde{\lambda}$ has compact support and continuous density.

Let u be a continuous function with $\chi_{\tilde{\mathcal{U}_0}} \leq u \leq \chi_{\tilde{\mathcal{V}}}$. With that let $\tilde{f}_t, \tilde{f} : \mathbb{R}^{n-1} \times \Gamma\backslash G \to \mathbb{R}$ be the continuous and compactly support functions

$$\begin{aligned} \tilde{f}_t(\tilde{\mathbf{x}}, \alpha) &= u(\tilde{\mathbf{x}}) f_t \left(\mathbf{x}, \alpha \begin{pmatrix} \mathbf{a} & e^{-t}\mathbf{b} \\ 0 & -\tilde{\mathbf{x}}\mathbf{b} + \mathbf{a}' \end{pmatrix} \right), && \tilde{\mathbf{x}} \in \tilde{\mathcal{V}} \\ \tilde{f}(\tilde{\mathbf{x}}, \alpha) &= u(\tilde{\mathbf{x}}) f \left(\mathbf{x}, \alpha \begin{pmatrix} \mathbf{a} & 0 \\ 0 & -\tilde{\mathbf{x}}\mathbf{b} + \mathbf{a}' \end{pmatrix} \right), && \tilde{\mathbf{x}} \in \tilde{\mathcal{V}} \\ \tilde{f}_t(\tilde{\mathbf{x}}, \alpha) &= \tilde{f}(\tilde{\mathbf{x}}, \alpha) = 0, && \tilde{\mathbf{x}} \notin \tilde{\mathcal{V}}. \end{aligned} \tag{6.5.6}$$

With that, we can apply Chapter 5, Theorem 5.6.3 to $\tilde{f}_t$,

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{R}^{n-1}} u(\tilde{\mathbf{x}}) f_t(\mathbf{x}, \Gamma g R(\mathbf{x}) a_t) d\lambda(\mathbf{x}) = \lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \int_{\mathbb{R}^{n-1}} \tilde{f}_t(\tilde{\mathbf{x}}, \Gamma g n_-(\tilde{\mathbf{x}}) a_t) d\tilde{\lambda}(\tilde{\mathbf{x}})$$
$$= \frac{1}{|\mathrm{m}^{BMS}|} \int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \tilde{\lambda}'(\tilde{\mathbf{x}}) \tilde{f}(\tilde{\mathbf{x}}, \alpha) d\mathrm{m}^{BR}(\alpha) d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}). \tag{6.5.7}$$

To complete the proof we have the following claim.

Claim:

$$\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \tilde{\lambda}'(\tilde{\mathbf{x}}) \tilde{f}(\tilde{\mathbf{x}}, \alpha) d\mathrm{m}^{BR}(\alpha) d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}) = \int_{\mathcal{U}\times\Gamma\backslash G} u(\tilde{\mathbf{x}}) \lambda'(\mathbf{x}) f(\mathbf{x}, \alpha) d\mathrm{m}^{BR}(\alpha) d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x}) \tag{6.5.8}$$

Accepting the claim for the moment, we have proved the Theorem 6.5.1 for a Borel subset $B \subset \mathcal{U}_0$. The full Theorem 6.5.1 follows in this case by a covering argument which is the same as the one presented in [MS10, Corollary 5.4].

Case 2: If $\mathbf{a}(\mathbf{x}_0) = 0$, then we can write

$$R(\mathbf{x}) = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ -\mathbf{b}' & \mathbf{a}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b}' & -\mathbf{a}' \\ \mathbf{a} & \mathbf{b} \end{pmatrix} =: \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} R_0(\mathbf{x}) \tag{6.5.9}$$

where $\mathbf{b}(\mathbf{x}_0) \neq 0$. Thus we can replace g in (6.5.7) with $g\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. From here the proof follows the same lines as Case 1.

Proof of Claim:

Step 1:

Expanding the left hand side of (6.5.8)

$$\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \tilde{\lambda}'(\tilde{\mathbf{x}})\tilde{f}(\tilde{\mathbf{x}},\alpha)dm^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}})$$
$$= \int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \tilde{\lambda}'(\tilde{\mathbf{x}})u(\tilde{\mathbf{x}})f(\mathbf{x},\alpha\left(\begin{smallmatrix} \mathbf{a} & 0 \\ 0 & -\tilde{\mathbf{x}}\mathbf{b}+\mathbf{a}' \end{smallmatrix}\right))dm^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}). \quad (6.5.10)$$

We may write $\left(\begin{smallmatrix} \mathbf{a} & 0 \\ 0 & -\tilde{\mathbf{x}}\mathbf{b}+\mathbf{a}' \end{smallmatrix}\right) = \left(\begin{smallmatrix} |\mathbf{a}(\mathbf{x})| & 0 \\ 0 & |\mathbf{a}(\mathbf{x})|^{-1} \end{smallmatrix}\right)M(\mathbf{x})$ where $M(\mathbf{x}) \in M$. Since f is right M-invariant, $M(\mathbf{x})$ can be ignored. Now note that the Burger-Roblin measure is 'quasi-invariant' for the geodesic flow (see [Moh13, (2)]) thus

$$\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \tilde{\lambda}'(\tilde{\mathbf{x}})\tilde{f}(\tilde{\mathbf{x}},\alpha)dm^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}})$$
$$= \int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} |\mathbf{a}(\mathbf{x})|^{(n-1-\delta_\Gamma)}\tilde{\lambda}'(\tilde{\mathbf{x}})u(\tilde{\mathbf{x}})f(\mathbf{x},\alpha)dm^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}). \quad (6.5.11)$$

Step 2:

First we note that since $R(\mathbf{x}) \in \overline{K}$, $\mathbf{a}\overline{\mathbf{a}} + \mathbf{b}\overline{\mathbf{b}} = 1$ and thus

$$R(\mathbf{x}) = n_-(\tilde{\mathbf{x}})\begin{pmatrix} \mathbf{a}(\mathbf{x}) & \mathbf{b}(\mathbf{x}) \\ 0 & \mathbf{a}^*(\mathbf{x})^{-1} \end{pmatrix}$$

Which we can further decompose

$$\begin{aligned} R(\mathbf{x}) &= n_-(\tilde{\mathbf{x}})n_+(\mathbf{b}(\mathbf{x})\mathbf{a}^*(\mathbf{x}))\begin{pmatrix} |\mathbf{a}(\mathbf{x})| & 0 \\ 0 & |\mathbf{a}(\mathbf{x})|^{-1} \end{pmatrix}\begin{pmatrix} \frac{\mathbf{a}(\mathbf{x})}{|\mathbf{a}(\mathbf{x})|} & 0 \\ 0 & \frac{\mathbf{a}^*(\mathbf{x})^{-1}}{|\mathbf{a}(\mathbf{x})|^{-1}} \end{pmatrix}, \\ &= n_-(\tilde{\mathbf{x}})A(\mathbf{x}), \end{aligned} \quad (6.5.12)$$

where we have defined $A(\mathbf{x}) := n_+(\mathbf{b}(\mathbf{x})\mathbf{a}^*(\mathbf{x}))\left(\begin{smallmatrix} |\mathbf{a}(\mathbf{x})| & 0 \\ 0 & |\mathbf{a}(\mathbf{x})|^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \frac{\mathbf{a}(\mathbf{x})}{|\mathbf{a}(\mathbf{x})|} & 0 \\ 0 & \frac{\mathbf{a}^*(\mathbf{x})^{-1}}{|\mathbf{a}(\mathbf{x})|^{-1}} \end{smallmatrix}\right)$. Note that the last matrix is in M. As we are working on $\overline{K}$ this last matrix can be ignored.

Now observe that by using (6.5.12)

$$\begin{aligned} gR(\mathbf{x})\mathbf{X}_\mathbf{i}^+ &= \lim_{t\to\infty} gR(\mathbf{x})a_t\mathbf{X}_\mathbf{i} \\ &= gn_-(\tilde{\mathbf{x}})\mathbf{X}_\mathbf{i}^+. \end{aligned}$$

Therefore using the definition of $\omega^{PS}_{\Gamma,g,\overline{H}}$ (5.5.12) we can write

$$
\begin{aligned}
d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}) &= d\mu^{PS}_{\Gamma g\overline{H}}(gn_-(\tilde{\mathbf{x}})) \\
&= e^{\delta_\Gamma \beta_{gn(\tilde{\mathbf{x}})\mathbf{X}^+_{\mathbf{i}}}(\mathbf{i}, gn_-(\tilde{\mathbf{x}})\mathbf{i})} d\nu_{\mathbf{i}}(gn_-(\tilde{\mathbf{x}})\mathbf{X}^+_{\mathbf{i}}) \\
&= e^{\delta_\Gamma \beta_{gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}}(\mathbf{i}, gR(\mathbf{x})A(\mathbf{x})^{-1}\mathbf{i})} d\nu_{\mathbf{i}}(gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}).
\end{aligned} \tag{6.5.13}
$$

Note that the Busemann function is both M and N_+ invariant (via right multiplication). Hence

$$
\begin{aligned}
\beta_{gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}}(\mathbf{i}, gR(\mathbf{x})A(\mathbf{x})^{-1}\mathbf{i}) &= \beta_{gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}}\left(\mathbf{i}, gR(\mathbf{x})\begin{pmatrix} |\mathbf{a}|^{-1} & 0 \\ 0 & |\mathbf{a}| \end{pmatrix}\mathbf{i}\right) \\
&= \ln|\mathbf{a}| + \beta_{gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}}(\mathbf{i}, gR(\mathbf{x})\mathbf{i})
\end{aligned} \tag{6.5.14}
$$

Therefore

$$
d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}) = |\mathbf{a}(\mathbf{x})|^{\delta_\Gamma} e^{\delta_\Gamma \beta_{gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}}(\mathbf{i}, gR(\mathbf{x})\mathbf{i})} d\nu_{\mathbf{i}}(gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}) \tag{6.5.15}
$$

Step 3:
Inserting $\tilde{\lambda}'(\tilde{\mathbf{x}}) = \left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right|^{-1}\lambda'(\mathbf{x})$ into (6.5.11) gives

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \tilde{\lambda}'(\tilde{\mathbf{x}})\tilde{f}(\tilde{\mathbf{x}},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}) \\
&\qquad = \int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} |\mathbf{a}(\mathbf{x})|^{(n-1-\delta_\Gamma)} \left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right|^{-1} \lambda'(\mathbf{x})u(\tilde{\mathbf{x}})f(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}).
\end{aligned} \tag{6.5.16}
$$

Now if we insert (6.5.15) into (6.5.16) we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \tilde{\lambda}'(\tilde{\mathbf{x}})\tilde{f}(\tilde{\mathbf{x}},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{H}}(\tilde{\mathbf{x}}) \\
&= \int_{\mathbb{R}^{n-1}\times\Gamma\backslash G} \lambda'(\mathbf{x})u(\tilde{\mathbf{x}})f(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)\left(\left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right|^{-1}|\mathbf{a}(\mathbf{x})|^{n-1}e^{\delta_\Gamma \beta_{gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}}}(\mathbf{i}, gR(\mathbf{x})\mathbf{i})} d\nu_{\mathbf{i}}(gR(\mathbf{x})\mathbf{X}^+_{\mathbf{i}})\right).
\end{aligned} \tag{6.5.17}
$$

Note that the final measure in the brackets is exactly the definition of $d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x})$, (5.5.20). Proving the claim.

□

We can extend Theorem 6.5.1 to sequences of characteristic functions in much the same way as for Chapter 5, Corollary 5.6.4

Corollary 6.5.2. *Under the assumptions of Theorem 6.5.1, for any $g \in \Gamma\backslash G$ and any bounded family of subsets $\mathcal{E}_t \subset \mathcal{U} \times \Gamma\backslash G$ with boundary of $\omega^{PS}_{\Gamma,g,\overline{K}} \times \mathrm{m}^{BR}$-measure 0*

$$\liminf_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathcal{U}} \chi_{\mathcal{E}_t}(\mathbf{x},\Gamma g R(\mathbf{x})a_t)d\lambda(\mathbf{x}) \geq \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathcal{U}\times\Gamma\backslash G} \lambda'(\mathbf{x})\chi_{\lim(\inf \mathcal{E}_t)^o}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x}) \quad (6.5.18)$$

and

$$\limsup_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathcal{U}} \chi_{\mathcal{E}_t}(\mathbf{x},\Gamma g R(\mathbf{x})a_t)d\lambda(\mathbf{x}) \leq \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathcal{U}\times\Gamma\backslash G} \lambda'(\mathbf{x})\chi_{\lim \overline{\sup \mathcal{E}_t}}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x}) \quad (6.5.19)$$

If furthermore $\lambda\times\mathrm{m}^{BR}$ *gives zero measure to* $\lim \overline{\sup \mathcal{E}_t}\backslash \lim(\inf \mathcal{E}_t)^o$

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathcal{U}} \chi_{\mathcal{E}_t}(\mathbf{x},\Gamma g R(\mathbf{x})a_t)d\lambda(\mathbf{x}) = \frac{1}{|\mathrm{m}^{BMS}|}\int_{\mathcal{U}\times\Gamma\backslash G} \lambda'(\mathbf{x})\chi_{\lim\sup \mathcal{E}_t}(\mathbf{x},\alpha)d\mathrm{m}^{BR}(\alpha)d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x}) \quad (6.5.20)$$

6.6 Projection Statistics for Observers in $\mathbb{H}^n$

Define the coordinate chart of a neighbourhood of the south pole of S_1^{n-1} in $\mathbb{H}^n$ given by the map

$$\mathbf{x} \mapsto E(\mathbf{x})^{-1}(e^{-1}\mathbf{i}) \quad (6.6.1)$$

where

$$E(\mathbf{x}) = \left(\exp\begin{pmatrix} 0 & \mathbf{x} \\ -\mathbf{x}' & 0 \end{pmatrix}\right) \quad (6.6.2)$$

Note that by [MV18, (6.3)] the map $\mathbf{x}\mapsto\tilde{\mathbf{x}} = E(\mathbf{x})^{-1}\mathbf{0}$ has a nonsingular differential for all $|\mathbf{x}| < \pi/2$ hence we can apply Corollary 6.5.2.

Define the shrinking test set

$$\mathcal{B}_{t,s}(\mathcal{A},0) := \{E(\mathbf{x})^{-1}(e^{-1}\mathbf{i}) : \mathbf{x}\in\rho_{t,s}\mathcal{A}\} \quad (6.6.3)$$

where $\mathcal{A}\subset\mathbb{R}^{n-1}$ is a set wih fixed boundary of Lesbegue measure 0 and $\rho_{t,s} > 0$ is chosen such that

$$\omega(\mathcal{B}_{t,s}(\mathcal{A},0)) = \frac{\mathrm{vol}_{\mathbb{R}^{n-1}}\mathcal{A}}{(\#\mathcal{P}_{t,s}(g\overline{\mathbf{w}}))^{\frac{n-1}{\delta_\Gamma}}}, \quad (6.6.4)$$

thus, for large t, $\rho_{t,s}\sim\vartheta^{-1/\delta_\Gamma}e^{-t}$. Now we replace the random translations which we considered for the cuspidal observer with random rotations on the sphere. Recall the map from Theorem 6.5.1 for an open $\mathcal{U}\subset\mathbb{R}^{n-1}$, $\mathbf{x}\mapsto R(\mathbf{x})$ and let

$$\mathcal{B}_{t,s}(\mathcal{A},\mathbf{x}) := R(\mathbf{x})^{-1}(\mathcal{B}_{t,s}(\mathcal{A},0)). \quad (6.6.5)$$

From which we define the random variable

$$\mathcal{N}_{t,s}(\mathcal{A}, \mathbf{x}, g\overline{\mathbf{w}}) := \#(\mathcal{P}_{t,s}(g\overline{\mathbf{w}}) \cap \mathcal{B}_{t,s}(\mathcal{A}, \mathbf{x})). \tag{6.6.6}$$

Finally, let

$$C_{\lambda,\mathcal{U}} := \int_{\mathcal{U}} \lambda'(\mathbf{x}) d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x}) \tag{6.6.7}$$

With that we can describe the joint distribution for several test sets: $\mathcal{A}_1, ..., \mathcal{A}_m$:

Theorem 6.6.1. *Let $\mathcal{U} \subset \mathbb{R}^{n-1}$ be a nonempty open subset and let $R : \mathcal{U} \to K$ be a map as in Theorem 6.5.1. Let λ be a compactly supported Borel probability measure on $\mathcal{U}$, absolutely continuous with respect to Lebesgue and with continuous density. Then for every $g \in G$, $s \in [0, \infty]$, $r = (r_1, ...r_m) \in \mathbb{Z}^m_{>0}$ and $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_m$ with $\mathcal{A}_j \subset \mathbb{R}^{n-1}$ bounded of Lesbegue measure 0:*

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t} \lambda(\{\mathbf{x} \in \mathcal{U} : \mathcal{N}_{t,s}(\mathcal{A}_j, \mathbf{x}; g\overline{\mathbf{w}}) = r_j \forall j\}) = E_s(r, \mathcal{A}; g\overline{\mathbf{w}}) \tag{6.6.8}$$

where $E_s(r, \mathcal{A}; g\overline{\mathbf{w}})$ is as in Theorem 6.3.2 with A_λ replaced by $C_{\lambda,\mathcal{U}}$.

The proof of this theorem follows the same steps as Theorem 6.3.2 replacing the horospherical averages with the spherical ones proved in the previous section and Lemma 6.3.4 replaced with the following:

Lemma 6.6.2. *Under the hypotheses of Theorem 6.6.1, given $\epsilon > 0$ there exists a $t_0 < \infty$ and bounded subsets $\mathcal{A}^-_j \subset \mathcal{A}^+_j \subset \mathbb{R}^{n-1}$ with boundary of measure 0, such that:*

$$\mathrm{vol}_{\mathbb{R}^{n-1}}(\mathcal{A}^+_j \setminus \mathcal{A}^-_j) < \epsilon \tag{6.6.9}$$

and for all $t \geq t_0$:

$$\#(a_t R(\mathbf{x}) a_t \overline{\mathbf{w}} \cap \mathcal{Z}(\epsilon, s^-, \mathcal{A}^-_j)) \leq \mathcal{N}_{t,s}(\mathcal{A}_j, \mathbf{x}; g\overline{\mathbf{w}}) \leq \#(a_t R(\mathbf{x}) a_t \overline{\mathbf{w}} \cap \mathcal{Z}(-\epsilon, s+\epsilon, \mathcal{A}^+_j)) \tag{6.6.10}$$

with

$$s^- = \begin{cases} s - \epsilon & (s < \infty) \\ \epsilon^{-1} & (s = \infty). \end{cases} \tag{6.6.11}$$

The proof of this Lemma is identical to that of [MV18, Lemma 16]. The one exception is the scaling in the definition of $\rho_{t,s}$ in (6.6.3). We therefore omit it.

Proof of Theorem 6.2.2. The proof is essentially an application of Theorem 6.6.1. Choose $m = 1$ and $\mathcal{A} \subset \mathbb{R}^{n-1}$ to be a Euclidean ball of volume σ. Then set

$$\mathcal{B}_{t,s}(\mathcal{A}, 0) := \{E(\mathbf{x})^{-1}(e^{-1}\mathbf{i}) : \mathbf{x} \in \rho_{t,s}\mathcal{A}\} = \mathcal{D}_{t,s}(\sigma, e^{-1}\mathbf{i}, g\overline{\mathbf{w}}) \tag{6.6.12}$$

Define the coordinate chart

$$\begin{aligned} &\mathcal{U} \to S^{n-1}_1 \\ &\mathbf{x} \mapsto \mathbf{v} = R(\mathbf{x})^{-1}(e^{-1}\mathbf{i}) \end{aligned} \tag{6.6.13}$$

for appropriate $\mathcal{U}$ and $R(\mathbf{x})$. Consider

$$\begin{aligned}E_s(r,\sigma;g\overline{\mathbf{w}}) &= \lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\lambda(\{\mathbf{v}\in S_1^{n-1}:\mathcal{N}_{t,s}(\sigma,\mathbf{v};g\overline{\mathbf{w}})=r\})\\ &= \lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\lambda(\{k\in K:\mathcal{N}_{t,s}(\sigma,ke^{-1}\mathbf{i};g\overline{\mathbf{w}})=r\})\end{aligned} \tag{6.6.14}$$

Applying the parameterisation $R:\mathcal{U}\to\overline{K}$ (and thus restricting the measure λ so that the new density is $\lambda'\chi_{R(\mathcal{U})}$) and using Lemma 5.5.1

$$E_{s,\mathcal{U}}(r,\sigma;g\overline{\mathbf{w}}) = \lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\int_{\mathbb{R}^{n-1}}\chi_{\mathcal{U}}(\mathbf{x})\lambda'(R(\mathbf{x}))\chi_{\mathcal{A}}(R(\mathbf{x}))\left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right||\mathbf{a}(\mathbf{x})|^{-(n-1)}d\mathbf{x} \tag{6.6.15}$$

Now applying Theorem 6.6.1 with $\tilde{\lambda}'(\mathbf{x})=\chi_{\mathcal{U}}(\mathbf{x})\lambda'(R(\mathbf{x}))\left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right||\mathbf{a}(\mathbf{x})|^{-(n-1)}$ implies

$$E_{s,\mathcal{U}}(r,\sigma;g\overline{\mathbf{w}}) = C_{\tilde{\lambda},\mathcal{U}}\mathrm{m}^{BR}(\{\alpha\in G/\Gamma:\#(\alpha^{-1}\overline{\mathbf{w}}\cap\mathcal{Z}_0(s,\sigma))=r\}) \tag{6.6.16}$$

With

$$\begin{aligned}C_{\tilde{\lambda},\mathcal{U}} &= \int_{\mathcal{U}}\lambda'(R(\mathbf{x}))\left|\frac{\partial\tilde{\mathbf{x}}}{\partial\mathbf{x}}\right||\mathbf{a}(\mathbf{x})|^{-(n-1)}d\omega^{PS}_{\Gamma,g,\overline{K}}(\mathbf{x})\\ &= \int_{\overline{K}}\chi_{R(\mathcal{U})}(k)\lambda'(k)d\mu^{PS}_{\Gamma g\overline{K}}(k).\end{aligned} \tag{6.6.17}$$

By choosing suitable $\mathcal{U}$, partitioning S_1^{n-1} we have thus proved Theorem 6.2.2. The continuity in s and σ and (6.2.10) follow from (6.3.10).

□

6.6.1 Moment Generating Function

Much like in Section 6.4 the convergence result Theorem 6.5.1 gives rise to a convergence result for the moment generating function for a non-cuspidal observer:

$$\mathbb{G}_{t,s}(\tau_1,...,\tau_m;\mathcal{A}) := \int_{S^{n-1}}\mathbb{1}(\mathcal{N}_{t,s}(\mathcal{A}_j,\mathbf{v};g\overline{\mathbf{w}})\neq 0;\forall j)\exp\left(\sum_{j=1}^m\tau_j\mathcal{N}_{t,s}(\mathcal{A}_j,\mathbf{v};g\overline{\mathbf{w}})\right)d\lambda(\mathbf{v}). \tag{6.6.18}$$

Theorem 6.6.3. *Let λ be a probability measure on S_1^{n-1} absolutely continuous with respect to Lebesgue and with continuous density. Then there exists a $c_0>0$ such that for all $\mathrm{Re}_+(\tau_1)+...+\mathrm{Re}_+(\tau_m)<c_0$ and $s\in(0,\infty]$:*

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)t}\mathbb{G}_{t,s}(\tau_1,...,\tau_m;\mathcal{A}) = \frac{C_\lambda}{|\mathrm{m}^{BMS}|}\mathbb{G}_s(\tau_1,...,\tau_m;\mathcal{A}). \tag{6.6.19}$$

The proof of Theorem 6.6.3 is very similar to the proof Theorem 6.4.1. The only difference is that Lemma 6.4.2 and Lemma 6.4.4 are replaced with Lemma 6.6.4 and Lemma 6.6.5 respectively. Recall the definition of the direction function $\varphi_{\mathbf{i}}(\mathbf{z})$ from the top of Section 6.2.1. For $B\subset S_1^{n-1}$ and $-\infty\leq a<b<\infty$ define the cone

$$\mathcal{C}(a,b,B) := \{z\in\mathbb{H}^n\setminus\{\mathbf{i}\},\varphi_{\mathbf{i}}(\mathbf{z}):a<d(\mathbf{i},\mathbf{z})\leq b\}. \tag{6.6.20}$$

Lemma 6.6.4. *Fix $a \in \mathbb{R}$ and a bounded $\mathcal{A} \subset \mathbb{R}^{n-1}$. Then there exist positive constants ζ, η, t_0 such that for all $g \in G$, $r \in \mathbb{N}_{>0}$, $t \geq t_0$*

$$[\#(g\overline{\mathbf{w}} \cap \mathcal{C}(0, t, \mathcal{B}_{t,\infty}(\mathcal{A}, 0))) \geq r] \Rightarrow [\#(g\overline{\mathbf{w}} \cap \mathcal{C}(0, t - \zeta r + \eta, \mathcal{B}_{t,\infty}(\mathcal{A}, 0))) \geq 1]\,. \tag{6.6.21}$$

As with Lemma 6.4.2, this theorem is stated identically to [MV18, Lemma 19], as the statement concerns only the definition of the spherical cone $\mathcal{C}$ and this is the same in both papers we omit the details.

Lemma 6.6.5. *Fix a bounded set $\mathcal{A} \subset \mathbb{R}^{n-1}$ and ζ and η as in Lemma 6.6.4. Let λ be a Borel probability measure on $\mathcal{U}$ as in Theorem 6.6.1. Then there exists a C such that for all $r \geq 0$*

$$\sup_{t>0} e^{(n-1-\delta_\Gamma)t} \int_{\mathcal{U}} \#(a_t R(\mathbf{x}) g\overline{\mathbf{w}} \cap \mathcal{C}(0, t - \zeta r + \eta, \mathcal{B}_{t,\infty}(\mathcal{A}, 0))) d\lambda(\mathbf{x}) \leq C e^{-\delta_\Gamma \zeta r}. \tag{6.6.22}$$

Proof. The proof of this lemma is identical to that of [MV18, Lemma 20] with the one exception that we use (6.3.3) rather than the analogous asymptotics.

Replace $\mathcal{B}_{t,\infty}(\mathcal{A}, 0)$ with the ball $\mathcal{D}_t \subset S_1^{n-1}$ contianing it of volume $\omega(\mathcal{D}_t) = \sigma_0 e^{-(n-1)t}$ for all $t \geq 0$ and some σ_0. We can bound this by

$$\begin{aligned} &\int_{\mathcal{U}} \#(a_t R(\mathbf{x}) g\overline{\mathbf{w}} \cap \mathcal{C}(0, t - \zeta r + \eta, \mathcal{B}_{t,\infty}(\mathcal{A}, 0))) d\lambda(\mathbf{x}) \\ &\qquad\qquad \leq C_2 \int_K \#(a_t k g\overline{\mathbf{w}} \cap \mathcal{C}(0, t - \zeta r + \eta, \mathcal{D}_t)) d\mu_K^{Haar}(k). \end{aligned} \tag{6.6.23}$$

Using the definition of $\mathcal{C}(\cdot, \cdot, \cdot)$,

$$C_2 \int_K \#(a_t k g\overline{\mathbf{w}} \cap \mathcal{C}(0, t - \zeta r + \eta, \mathcal{D}_t)) d\mu_K^{Haar}(k) \leq \sigma_0 e^{-(n-1)t} \#\{\gamma \in \Gamma/\Gamma_{\mathbf{w}}, : d(g\gamma\mathbf{w}) \leq e^{t - \zeta r + \eta}\}. \tag{6.6.24}$$

By (6.3.3) we conclude that

$$\int_{\mathcal{U}} \#(a_t R(\mathbf{x}) g\overline{\mathbf{w}} \cap \mathcal{C}(0, t - \zeta r + \eta, \mathcal{B}_{t,\infty}(\mathcal{A}, 0))) d\lambda(\mathbf{x}) \leq C\sigma_0 e^{-(n-1)t} \max(1, e^{\delta_\Gamma (t - \zeta r)}). \tag{6.6.25}$$

Lemma 6.6.5 follows from here.

□

6.7 Applications to Moments, Two Point Correlation Function and Gap Statistics

6.7.1 Convergence of Moments

Once again analogous to [MV18], we note that Theorem 6.4.1 and Theorem 6.6.3 each gives rise to a corollary concerning the convergence of moments (we state them here as one):

For an observer on the boundary observer consider the mixed-moment:

$$\mathbb{M}^{\infty}_{t,s}(\beta_1,...,\beta_m;\mathcal{A}) := \int_{\mathbb{T}^{n-1}} \prod_{j=1}^{m} (\mathcal{N}^{\infty}_{t,s}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}}))^{\beta_j} d\lambda(\mathbf{x}) \tag{6.7.1}$$

for all $\beta_j \in \mathbb{R}_{\geq 0}$ with limit moment:

$$\mathbb{M}_s(\beta_1,...,\beta_m;\mathcal{A}) := \frac{|\mathrm{m}^{BMS}|}{A_\lambda} \sum_{r_1,...,r_m=1}^{\infty} r_1^{\beta_1}...r_m^{\beta_m} E_s(r,\mathcal{A};\overline{\mathbf{w}}). \tag{6.7.2}$$

For a non-cuspidal observer we define:

$$\mathbb{M}_{t,s}(\beta_1,...,\beta_m;\mathcal{A}) := \int_{S_1^{n-1}} \prod_{j=1}^{m} (\mathcal{N}_{t,s}(\mathcal{A}_j,\mathbf{x};\overline{\mathbf{w}}))^{\beta_j} d\lambda(\mathbf{x}) \tag{6.7.3}$$

for all $\beta_j \in \mathbb{R}_{\geq 0}$ (the limit moment is the same). Hence the following corollary follows from Theorem 6.4.1 and Theorem 6.6.3.

Corollary 6.7.1. *Let λ be a probability measure on $\mathbb{T}^{n-1}$ absolutely continuous with respect to Lebesgue and with bounded continuous density, and $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_m$ with $\mathcal{A}_j \subset \mathbb{R}^{n-1}$ bounded with boundary of Lebesgue measure zero. Then for all $\beta_1,...,\beta_m \in \mathbb{R}_{\geq 0}$, $s \in [0,\infty]$:*

$$\mathbb{M}_s(\beta_1,...,\beta_m;\mathcal{A}) < \infty \tag{6.7.4}$$

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)}\mathbb{M}^{\infty}_{t,s}(\beta_1,...,\beta_m;\mathcal{A}) = \frac{A_\lambda}{|\mathrm{m}^{BMS}|}\mathbb{M}_s(\beta_1,...,\beta_m;\mathcal{A}). \tag{6.7.5}$$

For an observer in $\mathbb{H}^n$ (6.7.5) is replaced with

$$\lim_{t\to\infty} e^{(n-1-\delta_\Gamma)}\mathbb{M}_{t,s}(\beta_1,...,\beta_m;\mathcal{A}) = \frac{C_\lambda}{|\mathrm{m}^{BMS}|}\mathbb{M}_s(\beta_1,...,\beta_m;\mathcal{A}). \tag{6.7.6}$$

With that, there is an explicit formula for each of these moments. For example, if we take $\beta_1 = \cdots = \beta_m = 1$, then

$$\begin{aligned}\mathbb{M}_s(1,...,1;\mathcal{A}) &= \frac{|\mathrm{m}^{BMS}|}{A_\lambda} \sum_{r_1,...,r_m}^{\infty} r_1 \dots r_m E_s(r,\mathcal{A};\overline{\mathbf{w}}) \\ &= \int_{G/\Gamma} \sum_{\gamma_1,...,\gamma_m \in \Gamma/\Gamma_{\mathbf{w}}} \prod_{j=1}^{m} \mathbb{1}(\alpha^{-1}\gamma_j \mathbf{w} \in \mathcal{Z}(s,\mathcal{A}_j)) d\mathrm{m}^{BR}(\alpha).\end{aligned} \tag{6.7.7}$$

6.7.2 Two-Point Correlation Function

We will work in the case of an observer on the boundary (thus w.l.o.g at ∞), note that this then applies to the sphere packing case. The case of an observer in the interior can be treated similarly however working on S_1^{n-1} rather than $\mathbb{T}^{n-1}$ makes the problem more complex. Furthermore we will work in the special case of $\mathbb{T}^{n-1}$ rather than $\mathbb{T}^l \times \mathbb{R}^{n-1-l}$, however that case follows similarly. As we will use it throughout recall that $\mathcal{B}_r(\mathbf{x}) \subset \mathbb{T}^{n-1}$ denotes the ball of size r around $\mathbf{x}$.

Consider the points in $\mathcal{P}^{\infty}_t(\overline{\mathbf{w}})$ and label them $\{\mathbf{x}_i\}_{i=1}^{N_t} \subset \mathbb{T}^{n-1}$ where $N_t = \#\mathcal{P}^{\infty}_t(\overline{\mathbf{w}}) \sim c_0^{-1} e^{\delta_\Gamma t}$ (in the notation of Theorem 6.3.1 $c_0^{-1} = \vartheta|\mu^{PS}_{\Gamma g \overline{H}}|$). We consider first the two-point correlation function, for $f \in \mathcal{C}_0(\mathbb{T}^{n-1})$,

$$R_2(f)(t) := \frac{c_0}{e^{\delta_\Gamma t}} \sum_{\substack{i,j=1,\\ i\neq j}}^{N_t} f(e^t(\mathbf{x}_i - \mathbf{x}_j)). \tag{6.7.8}$$

As explained in [EBMV15, Appendix A], we can approximate f from above and below by a finite linear combination of functions of the form

$$\tilde{f}(\mathbf{z}) = \sum_{k=1}^{p} \gamma_k \int_{\mathbf{x}\in\mathbb{T}^{n-1}} \left(\chi_{\mathcal{R}_{1,k}}(\mathbf{z}+\mathbf{x})\chi_{\mathcal{R}_{2,k}}(\mathbf{x})\right) d\mathbf{x} \tag{6.7.9}$$

where $\mathcal{R}_{i,k}$ are rectangular boxes. That is, in dimension 2 we can approximate the function by a Riemann sum, in higher dimensions we approximate f by a linear combination of step functions supported on boxes. In other words, for any ϵ, there exists a $p < \infty$, a set of boxes $\{\mathcal{R}_{i,k}\}_{k=1}^{p}$, and bounded constants $\{\gamma_k^u\}_{k=1}^{p}, \{\gamma_k^l\}_{k=1}^{p}$ such that

$$\sum_{k=1}^{p} \gamma_k^l \int_{\mathbf{x}\in\mathbb{T}^{n-1}} (\chi_{\mathcal{R}_{1,k}}(\mathbf{z}+\mathbf{x})\chi_{\mathcal{R}_{2,k}}(\mathbf{x}))d\mathbf{x} \leq f(\mathbf{z}) \leq \sum_{k=1}^{p} \gamma_k^u \int_{\mathbf{x}\in\mathbb{T}^{n-1}} \left(\chi_{\mathcal{R}_{1,k}}(\mathbf{z}+\mathbf{x})\chi_{\mathcal{R}_{2,k}}(\mathbf{x})\right) d\mathbf{x}, \tag{6.7.10}$$

and

$$\sum_{k=1}^{p} (\gamma_k^u - \gamma_k^l) \int_{\mathbf{x}\in\mathbb{T}^{n-1}} \left(\chi_{\mathcal{R}_{1,k}}(\mathbf{z}+\mathbf{x})\chi_{\mathcal{R}_{2,k}}(\mathbf{x})\right) d\mathbf{x} \leq \epsilon. \tag{6.7.11}$$

Hence we can approximate $R_2(f)(t)$ by functions of the form

$$\begin{aligned}
& c_0 e^{(-\delta_\Gamma)t} \sum_{k=1}^{p} \gamma_k \int_{\mathbf{x}\in\mathbb{T}^{n-1}} \left(\sum_{\substack{i,j=1,\\ i\neq j}}^{N_t} \chi_{\mathcal{R}_{1,k}}(e^t(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{x})\chi_{\mathcal{R}_{2,k}}(\mathbf{x}) \right) d\mathbf{x} \\
= \quad & c_0 e^{(n-1-\delta_\Gamma)t} \sum_{k=1}^{p} \gamma_k \int_{\mathbf{x}\in\mathbb{T}^{n-1}} \left(\sum_{\substack{i,j=1,\\ i\neq j}}^{N_t} \chi_{e^{-t}\mathcal{R}_{1,k}}(\mathbf{x}_i + \mathbf{x})\chi_{e^{-t}\mathcal{R}_{2,k}}(\mathbf{x}_j + \mathbf{x}) \right) d\mathbf{x} \qquad (6.7.12) \\
= \quad & c_0 e^{(n-1-\delta_\Gamma)t} \sum_{k=1}^{p} \gamma_k \int_{\mathbf{x}\in\mathbb{T}^{n-1}} \left(\mathcal{N}_{t,\infty}^{\infty}(\mathcal{R}_{1,k}, \mathbf{x}; \overline{\mathbf{w}})\mathcal{N}_{t,\infty}^{\infty}(\mathcal{R}_{2,k}, \mathbf{x}; \overline{\mathbf{w}}) - \mathcal{N}_{t,\infty}^{\infty}(\mathcal{R}_{1,k} \cap \mathcal{R}_{2,k}, \mathbf{x}; \overline{\mathbf{w}}) \right) d\mathbf{x}.
\end{aligned}$$

Using Corollary 6.7.1 we know

$$\begin{aligned}
\lim_{t\to\infty} c_0 e^{(n-1-\delta_\Gamma)t} \sum_{k=1}^{p} \gamma_k \int_{\mathbf{x}\in\mathbb{T}^{n-1}} \left(\mathcal{N}_{t,\infty}^{\infty}(\mathcal{R}_{1,k}, \mathbf{x}; \overline{\mathbf{w}})\mathcal{N}_{t,\infty}^{\infty}(\mathcal{R}_{2,k}, \mathbf{x}; \overline{\mathbf{w}}) - \mathcal{N}_{t,\infty}^{\infty}(\mathcal{R}_{1,k} \cap \mathcal{R}_{2,k}, \mathbf{x}; \overline{\mathbf{w}}) \right) d\mathbf{x} \\
= \frac{A_\lambda c_0}{|\mathrm{m}^{BMS}|} \sum_{k=1}^{p} \left(\gamma_k (\mathbb{M}_\infty(1,1;\mathcal{R}_{1,k} \times \mathcal{R}_{2,k}) - \mathbb{M}_\infty(1, \mathcal{R}_{1,k} \cap \mathcal{R}_{2,k})) \right). \quad (6.7.13)
\end{aligned}$$

If the sets $\mathcal{A}$ have finite area, then for any $\beta_1, \ldots \beta_n$, $\mathbb{M}_\infty(\beta_1, \ldots, \beta_n, \mathcal{A})$ is finite. Therefore for any $\varrho > 0$ there exist $p, \{\mathcal{R}_k\}_{k=1}^{p}, \{\gamma_k^u\}_{k=1}^{p}, \{\gamma_k^u\}_{k=1}^{p}$ such that

$$\sum_{k=1}^{p} \left((\gamma_k^u - \gamma_k^l)(\mathbb{M}_\infty(1,1;\mathcal{R}_{1,k} \times \mathcal{R}_{2,k}) - \mathbb{M}_\infty(1, \mathcal{R}_{1,k} \cap \mathcal{R}_{2,k})) \right) \leq \varrho. \tag{6.7.14}$$

Hence the approximations from above and below converge in the limit $t \to \infty$ as well. Hence the limit $\lim_{t\to\infty} R_2(f)(t)$ exists. By an approximation argument if f is an indicator function $\lim_{t\to\infty} R_2(f)(t)$

also exists. Thus

$$R_2(\xi) := \lim_{t\to\infty} \frac{c_0}{e^{\delta_\Gamma t}} \sum_{i,j=1,\ i\neq j}^{N_t} \mathbb{1}\left(\mathbf{x}_j \in \mathcal{B}_{\xi e^{-t}}(\mathbf{x}_i)\right) \tag{6.7.15}$$

has a limit for every fixed ξ. Thus, as in [MV18] (for lattices), we have

$$R_2(\xi) = \lim_{\epsilon\to 0} \frac{c}{\epsilon^{n-1}} \left[\mathbb{M}_\infty(1,1;\mathcal{B}_\xi(0)\times\mathcal{B}_\epsilon(0)) - \mathbb{M}_\infty(1;\mathcal{B}_\epsilon(0))\right] \tag{6.7.16}$$

where we have again used Corollary 6.7.1 and set $c = c_0 \frac{A_\lambda}{|\mathrm{m}^{BMS}|}$.

Moreover, using (6.7.7), we can write

$$\mathbb{M}_\infty(1,1;\mathcal{B}_\xi(0)\times\mathcal{B}_\epsilon(0)) - \mathbb{M}_\infty(1;\mathcal{B}_\epsilon(0)) = \frac{1}{\#\Gamma_{\mathbf{w}}} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} F_{\gamma,\epsilon}(\vartheta^{-1}\xi) \tag{6.7.17}$$

where

$$F_{\gamma,\epsilon}(\vartheta^{-1}\xi) := \int_G \mathbb{1}(\alpha^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty,\mathcal{B}_\xi(0)))\mathbb{1}(\alpha^{-1}\mathbf{w} \in \mathcal{Z}(\infty,\mathcal{B}_\epsilon(0)))d\mathrm{m}^{BR}(\alpha), \tag{6.7.18}$$

here $\mathcal{B}_r(0)$ is the ball of radius r around 0 in $\partial\mathbb{H}^n$.

Applying the same Iwasawa decomposition and change of coordinates as was done in the proof of Proposition 6.3.3 gives

$$F_{\gamma,\epsilon}(\vartheta^{-1}\xi) = \int_{KAN_+} \mathbb{1}(g_{\mathbf{w}}\alpha^{-1}g_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty,\mathcal{B}_\xi(0)))\cdot \\ \mathbb{1}(g_{\mathbf{w}}n_+a_{-r}k\mathbf{i} \in \mathcal{Z}(\infty,\mathcal{B}_\epsilon(0)))e^{-\delta_\Gamma r}d\mu_{N_+}^{Haar}drd\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-), \tag{6.7.19}$$

recall $\nu^{\mathbf{w}}$ is the conformal density associated to the subgroup $\Gamma^{\mathbf{w}}$ (see the proof of Proposition 6.3.3). Note that $g_{\mathbf{w}} \in G/K \cong AN_+$ which we write as $a_{r_{\mathbf{w}}}n_+(\mathbf{x}_{\mathbf{w}})$. Hence

$$g_{\mathbf{w}}n_+(\mathbf{x})a_{-r}k\mathbf{i} = g_{\mathbf{w}}a_{-r}\mathbf{i} + e^{-r_{\mathbf{w}}}\mathbf{x}. \tag{6.7.20}$$

Hence

$$F_{\gamma,\epsilon}(\vartheta^{-1}\xi) = \int_{KA\mathbb{R}^{n-1}} \mathbb{1}(g_{\mathbf{w}}a_{-r}g_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty,\mathcal{B}_\xi(0)) - \mathbf{x}e^{-r_{\mathbf{w}}})\cdot \\ \mathbb{1}(g_{\mathbf{w}}a_{-r}\mathbf{i} \in \mathcal{Z}(\infty,\mathcal{B}_\epsilon(0)) - \mathbf{x}e^{-r_{\mathbf{w}}})e^{-\delta_\Gamma r}d\mathbf{x}drd\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-), \tag{6.7.21}$$

Hence in the limit as $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\frac{c}{\epsilon^{n-1}}F_{\gamma,\epsilon}(\vartheta^{-1}\xi) = c\int_{KA} \mathbb{1}(a_{r_{\mathbf{w}}-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty,\mathcal{B}_\xi(0)))\cdot \\ \mathbb{1}(r_{\mathbf{w}} - r > 0)e^{(n-1)r_{\mathbf{w}}-\delta_\Gamma r}drd\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-), \tag{6.7.22}$$

Simplifying then gives

$$\lim_{\epsilon\to 0}\frac{c}{\epsilon^{n-1}}F_{\gamma,\epsilon}(\vartheta^{-1}\xi) = c\int_{K\mathbb{R}_{>0}} \mathbb{1}(a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty,\mathcal{B}_\xi(0)))e^{(n-1-\delta_\Gamma)r_{\mathbf{w}}-\delta_\Gamma r}drd\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-). \tag{6.7.23}$$

Hence

$$R_2(\xi) = \frac{2c}{\#\Gamma_{\mathbf{w}}} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \int_{K\mathbb{R}_{>0}} \mathbb{1}(a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty, \mathcal{B}_\xi(0)))e^{(n-1-\delta_\Gamma)r_{\mathbf{w}}-\delta_\Gamma r}drd\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-). \tag{6.7.24}$$

Now, to evaluate whether R_2 is continuous in ξ, take $\xi > \xi'$ and consider the difference

$$|R_2(\xi) - R_2(\xi')| = \frac{2c}{\#\Gamma_{\mathbf{w}}} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \int_{K\mathbb{R}_{>0}} \mathbb{1}(a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty, \mathcal{B}_\xi(0)\setminus\mathcal{B}_{\xi'}(0)))\cdot$$
$$e^{(n-1-\delta_\Gamma)r_{\mathbf{w}}-\delta_\Gamma r}drd\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-). \tag{6.7.25}$$

Suppose we are working in dimension $n = 2$. In that case $\mathcal{Z}(\infty, \mathcal{B}_\xi(0)\setminus\mathcal{B}_{\xi'}(0))$ converges to 2 vertical line segments. Hence in the limit as $\xi' \to \xi$ for fixed r there are at most 4 rotations such that the point hits these four line segments. However since the measure $\nu_{\mathbf{i}}$ is non-atomic (see [Sul84]), the measure of these four rotations must be 0 mass. Hence the difference in the left hand side of (6.7.25) converges to 0 and the two-point correlation function is continuous.

A similar argument implies, in general dimension $n > 2$, if $\delta_\Gamma > n-2$ then the difference in (6.7.25) also goes to 0 and the two-point correlation function is continuous. The argument is essentially the same: the projection of the set $\mathcal{Z}(\infty, \mathcal{B}_\xi(0)\setminus\mathcal{B}_{\xi'}(0))$ to the boundary will be an $(n-2)$-sphere. Hence since the dimension of the limit set is larger than $n-2$ and the conformal density $\nu_{\mathbf{i}}$ is supported on the limit set (and finite), the above difference must go to 0.

However, if $\delta_\Gamma \leq n-2$ the continuity of R_2 will depend on the geometry of the limit set.

6.7.3 Nearest Neighbour Statistics

We will now use a similar method as for the two-point correlation function to write down an explicit formula for the nearest neighbour statistics of the point set $\mathcal{P}_t^\infty(\overline{\mathbf{w}})$. In Subsection 6.7.4 we will use a trick which works only in 2 dimensions to say something more about the gap statistics (i.e about the nearest neighbour *to the right* statistics) however here we continue to work in general dimension n.

Define the limiting cumulative nearest neighbour distribution to be

$$\mathcal{J}(L) := \lim_{t\to\infty} \mathcal{J}_t(L) := \lim_{t\to\infty} \frac{1}{N_t}\sum_{i=1}^{N_t} \mathbb{1}(\#(\mathcal{B}_{Le^{-t}}(\mathbf{x}_i)\cap\mathcal{P}_t^\infty(\overline{\mathbf{w}})) = 1), \tag{6.7.26}$$

that is, we want to calculate the proportion of points $\mathbf{x}_i$ such that a ball of radius Le^{-t} contains *no other points* of $\mathcal{P}_t^\infty(\overline{\mathbf{w}})$.

To determine the limiting behaviour we will perform a similar trick as was used for the two-point correlation function. Again, writing $N_t \sim c_0^{-1}e^{\delta_\Gamma t}$

$$\lim_{t\to\infty}\mathcal{J}_t(L) = \lim_{t\to\infty}\lim_{\epsilon\to 0}\frac{c_0}{e^{t\delta_\Gamma}\epsilon^{n-1}}\int_{\mathbf{x}\in\mathbb{T}^{n-1}} \mathbb{1}(\#(\mathcal{B}_\epsilon(\mathbf{x})\cap\mathcal{P}_t^\infty(\overline{\mathbf{w}})) = 1)\mathbb{1}(\#(\mathcal{B}_{Le^{-t}}(\mathbf{x})\cap\mathcal{P}_t^\infty(\overline{\mathbf{w}})) = 1)d\mathbf{x}$$
$$= \lim_{t\to\infty}\lim_{\epsilon\to 0}\frac{c_0e^{(n-1-\delta_\Gamma)t}}{\epsilon^{n-1}}\int_{\mathbf{x}\in\mathbb{T}^{n-1}} \mathbb{1}(\#(\mathcal{B}_{\epsilon e^{-t}}(\mathbf{x})\cap\mathcal{P}_t^\infty(\overline{\mathbf{w}})) = 1)\mathbb{1}(\#(\mathcal{B}_{Le^{-t}}(\mathbf{x})\cap\mathcal{P}_t^\infty(\overline{\mathbf{w}}) = 1)d\mathbf{x}. \tag{6.7.27}$$

Using the fact that our test set $\mathcal{B}_{e^{-t}L}(\mathbf{x})$ and $\mathcal{B}_{e^{-t}\epsilon}(\mathbf{x})$ have the same scaling as $\mathcal{B}_{t,s}$ (6.3.4) together with the asymptotic $\#\mathcal{P}_t^\infty(\overline{\mathbf{w}}) \sim c_0^{-1} e^{\delta_\Gamma t}$ we can apply Theorem 6.3.2 to take the limit $t \to \infty$ (and as above, using the linearity in ϵ to exchange the limits), giving

$$\mathcal{J}(L) = \lim_{\epsilon \to 0} \frac{c_0}{\epsilon^{n-1}} E_\infty((1,1), \mathcal{B}_\epsilon(0) \times \mathcal{B}_L(0); \overline{\mathbf{w}}), \tag{6.7.28}$$

which is then equal

$$\begin{aligned}
\mathcal{J}(L) &= \lim_{\epsilon \to 0} \frac{\vartheta}{|\mathrm{m}^{BMS}|\epsilon^{n-1}} \mathrm{m}^{BR}\left(\{\alpha \in \Gamma\backslash G : \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(\infty, \mathcal{B}_\epsilon(0))) = 1,\ \#(\alpha^{-1}\overline{\mathbf{w}} \cap \mathcal{Z}(\infty, \mathcal{B}_L(0))) = 1\}\right) \\
&= \lim_{\epsilon \to 0} \frac{\vartheta}{|\mathrm{m}^{BMS}|\epsilon^{n-1}} \int_G \mathbb{1}(\alpha^{-1}\mathbf{w} \in \mathcal{Z}(\infty, \mathcal{B}_\epsilon(0))) \prod_{\substack{\gamma \in \Gamma/\Gamma_{\mathbf{w}} \\ \gamma \neq \Gamma_{\mathbf{w}}}} \left(1 - \mathbb{1}(\alpha^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty, \mathcal{B}_L(0)))\right) d\mathrm{m}^{BR}(\alpha)
\end{aligned}$$

Hence, using the same trick as we used to prove (6.7.19) we can write this

$$\mathcal{J}(L) = \frac{\vartheta}{|\mathrm{m}^{BMS}|} \int_{K\mathbb{R}_{>0}} \prod_{\substack{\gamma \in \Gamma/\Gamma_{\mathbf{w}} \\ \gamma \neq \Gamma_{\mathbf{w}}}} \left(1 - \mathbb{1}(a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty, \mathcal{B}_L(0)))\right) e^{(n-1-\delta_\Gamma)r_{\mathbf{w}}} e^{\delta_\Gamma r} dr d\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-). \tag{6.7.29}$$

6.7.4 Gap Statistics

In this last section we prove, for the discrete subgroups considered here, the same result as is found in [Zha17] for Schottky groups. That is, we prove Theorem 6.1.1 from the introduction. In the notation of the introduction define the gap distribution to be

$$P_t(s) := \frac{1}{N_t} \sum_{j=1}^{N_t} \delta(s - s_j) \tag{6.7.30}$$

where δ denotes a Dirac mass at the origin.

Using the same argument we used above for the nearest neighbour distribution we can write

$$\begin{aligned}
F(L) &:= \int_L^\infty P(s) ds \\
&= \lim_{t \to \infty} \frac{1}{N_t} \sum_{i=1}^{N_t} \mathbb{1}(\#([\mathbf{x}_i, \mathbf{x}_i + Le^t) \cap \mathcal{P}_t^\infty(\overline{\mathbf{w}})) = 1) \\
&= \lim_{\epsilon \to 0} \frac{c_0}{\epsilon} E_\infty\left((1,1), [0,\epsilon) \times [0, L); \overline{\mathbf{w}}\right) \\
&= \frac{\vartheta}{|\mathrm{m}^{BMS}|} \int_{K\mathbb{R}_{>0}} \prod_{\substack{\gamma \in \Gamma/\Gamma_{\mathbf{w}} \\ \gamma \neq \Gamma_{\mathbf{w}}}} \left(\mathbb{1}(a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w} \notin \mathcal{Z}(\infty, [0, L)))\right) e^{(1-\delta_\Gamma)r_{\mathbf{w}}} e^{\delta_\Gamma r} dr d\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-)
\end{aligned} \tag{6.7.31}$$

A classical argument (explained in some detail in [Mar07]) shows that the gap distribution is the derivative of the $E_s(r, \sigma, \overline{\mathbf{w}})$ for $r = 0$. As we have not treated the case $r = 0$ let

$$E(L, \overline{\mathbf{w}}) := \sum_{r=1}^{\infty} E_s(r, L; \overline{\mathbf{w}}). \tag{6.7.32}$$

Thus the following lemma is a direct consequence of the argument in [Mar07], where we write $E_s(0, L; \overline{\mathbf{w}}) = 1 - E(L, \overline{\mathbf{w}})$

Lemma 6.7.2. *In the present setting, for any $L > 0$*

$$F(L) := \int_L^\infty P(s)ds = -\frac{d}{dL}E(L, \overline{\mathbf{w}}). \tag{6.7.33}$$

With that we prove Theorem 6.1.1 restated here for convenience.

Theorem 6.1.1 . *The limiting function $F(L)$ exists, is monotone decreasing and continuous (including at 0). Moreover if the fundamental domain of Γ is bounded by non-intersecting half-circles and the boundary $\partial\mathbb{H}^2$ then there exists an $L_0 > 0$ such that*

$$F(L) = 1 \tag{6.7.34}$$

for all $L < L_0$ (i.e $1 - F$ is supported away from the origin).

Proof. The calculation above Lemma 6.7.2 establishes the existence of F and the fact that it is monotone decreasing follows from Lemma 6.7.2.

Moreover the argument for continuity follows from the comment at the end of Section 6.7.2 for the two point correlation function. I.e for $L > L'$ we consider the difference

$$F(L') - F(L) = \frac{\vartheta}{|\mathrm{m}^{BMS}|}\int_{K\mathbb{R}_{>0}} \prod_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \left(\mathbb{1}(a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w} \in \mathcal{Z}(\infty, [L', L)))\right) e^{(1-\delta_\Gamma)r_{\mathbf{w}}} e^{\delta_\Gamma r} dr d\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-). \tag{6.7.35}$$

Again, as $L \to L'$ the indicator function inside the integral becomes the indicator function that the point $a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{w}$ lies on a line segment. Since the line segment is transversal to the rotation, for a_r fixed this can only happen for (at most) 2 rotations. Since $\nu_{\mathbf{i}}$ is non-atomic this event has measure 0.

Suppose the fundamental domain for Γ is composed of non-intersecting half-circles. To prove that the cumulative gap distribution is supported as described we use the argument in [Zha19]. Namely: suppose $x_1(t)$ and $x_2(t)$ are neighbours at t and that each x_i is associated to a point in $\mathbb{H}^2$, $\gamma_i\mathbf{w}$. For large t we can assume the associated $\gamma_1\mathbf{w}$ and $\gamma_2\mathbf{w}$ belong to adjacent half-circles. Because these half-circles have finite radius, the distance between $x_1(t)$ and $x_2(t)$ is of the order e^{-t}. Which gives a constant order with our scaling.

□

6.7.5 Explicit Calculations for the Gap Distribution

In (6.7.31) we used the Iwasawa decomposition and $\mathbf{w} = g_{\mathbf{w}}\mathbf{i}$. In fact, since $\mathbf{i}$ is K invariant, we had a choice of $g_{\mathbf{w}} \in G$. Thus in the equation

$$F(L) = \frac{\vartheta}{|\mathrm{m}^{BMS}|}\int_{K\mathbb{R}_{>0}} \prod_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \mathbb{1}(a_{-r}kg_{\mathbf{w}}^{-1}\gamma\mathbf{i} \notin \mathcal{Z}(\infty, [0, L)))e^{\delta_\Gamma r}e^{(1-\delta_\Gamma)r_{\mathbf{w}}} dr d\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-). \tag{6.7.36}$$

choose $g_{\mathbf{w}}^{-1}$ such that, in polar coordinates $g_{\mathbf{w}}^{-1}\gamma\mathbf{i} = \kappa(\gamma)(e^{l(\gamma)}\mathbf{i})$ where $l(\gamma) = d(\mathbf{w}, \gamma\mathbf{w})$ and $\kappa(\gamma)$ is a rotation. In which case (6.7.36) becomes

$$F(L) = \frac{\vartheta}{|\mathrm{m}^{BMS}|}\int_{K\mathbb{R}_{>0}} \prod_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \mathbb{1}(a_{-r}k\kappa(\gamma)(e^{l(\gamma)}\mathbf{i}) \notin \mathcal{Z}(\infty, [0, L)))e^{\delta_\Gamma r}e^{(1-\delta_\Gamma)r_{\mathbf{w}}} dr d\nu_{\mathbf{i}}^{\mathbf{w}}(k\mathbf{X}_{\mathbf{i}}^-). \tag{6.7.37}$$

Unfortunately we cannot remove the factor $\kappa(\gamma)$, while the conformal density is invariant under the action of Γ the terms in the product inside the integral are not independent. However, given the group element, $\kappa(\gamma)$ and $l(\gamma)$ are explicit. We can now use a change of variables as in the appendix of [MV18] with

$$k = k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{6.7.38}$$

With that, and writing $\kappa(\gamma) = k(\theta(\gamma))$, the constraint

$$\mathcal{D}(\gamma) := \{(r,\theta) : a_{-r}k(\theta+\theta(\gamma))(e^{l(\gamma)}\mathbf{i}) \in \mathcal{Z}(\infty,[0,L))\} \tag{6.7.39}$$

is equal

$$\mathcal{E}(\gamma) = \left\{(r,\theta) : \frac{e^{-r}}{\sinh l(\gamma)\cos 2(\theta+\theta(\gamma)) - \cosh l(\gamma)} > 1, \right.$$
$$\left. 0 \leq \frac{e^{-r}\sinh l(\gamma)\sin 2(\theta+\theta(\gamma))}{\cosh l(\gamma) - \sinh l(\gamma)\cos 2(\theta+\theta(\gamma))} < \vartheta^{-1/\delta_\Gamma} L\right\}. \tag{6.7.40}$$

In which case we have the following theorem

Theorem 6.7.3. *For $L > 0$, the cumulative gap distribution can be written*

$$F(L) = \frac{\vartheta e^{(1-\delta_\Gamma)r_{\mathbf{w}}}}{|\mathrm{m}^{BMS}|}\int_0^\infty e^{\delta_\Gamma r}\int_0^\pi \prod_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \left(1-\chi_{\mathcal{E}(\gamma)}(r,\theta)\right) d\nu_{\mathbf{i}}^{\mathbf{w}}(\theta)dr. \tag{6.7.41}$$

Given γ one can compute $\mathcal{E}(\gamma)$ explicitly, however the conformal density $\nu_{\mathbf{i}}$ is defined as the weak limit of a sequence of measures. When Γ is a lattice, (6.7.41) can be written

$$F(L) = \frac{\vartheta}{\mathrm{vol}_{\mathbb{H}^2}(\mathbb{H}^2/\Gamma)}\int_0^\infty e^{r}\int_0^\pi \prod_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma\neq\Gamma_{\mathbf{w}}}} \left(1-\chi_{\mathcal{E}(\gamma)}(r,\theta)\right) d\theta dr. \tag{6.7.42}$$

To our knowledge, even in the lattice case, this is the first general explicit formula for the gap distribution. The gap distribution has been calculated explicitly for specific examples (notably [RZ17] who study the problem in certain circle packings). (6.7.42) can be derived from [MV18], where the authors perform a similar calculation for the pair correlation.

Finally one can ask about the derivative of the cumulative gap distribution. Given γ, L, and θ let

$$e^{-r(\gamma,L,\theta)} := \frac{\vartheta^{-1/\delta_\Gamma} L(\cosh(l(\gamma)) - \sinh(l(\gamma)))\cos(2(\theta+\theta(\gamma)))}{\sinh(l(\gamma))\sin(2(\theta+\theta(\gamma)))}, \tag{6.7.43}$$

let $r_{\min}(L,\theta) = \min_{\gamma\in\Gamma/\Gamma_{\mathbf{w}},\gamma\neq\Gamma_{\mathbf{w}}} r(\gamma,L,\theta)$ and let $\gamma_{\max}(L,\theta)$ be the γ maximising that equation. In this case, recall that $P(L) = -F'(L)$, then

$$P(L) = \frac{\vartheta e^{(1-\delta_\Gamma)r_{\mathbf{w}}}}{|\mathrm{m}^{BMS}|}\int_0^\pi \mathbb{1}(r_{\min}(L,\theta)<0)e^{-\delta_\Gamma r_{\min}(L,\theta)} \prod_{\substack{\gamma\in\Gamma/\Gamma_{\mathbf{w}} \\ \gamma_{\max}(L,\theta)\neq\gamma\neq\Gamma_{\mathbf{w}}}} \left(1-\chi_{\mathcal{E}(\gamma)}(r_{\min}(L,\theta),\theta)\right) d\nu_{\mathbf{i}}^{\mathbf{w}}(\theta). \tag{6.7.44}$$

The conditions on θ are now equivalent to

$$I(\gamma) = \left\{ \theta \in [0, \pi) : \frac{e^{-r_{\min}(L,\theta)}}{\sinh(l(\gamma))\cos(\theta + \theta(\gamma)) - \cosh(l(\gamma))} < 1, \quad r_{\min}(L, \theta) < 0 \right\} \tag{6.7.45}$$

In which case

$$P(L) = \frac{\vartheta e^{(1-\delta_\Gamma) r_{\mathbf{w}}}}{|\mathrm{m}^{BMS}|} \int_0^\pi e^{-\delta_\Gamma r_{\min}(L,\theta)} \prod_{\substack{\gamma \in \Gamma/\Gamma_{\mathbf{w}} \\ \gamma_{\max}(L,\theta) \neq \gamma \neq \Gamma_{\mathbf{w}}}} \chi_{I(\gamma)}(\theta) d\nu_{\mathbf{i}}^{\mathbf{w}}(\theta). \tag{6.7.46}$$

Chapter 7

Generalised Farey Sequences

7.1 Introduction

Consider the classical *Farey sequence* of height Q:

$$\tilde{\mathcal{F}}_Q := \left\{ \frac{p}{q} \in [0,1) \ : \ (p,q) \in \hat{\mathbb{Z}}^2, 0 < q < Q \right\}, \tag{7.1.1}$$

where $\hat{\mathbb{Z}}^2$ denotes the set of primitive vectors in $\mathbb{Z}^2$. Naturally this sequence is a fundamental object in number theory dating back to 1802 with its introduction by Haros and subsequent work by Farey and Cauchy. For example, this sequence has connections to the Riemann hypothesis (see for example [LM17, Yos98]) and plays a fundamental role in Diophantine approximation.

In this chapter we generalise the Farey sequence. For concreteness, one example of such a generalised Farey sequence is given by the following: throughout this chapter we use the standard continued fraction notation

$$[a_0; a_1, \dots a_n] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{\dots}{a_n}}} \tag{7.1.2}$$

(see for example [Khi03]) then denote

$$\mathcal{Q}_4 := \{[0; a_1, \dots a_k] : k \in \mathbb{N}\ ,\ a_i \in 4\mathbb{Z}_{\neq 0}\ \forall i\}\,, \tag{7.1.3}$$

that is, rationals whose continued fraction expansions involve only multiples (*possibly negative*) of 4. The generalised Farey sequence in this context is

$$\widehat{\mathcal{F}}_Q = \{\frac{p}{q} \in \mathcal{Q}_4 : 0 < q < Q,\ \gcd(p,q) = 1\}. \tag{7.1.4}$$

Thus, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We return to this example in Section 7.1.1 where we give a geometric interpretation of these sets. To see some of the points of $\mathcal{Q}_4$ see Figure 7.1 on page 146.

There is a geometric interpretation of the classical Farey sequence which will play an integral role in this paper. Let $G := \mathrm{PSL}(2,\mathbb{R})$ and $\Lambda := \mathrm{PSL}(2,\mathbb{Z}) < G$. Consider the action of G on $\mathbb{H}$ via Möbius transformations (see Chapter 5, Section 5.1.1). As Λ is a lattice, the Λ-action on $\mathbb{H}$ tessellates $\mathbb{H}$ into disjoint fundamental domains. These fundamental domains are not compact as each one contains a point on the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, at the end of a cusp. The set of such cuspidal points is exactly

$$(\Lambda/\Lambda_\infty)\infty = \mathbb{Q} \tag{7.1.5}$$

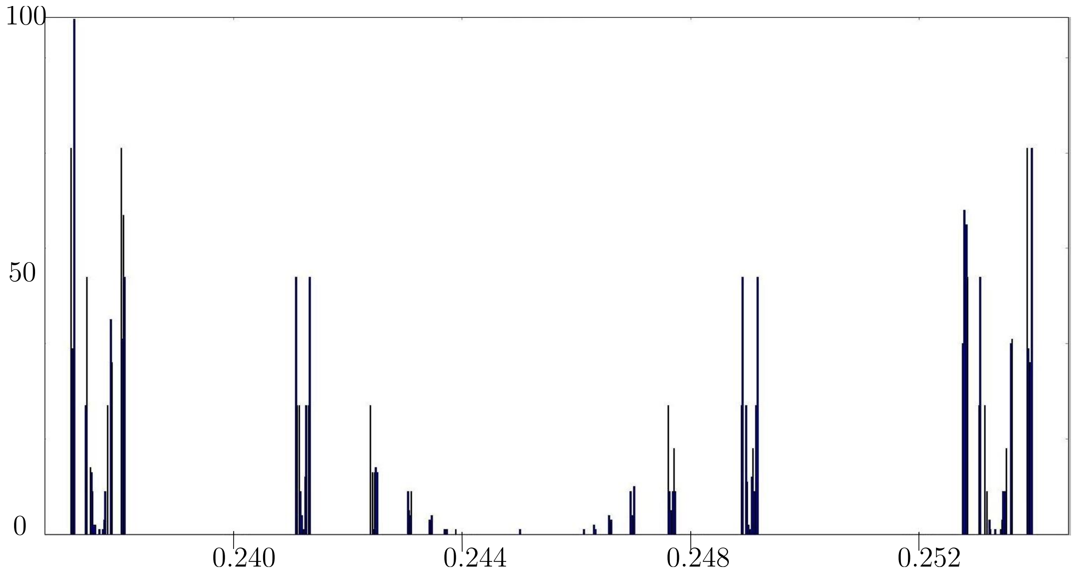

Figure 7.1: Above we show some of the points in $\mathcal{Q}_4$. The graph was generated as follows: we generated all words of length 10 (with respect to the two generators given in (7.1.7) applied to ∞). Then separated the interval $[0,1)$ into bins of size 10^{-5}. The above is a bar chart showing the number of points in each bin. Note that the sequence is supported on a fractal subset of the interval. This does not show $\widehat{\mathcal{F}}_Q$ (as the cut-off is with respect to word length), however will suffice for a qualitative picture.

(we use G_x to denote the stabiliser of x in a group G). That is, the set of cuspidal points can be written as the Λ-orbit of the point at $\infty \in \partial\mathbb{H}$ - this orbit corresponds to the rationals. Thus the Farey sequence of height Q can be written

$$\tilde{\mathcal{F}}_Q = \left\{ \frac{p}{q} \in (\Lambda/\Lambda_\infty)\infty : (p,q) \in \hat{\mathbb{Z}}^2, 0 < q < Q \right\} \tag{7.1.6}$$

i.e the points in the Λ-orbit of the point at $\infty \in \partial\mathbb{H}$ with denominator less than Q. The goal of this chapter is to consider a generalisation of this setup, where we replace Λ by a general (possibly infinite covolume) discrete subgroup. For our example (7.1.4) the corresponding subgroup is the Hecke group

$$\widehat{\Gamma} = \left\langle \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle. \tag{7.1.7}$$

Most of our theorems hold for general subgroups. Hence, let $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ be a *general* non-elementary, finitely generated subgroup in G with critical exponent δ_Γ (see Chapter 5, Section 5.5). In our context $1/2 < \delta_\Gamma \leq 1$. Furthermore assume Γ has a cusp at ∞ and let $\Gamma^\infty = (\Gamma/\Gamma_\infty)\infty \subset \partial\mathbb{H}$ denote the orbit of ∞. Hence, Γ^∞ is the set of the cusps located at points on the boundary, isomorphic to ∞. Finally we assume that $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. I.e that the fundamental domain is periodic with period 1 along the real line. Note that $\widehat{\Gamma}$ has period 4. A scaling could be applied to give it period 1 (in order to preserve the continued fraction description - and since it serves only as an example - we refrain from doing so).

Let

$$\mathcal{Z} := \{(p,q) \in (0,1)\Gamma\} \subset \mathbb{R}^2, \tag{7.1.8}$$

denote the analogue of primitive vectors and define

$$\begin{aligned}\mathcal{F}_Q &:= \left\{\frac{p}{q} \in [0,1) : (p,q) \in \mathcal{Z}, 0 < q < Q\right\} \\ &= \left\{\frac{p}{q} \in \Gamma^{\infty} : 0 \le p < q < Q\right\}.\end{aligned} \tag{7.1.9}$$

$\mathcal{F}_Q$ is the primary object of study for this chapter, which we call a *generalised Farey sequence* (or *gFs*). In Subsection 7.2.1 we show that asymptotically there exists a constant $0 < c_\Gamma < \infty$ such that

$$|\mathcal{F}_Q| \sim c_\Gamma Q^{2\delta_\Gamma}. \tag{7.1.10}$$

The goal of the chapter is to establish the Theorems in Sections 7.3 - 7.8 which we describe briefly here. Section 7.2 presents some preliminary theorems which we make use of later. Subsequently the **main results of the chapter** are:

- **Counting primitive points:** In *Section 7.3* we show how the equidistribution result of Oh-Shah [OS13] stated in Chapter 5 Section 5.6 can be used to prove a technical theorem about counting primitive points in a sheared set (Theorem 7.3.3) and another technical theorem about counting primitive points in a rotated set (Theorem 7.3.5). These theorems generalise the analogous result for lattices in [MS10].

- **Diophantine approximation by parabolics:** We prove two theorems in metric Diophantine approximation in Fuchsian groups. These are the analogues of the Erdős-Szűsz-Turán and Kesten problems in the infinite volume setting. In the classical setting, these problems were solved using homogeneous dynamics by Marklof in [Mar00, Theorem 4.4] and Athreya and Ghosh [AG18]. Moreover Xiong and Zaharescu [XZ06] and Boca [Boc08] solved the problem using number theoretic methods (by applying the BCZ map). Extending classical results in metric Diophantine approximation to the setting of Fuchsian groups is not new and was done by Patterson [Pat76] who proved Dirichlet and Khintchine type theorems for such parabolic points. More recently, for example Beresnevich et. al. [BGSV18] studied the equivalent problems for Kleinian groups.

  In the same section we show that Theorem 7.3.5 allows us to prove that there is a limiting distribution for the direction of primitive points, $\mathcal{Z}$, as viewed from the origin. This problem has not been addressed in the Euclidean setting except for lattices ([MS10]).

- **Equidistribution of gFs:** Theorem 7.5.1 states that the gFs equidistributes over a horospherical section. In a series of papers ([Mar10], [Mar13]), Marklof showed that the (classical) Farey sequence, when embedded into a horosphere equidistributes on a particular section. This equidistribution theorem was then used to show that the spatial statistics of the Farey sequence converge. This was followed by work of Athreya and Cheung [AC14] who (in dimension $d = 2$) were able to construct a Poincaré section for the horocycle flow such that the return time map generates Farey points. We restrict our attention to proving the equidistribution result in this more general setting. Heersink [Hee19] generalised [Mar10] to certain congruence subgroups of Λ (still in the finite covolume setting). Furthermore, the method of [AC14] has been generalised to more general subgroups such as Hecke triangle groups (e.g [Tah19]). However we will not discuss this approach here.

- **Convergence of local statistics:** Theorem 7.6.1, as a consequence of Theorem 7.3.3 and Theorem 7.5.1, states that two sorts of local statistics converge. A corollary of one of these is that the limiting gap distribution exists. This distribution in the classical setting was originally calculated by Hall [Hal70] (and is known as the Hall distribution) and has been studied by many people since. The Hall distribution was originally put into the context of ergodic theory in [BCZ01].

- **An explicit formula for the gap distribution:** In *Section 7.7* we restrict to the example $\widehat{\Gamma}$. For this example we show that the limiting gap distribution can be explicitly written as an integral over a compact region. While the integral involves a fractal measure this is the first time such an explicit formula has been calculated in the infinite volume setting. There is much interest in finding explicit formula for limiting gap distributions for projected lattice point sets and the infinite covolume analogue. The only instance (to our knowledge) of such explicit examples are those covered in [RZ17]. In that paper Rudnick and Zhang used the relation between Farey points and Ford circles to produce examples for which they could express the limiting gap distribution explicitly (recovering, in one instance, the Hall distribution). In Section 7.1.1 we show that the Farey sequence for $\widehat{\Gamma}$ can also be used to generate a (sparse) Ford Configuration which leads to our result.

- **Ergodicity of a new Gauss-like measure:** Continuing to work with the example $\widehat{\Gamma}$, we show that a new fractal measure takes on the role of the Gauss measure (Theorem 7.8.2). That is, this measure is ergodic for the Gauss map. As an application, using this ergodicity we show that the Gauss-Kuzmin statistics converge to an explicit function. This section takes inspiration from [Ser85] where Series showed how the Gauss measure can be viewed as a projection of the Haar measure on a particular cross-section.

7.1.1 Ford Configurations for $\widehat{\Gamma}$

To give some further intuition for generalised Farey sequences, in this section we show that the gFs for $\widehat{\Gamma}$ admits a simple geometric interpretation which we shall return to in *Section 7.7*. Returning to our example $\widehat{\mathcal{F}}_Q$ – (7.1.4), note that

$$\widehat{\Gamma}^{\infty} = \mathcal{Q}_4. \tag{7.1.11}$$

To see this, simply note that the two generators in (7.1.7) correspond to the maps $f(x) = x + 4$ and $g(x) = \frac{-1}{x}$ which generate these continued fractions.

Consider the action of $\widehat{\Gamma}$ on an initial configuration of circles in the closure $\overline{\mathbb{H}}$:

$$\begin{gathered}\mathcal{K}_0 := (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)\\ \mathcal{C}_0 = \mathbb{R} \quad , \quad \mathcal{C}_1 = \mathbb{R} + i \quad , \quad \mathcal{C}_2 = C(i/2, 1/2) \quad , \quad \mathcal{C}_3 = C(i/2 + 4, 1/2)\end{gathered} \tag{7.1.12}$$

where $C(z, r)$ is a circle located at $z \in \overline{\mathbb{H}}$ of radius r. We are interested in the resulting sparse Ford configuration, $\mathcal{K} := \widehat{\Gamma}\mathcal{K}_0$, shown in Figure 7.2. Any group element in $\widehat{\Gamma}$ can be decomposed into a composition of circle inversions through vertical lines at 0 and 4 and $C(0, 1)$ and $C(4, 1)$ (these are also shown in Figure 7.2).

Let $\mathcal{A}_T$ denote the set of tangencies with $\mathcal{C}_0$ in $[0, 1]$ such that the circle tangent to $\mathcal{C}_0$ has diameter larger than T^{-1}. The way we have constructed the packing $\mathcal{K}$, these tangencies are exactly the cuspidal points of the group (i.e the tangencies are located on the orbit $\widehat{\Gamma}\infty$). Moreover one can easily show if a circle in this packing is tangent to $\mathcal{C}_0$ at p/q in reduced form then the diameter is given by $1/q^2$. Hence $\mathcal{A}_{Q^2} = \widehat{\mathcal{F}}_Q$, i.e the set of tangencies of circles with diameter greater than Q^2 is exactly the gFs

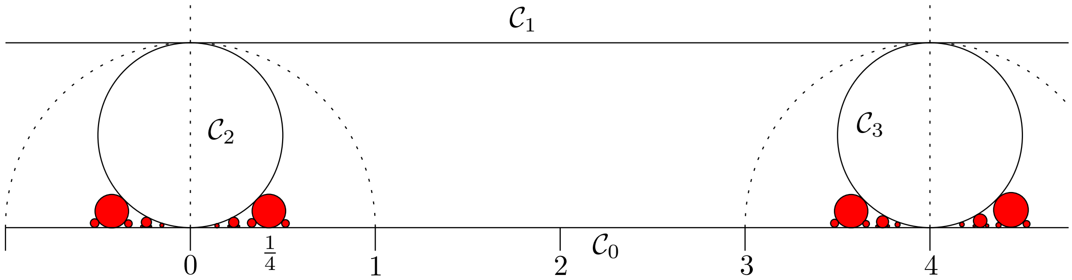

Figure 7.2: Diagram of a portion of $\mathcal{K}$. The dotted lines represent the circle inversions corresponding to the subgroup $\widehat{\Gamma}$. The white circles (including the x-axis and horizontal line above) represent the initial configuration $\mathcal{K}_0 = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. The filled-in circles represent some of the images.

of height Q.

Given an interval $\mathcal{I} \subset [0,1]$, let $\mathcal{A}_{T,\mathcal{I}} = \mathcal{A}_T \cap \mathcal{I}$. We label the elements of $\mathcal{A}_T = \{x_{T,\mathcal{I}}^j\}_{j=1}^{\#\mathcal{A}_{T,\mathcal{I}}}$ such that $x_{T,\mathcal{I}}^j < x_{T,\mathcal{I}}^{j+1}$ for all j. The gap distribution is then

$$\widehat{F}_{T,\mathcal{I}}(s) := \frac{\#\left\{i \in [1, \#\mathcal{A}_{T,\mathcal{I}}) : T(x_{T,\mathcal{I}}^{j+1} - x_{T,\mathcal{I}}^j) \leq s\right\}}{T^{\delta_{\widehat{\Gamma}}}} \tag{7.1.13}$$

for $s > 0$.

In Section 7.7 we show that the limiting gap distribution can be explicitly calculated as a sum of integrals over compact regions involving a fractal measure presented below. This allows us to show that all gaps have size bigger than $s < 2$ (not just in the limiting case), and to say something more about the regularity of F and the growth of the derivative.

Remark. Of course different subgroups generate different sparse Ford configurations and have other interesting relations to continued fractions (and hence Diophantine approximation). We only address this (simplest) example here. That said, our methods generalise without additional effort to any Hecke subgroup of the form $\Gamma_c = \left\langle \left(\begin{smallmatrix} 0 & c \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle$ for $c \in \mathbb{R}_{>2}$ (the corresponding continued fraction description will involve c rather than 4 and this loses some elegance for non-integer c).

7.2 Preliminary Results

7.2.1 Proof of (7.1.10)

Proof of (7.1.10). A rational $\frac{a}{b}$ belongs to $\mathcal{F}_Q$ if and only if there exists a $\gamma = \left(\begin{smallmatrix} a & * \\ b & * \end{smallmatrix}\right) \in \Gamma/\Gamma_\infty$ and $0 < a < b < Q$. Using the standard Iwasawa decomposition one can write

$$\gamma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \tag{7.2.1}$$

where $a = \cos\theta y^{1/2}$ and $b = \sin\theta y^{1/2}$. Therefore the problem is equivalent to counting

$$\#\left\{\gamma \in \Gamma/\Gamma_\infty : (\theta, y) \in \Omega\right\}, \tag{7.2.2}$$

where $\Omega := \{(\theta, y) : 0 < y^{1/2}\cos\theta < y^{1/2}\sin\theta < Q\}$. Counting the asymptotic number of points in such a sector is the content of [BKS10] (see Theorem 7.7.5 below).

Below, to prove Proposition 7.7.6 we perform this calculation more carefully (and will calculate the constant in that context, thus we leave the details till then). □

7.2.2 Gauss-Type Decomposition

Let $M_{\mathbf{y}} := \begin{pmatrix} y_2^{-1} & 0 \\ y_1 & y_2 \end{pmatrix}$, for $\mathbf{y} \in \mathbb{R}^2$. In what follows we will need the following decomposition of $T^1(\mathbb{H})$. For the remainder of the chapter, to simplify notation we let $d\mu^{PS}(x) := d\mu^{PS}_{N_-}(n_-(x))$.

Proposition 7.2.1. *For any $\phi \in C_c(T^1(\mathbb{H}))$ and any set $\mathscr{A} \subset \mathbb{R}^2$*

$$\int_{N_-\{M_{\mathbf{y}}:\mathbf{y}\in\mathscr{A}\}} \phi(hM_{\mathbf{y}})dm^{BR}(hM_{\mathbf{y}}) = 2\int_{\mathbb{R}\times\mathscr{A}} \phi(n_-(x)M_{\mathbf{y}})y_2^{2\delta_\Gamma - 2}dy_2dy_1d\mu^{PS}(x). \tag{7.2.3}$$

Proof. The goal is to understand the forwards and backwards orbits of $u = hM_{\mathbf{y}}X_i$. First we note that

$$u^- = (hM_{\mathbf{y}}X_i)^- = hX_i^- \tag{7.2.4}$$

(this follows from the definition of the stable and unstable directions of the geodesic flow). Hence we can write:

$$\begin{aligned} s &:= \beta_{u^-}(i, \pi(u)) \\ &= \beta_{X_i^-}(h^{-1}i, M_{\mathbf{y}}i). \end{aligned} \tag{7.2.5}$$

Inserting the definition of the Busemann function and using its invariance properties then gives

$$\begin{aligned} s &= \lim_{t\to\infty} d(h^{-1}i, a_{-t}i) - d(M_{\mathbf{y}}i, a_{-t}i) \\ &= \lim_{t\to\infty} d(i, a_{-t}i) - d(M_{\mathbf{y}}i, a_{-t}i) + d(h^{-1}i, a_{-t}i) - d(i, a_{-t}i). \end{aligned} \tag{7.2.6}$$

Now setting $r_0(h) = \beta_{hX_i^-}(i, hi)$ gives

$$\begin{aligned} s &= \lim_{t\to\infty} t - d(\begin{pmatrix} y_2^{-1} & 0 \\ 0 & y_2 \end{pmatrix} i, a_{-t}i) + r_0(h) \\ &= \lim_{t\to\infty} t - t + 2\ln y_2 + r_0(h) \\ &= 2\ln y_2 + r_0(h). \end{aligned} \tag{7.2.7}$$

Thus

$$ds = \frac{2dy_2}{y_2}. \tag{7.2.8}$$

Note also, by definition

$$e^{\delta_\Gamma r_0(n_-(x))}d\nu_i(n_-(x)X_i) = d\mu^{PS}(x). \tag{7.2.9}$$

Now consider the measure

$$d\lambda_g(z) = e^{\beta_{(hM_{\mathbf{y}}X_i)^+}(i, hM_{\mathbf{y}}i)}dm_i((hM_{\mathbf{y}}X_i)^+), \tag{7.2.10}$$

with $g = h\begin{pmatrix} y_2^{-1} & 0 \\ 0 & y_2 \end{pmatrix}$ and $z = n_+(y_2^{-1}y_1)$. Using the G-invariance of m we can write $d\lambda_g(z)$ as

$$
\begin{aligned}
&= e^{\beta_{(gzX_i)^+}(i,gzi)} dm_i((gzX_i)^+) \\
&= e^{\beta_{(gzX_i)^+}(i,gzi)} dm_{g^{-1}i}((zX_i)^+)
\end{aligned} \tag{7.2.11}
$$

and then using the invariance properties of conformal densities (Chapter 5, (5.5.5)):

$$
\begin{aligned}
&= e^{(\beta_{(gzX_i)^+}(i,gzi)+\beta_{(zX_i)^+}(i,g^{-1}i))} dm_i((zX_i)^+) \\
&= e^{\beta_{(zX_i)^+}(i,zi)} dm_i((zX_i)^+).
\end{aligned} \tag{7.2.12}
$$

Hence $d\lambda_g = d\lambda_e$ and in particular λ_e is N^+-invariant. Hence it is the Haar measure on N_+. Thus we have (for y_2 fixed)

$$
d\lambda_g(z) = dz = y_2^{-1} dy_1. \tag{7.2.13}
$$

Inserting (7.2.4), (7.2.7), (7.2.8), (7.2.9), and (7.2.13) into the definition of the BR-measure we get 7.2.3.

□

7.2.3 Global Measure Formula

The last theorem from the literature we require is the so-called global measure formula, stated in [SV95, Theorem 2], which requires some set up. In actuality we only use the simpler Corollary 7.2.3. As stated in [SV95], there exists a disjoint, Γ-invariant collection of horoballs $\mathscr{H}$ such that $(\mathcal{C}_\Gamma \setminus \mathscr{H})/\Gamma$ is compact, where $\mathcal{C}_\Gamma$ is the convex hull of $\mathcal{L}(\Gamma)$.

We let $\eta \in \mathcal{L}(\Gamma)$ be a *parabolic limit point*. Define η_t to be the unique point along the geodesic connecting i to η whose hyperbolic distance from i is t. And define

$$
b(x) = \begin{cases} 0 & \text{if } x \in \mathbb{H} \setminus \mathscr{H} \\ d(x, \partial H_\eta) & \text{if } x \in H_\eta \in \mathscr{H} \end{cases}, \tag{7.2.14}
$$

where H_η is the horoball at η.

Theorem 7.2.2 ([SV95, Theorem 2])**.** *There exists a constant $0 < C < \infty$ such that for any $\eta \in \mathcal{L}(\Gamma)$ - a parabolic cusp and for any $t > 0$,*

$$
C^{-1} e^{-\delta_\Gamma t} e^{b(\eta_t)(1-\delta_\Gamma)} \le \nu_i(\mathcal{B}(\eta, e^{-t})) \le C e^{-\delta_\Gamma t} e^{b(\eta_t)(1-\delta_\Gamma)} \tag{7.2.15}
$$

where $\mathcal{B}(\eta, e^{-t}) \subset \partial\mathbb{H}$ is the ball centered at η of radius e^{-t}

Corollary 7.2.3. *Assume $\eta \in \mathcal{L}(\Gamma)$ is a parabolic cusp, in a small ball around η we can approximate the measure:*

$$
d\nu_i(\eta + h) \le h^{2\delta_\Gamma - 2} dh. \tag{7.2.16}
$$

This corollary follows by differentiating (7.2.15) with $h = e^{-t}$ and by noting $b(\eta_t) \le t$.

7.3 Horospherical Equidistribution

Consider an unstable horosphere for the geodesic flow a_t, N_+. We parameterise the projection by $n_+ : \mathbb{T} \to \Gamma \cap N_+ \backslash \Gamma N_+$. Recall, Chapter 5, Theorem 5.6.3, we state a simplified restriction (which will suffice for this chapter) here to aid the reader:

Theorem 7.3.1. *Let λ be a Borel probability measure on $\mathbb{T}$ absolutely continuous with respect to Lebesgue and with continuous density. Then for every $f : \mathbb{T} \times \Gamma\backslash G \to \mathbb{R}$ compactly supported and continuous*

$$\lim_{t\to\infty} e^{(1-\delta_\Gamma)t} \int_{\mathbb{T}} f(x, n_+(x)a_t)d\lambda(x) = \frac{1}{|m^{BMS}|} \int_{\mathbb{T}\times\Gamma\backslash G} f(x,\alpha)\lambda'(x)d\mu^{PS}_{N_+}(x)dm^{BR}(\alpha). \tag{7.3.1}$$

Furthermore this theorem can be applied to characteristic functions as with Chapter 5, Corollary 5.6.4 (again, we present a restriction here which will suffice).

Corollary 7.3.2. *Let λ be a Borel probability measure on $\mathbb{T}$ absolutely continuous with respect to Lebesgue and with continuous density. Let $\mathcal{E} \subset \mathbb{T}\times\Gamma\backslash G$ be a compact set with boundary of $(\mu^{PS}_{N_+}\times m^{BR})$-measure 0. Then*

$$\lim_{t\to\infty} e^{(1-\delta_\Gamma)t} \int_{\mathbb{T}} \chi_{\mathcal{E}}(x, n_+(x)a_t)d\lambda(x) = \frac{1}{|m^{BMS}|} \int_{\mathbb{T}\times\Gamma\backslash G} \chi_{\mathcal{E}}(x,\alpha)\lambda'(x)d\mu^{PS}_{N_+}(x)dm^{BR}(\alpha). \tag{7.3.2}$$

7.3.1 Counting Primitive Points in Sheared Sets

As a straightforward consequence of Corollary 7.3.2 we have the following theorem, which (in Sections 7.4 and 7.6) we show has a number of important consequences.

Theorem 7.3.3. *Let λ be a Borel probability measure on $\mathbb{T}$ absolutely continuous with respect to Lebesgue and with continuous density. Let $\mathcal{A} \subset \mathbb{R}^2$ be a compact set with boundary of Lebesgue measure* 0*. Then for every $k \geq 1$:*

$$\lim_{t\to\infty} e^{(1-\delta_\Gamma)t}\lambda\left(\{x\in\mathbb{T} : |\mathcal{Z}n_+(x)a_t\cap\mathcal{A}| = k\}\right) = \frac{C_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}(\{\alpha\in\Gamma\backslash G : |\mathcal{Z}\alpha\cap\mathcal{A}| = k\}), \tag{7.3.3}$$

where $C_\lambda = \mu^{PS}_{N_+}(\lambda')$.

Theorem 7.3.3 is an infinite covolume version of [MS10, Theorem 6.7]. The proof is a straightforward consequence of Corollary 7.3.2 and the fact that if $\mathcal{A}$ is compact and has boundary of Lebesgue measure 0, then

$$\{g\in\Gamma\backslash G : \mathcal{Z}g\cap\mathcal{A} = k\} \tag{7.3.4}$$

is compact and has boundary of volume 0, and the Burger-Roblin measure of a 0 volume set is 0.

Using [MO15, Theorem 6.10] in the same way we used [OS13, Theorem 3.6] to derive Theorem 7.3.1, we have

Theorem 7.3.4. *Let $\mathcal{A} \subset \mathbb{R}^2$ be a compact set with boundary of Lebesgue measure* 0*. Then for every $k \geq 1$:*

$$\lim_{t\to\infty} \mu^{PS}_{N_+}\left(\{x\in\mathbb{T} : |\mathcal{Z}n_+(x)a_t\cap\mathcal{A}| = k\}\right) = \frac{|\mu^{PS}_{N_+}|}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BMS}(\{\alpha\in\Gamma\backslash G : |\mathcal{Z}\alpha\cap\mathcal{A}| = k\}). \tag{7.3.5}$$

In words each of these two theorems is asking for the limiting probability that a randomly sheared set contains k points. In one instance (Theorem 7.3.3) we randomly shear the set with measure λ and in the other (Theorem 7.3.4) we use the measure $\mu^{PS}_{N_+}$.

7.3.2 Counting Primitive Points in Rotated Sets

Similarly to Section 7.3.1 one can ask about the probability of finding k primitive points in a randomly rotated set (as oppose to a randomly sheared one). In Chapter 6, Section 6.5 we show that similar equidistribution results to Theorem 7.3.1 and Corollary 7.3.2 also hold when the horospherical subgroup N_+ is replaced with the rotational subgroup, K. In keeping with the notation of Chapter 6, Section 6.5 let $x \mapsto R(x)$ be the standard parameterisation of the rotation group:

$$R(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}. \tag{7.3.6}$$

Then the rotational Patterson-Sullivan measure (see (5.5.20) is

$$d\mu_K^{PS}(x) = e^{\beta_x(i, R(x)(ei))} d\nu_i(x). \tag{7.3.7}$$

Note μ_K^{PS} is supported on $\mathcal{L}(\Gamma)$. Hence, the analogous theorem to Theorem 7.3.3 follows from Chapter 6, Corollary 6.5.2 (in the same way that Theorem 7.3.3 follows from Corollary 7.3.2):

Theorem 7.3.5. *Let λ be a Borel probability measure on $\mathbb{T}$ absolutely continuous with respect to Lebesgue and with continuous density. Let $\mathcal{A} \subset \mathbb{R}^2$ be a compact subset with boundary of Lebesgue measure 0. Then for every $k \geq 1$*

$$\lim_{t\to\infty} e^{(1-\delta_\Gamma)t} \lambda\left(\{x \in \mathbb{T} : |\mathcal{Z}R(x)a_t \cap \mathcal{A}| = k\}\right) = \frac{D_\lambda}{|\mathrm{m}^{BMS}|} \mathrm{m}^{BR}(\{\alpha \in \Gamma\backslash G : |\mathcal{Z}\alpha \cap \mathcal{A}| = k\}) \tag{7.3.8}$$

where $D_\lambda = \mu_K^{PS}(\lambda')$.

7.4 Consequences of Theorems 7.3.3 and 7.3.5

7.4.1 Diophantine Approximation in Fuchsian Groups

Theorem 7.3.3 can be used to prove several statements about the set of numbers which can be approximated by parabolic points in the limit set of the Fuchsian groups studied here. For example, as discussed in [AG18], Erdős-Szűsz-Turán (henceforth abbreviated EST) introduced the following problem in Diophantine approximation: what is the probability that a uniformly chosen point, $x \in [0,1]$, satisfies

$$\left| x - \frac{p}{q} \right| \leq \frac{A}{q^2} \tag{7.4.1}$$

for $\frac{p}{q} \in \mathbb{Q}$ with $q \in [\theta Q, Q]$ for a fixed triple $(A, \theta, Q) \in \mathbb{R}_{>0} \times (0,1) \times \mathbb{R}_{>0}$? Hence if we let $EST(A, \theta, Q)$ be the random variable: the number of solutions to (7.4.1), the EST problem is to prove the existence of

$$\lim_{Q\to\infty} \mathbb{P}(EST(A, \theta, Q) > 0). \tag{7.4.2}$$

The limiting distribution for this random variable is given in [AG18] in great generality. Our goal in this section is to understand the same problem with the rationals replaced by Γ^∞.

Given a triple (A, θ, Q) as above and a number x, define (the analogue of the random variable EST), $E(A, \theta, Q)$ to be the number of solutions, $(p, q) \in \mathcal{Z}$, to

$$|p - qx| \leq \frac{A}{q}. \tag{7.4.3}$$

Theorem 7.4.1. *Given $(A, \theta) \in \mathbb{R}_{>0} \times (0, 1)$. Let λ be a Borel probability measure on $[0, 1)$, absolutely continuous with respect to Lebesgue with continuous density. Then*

$$\lim_{Q \to \infty} Q^{2(1-\delta_\Gamma)} \lambda(\{x \in [0,1) : E(A, \theta, Q) = k\}) = \frac{C_\lambda}{|\mathrm{m}^{BMS}|} \mathrm{m}^{BR}(\{\alpha \in \Gamma \backslash G : |\mathcal{Z}\alpha \cap \mathfrak{C}_{A,\theta}| = k\}), \tag{7.4.4}$$

where

$$\mathfrak{C}_{A,\theta} := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : |x_1| x_2 \leq A : \theta < x_2 < 1\}. \tag{7.4.5}$$

Moreover,

$$\lim_{Q \to \infty} \mu^{PS}_{N_+}(\{x \in \mathcal{L}(\Gamma) \cap [0,1) : E(A, \theta, Q) = k\}) = \frac{1}{|\mathrm{m}^{BMS}|} \mathrm{m}^{BMS}(\{\alpha \in \Gamma \backslash G : |\mathcal{Z}\alpha \cap \mathfrak{C}_{A,\theta}| = k\}). \tag{7.4.6}$$

Proof. Write the left-hand-side of (7.4.4) as (with $Q = e^{t/2}$)

$$\begin{aligned}\lim_{t \to \infty} e^{(1-\delta_\Gamma)t} \lambda\left(\left\{x \in [0,1] : \#\left\{(p,q) \in \mathcal{Z} : (p,q)\left(\begin{smallmatrix} 1 & 0 \\ -x & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} Q & 0 \\ 0 & Q^{-1} \end{smallmatrix}\right) \in \mathfrak{C}_{A,\theta}\right\} = k\right\}\right) \\ = \lim_{t \to \infty} e^{(1-\delta_\Gamma)t} \lambda\left(\{x \in [0,1] : \#\left(\mathcal{Z}n_+(-x)a_t \cap \mathfrak{C}_{A,\theta}\right) = k\}\right).\end{aligned} \tag{7.4.7}$$

To which we apply Theorem 7.3.3 to get (7.4.4).

(7.4.6) follows in the same way except, in the last step, we apply Theorem 7.3.4 instead of Theorem 7.3.3.

□

Moreover, the same proof allows one to prove the *Kesten problem* in our context, stated as follows: for $A > 0$ and Q fixed let $K(A, Q)$ denote the number of solutions to

$$|\alpha q - p| \leq \frac{A}{Q} \quad , \quad 1 \leq q \leq Q. \tag{7.4.8}$$

In this case the following theorem holds:

Theorem 7.4.2. *Given $A > 0$ Theorem 7.4.1 holds with $E(A, \theta, Q)$ replaced by $K(A, Q)$ and $\mathfrak{C}_{A,\theta}$ replaced by*

$$R_A = \left\{(x, y) \in \mathbb{R}^2 : |x| \leq A, 0 \leq y \leq 1\right\}. \tag{7.4.9}$$

7.4.2 Directions of Primitive Points

Given a point in $\mathbb{R}^2$ (taken here to be the origin, however this is not necessary), one can ask how the directions of primitive points $\mathcal{Z}$ distribute for an observer at that point, this is in some sense the Euclidean version of the main theorem in Chapter 6. The corollary of Theorem 7.3.5 below answers this question.

Let $\mathcal{D}_t(\sigma, v) \subset S^1_1$ be the interval in the unit sphere with centre v and length σe^{-t}, and set

$$\mathcal{N}_t(\sigma, v; \mathcal{Z}) := \#\left\{\mathbf{y} \in \mathcal{Z}_t : \|\mathbf{y}\|^{-1}\mathbf{y} \in \mathcal{D}_t(\sigma, v)\right\}, \tag{7.4.10}$$

where $\mathcal{Z}_t = \{z \in \mathcal{Z} : \|z\| \leq e^t\}$.

Corollary 7.4.3. *Let λ be a probability measure on $\mathbb{T}$, absolutely continuous with respect to Lebesgue and with continuous density. For $k \in \mathbb{N}_{>0}$ we have*

$$\lim_{t\to\infty} e^{(1-\delta_\Gamma)t}\lambda\left(\{v \in \mathbb{T} : \mathcal{N}_t(\sigma, v; \mathcal{Z}) = k\}\right) = \frac{D_\lambda}{|\mathrm{m}^{BMS}|}\mathrm{m}^{BR}\left(\{\alpha \in \Gamma\backslash G : |\mathcal{Z}\alpha \cap \mathfrak{C}_\sigma| = k\}\right) \tag{7.4.11}$$

where, in polar coordinates

$$\mathfrak{C}_\sigma = \{x = (r, \theta) \in \mathbb{R}^2 : r < 1, |\theta| < \sigma\pi\}. \tag{7.4.12}$$

This corollary follows directly from Theorem 7.3.5.

7.5 Equidistribution of gFs

7.5.1 Statement

In addition to Theorem 7.3.3 another important consequence of the equidistribution statements in Section 7.3, is the following theorem, stating that the gFs equidistributes on a horospherical section. This is a generalisation of [Mar10, Theorem 6], to the infinite covolume setting.

Theorem 7.5.1. *Let $\sigma \in \mathbb{R}$ and $Q = e^{(t-\sigma)/2}$. Let $f : \mathbb{T} \times \Gamma\backslash G \to \mathbb{R}$ be bounded continuous and supported on a set with finite volume. Then*

$$\lim_{t\to\infty} e^{-\delta_\Gamma t}\sum_{r\in\mathcal{F}_Q} f(r, n_-(r)a_{-t}) = \frac{e^{(\delta_\Gamma-1)\sigma}}{|m^{BMS}|}\int_{\mathbb{T}\times\mathbb{T}}\int_\sigma^\infty \tilde{f}(x, n_-(w)a_{-r})e^{\delta_\Gamma r}dr d\mu^{PS}(w)d\mu^{PS}_{N_+}(x) \tag{7.5.1}$$

where $\tilde{f}(x, \alpha) := f(x, {}^t\alpha^{-1})$.

Remark. [Mar10] and [Mar13] treat Farey sequences in general dimension. However in the infinite covolume setting equidistribution results for $\mathrm{SL}(d, \mathbb{R})$ have not yet been proved (to our knowledge).

7.5.2 Proof

Proof of Theorem 7.5.1. The proof will follow the same lines as [Mar10, Proof of Theorem 6] with several exceptions as we are not working with Haar measure. In particular, since the Patterson-Sullivan measure does not satisfy the same invariance properties as the Haar measure, some care is needed when approximating f by compactly supported functions (step 1), and we will make use of the Gauss type decomposition of the Burger Roblin measure (Proposition 7.2.1).

Note first that by setting $f(x, \alpha) = f_0(x, \alpha a_{-\sigma})$ for f_0 bounded and continuous we may assume that $\sigma = 0$.

Step 1 First we show that we can reduce the theorem to f compactly supported via a standard approximation argument. Assume for the sake of notation that f is x-invariant. Assume further the theorem holds for compactly supported functions. Now consider a bounded, continuous function, f supported on a finite-volume set. Fix $\epsilon > 0$ and consider (for some t) the difference

$$\left| e^{-\delta_\Gamma t} \sum_{r \in \mathcal{F}_Q} f(n_-(r)a_{-t})) - \frac{1}{|m^{BMS}|} \int_{\mathbb{T}} \int_0^\infty \tilde{f}(n_-(w)a_{-r}) e^{\delta_\Gamma r} dr d\mu^{PS}(w) \right|. \tag{7.5.2}$$

Now decompose $f = f_1 + f_2$ such that f_1 is supported on a compact set and f_2 is supported on a set of volume $\varrho > 0$ (as $\text{supp}(f)$ has finite volume ϱ can be chosen arbitrarily small) and both are bounded and continuous. Hence the difference (7.5.2) is bounded above by

$$\left| e^{-\delta_\Gamma t} \sum_{r \in \mathcal{F}_Q} f_1(n_-(r)a_{-t})) - \frac{1}{|m^{BMS}|} \int_{\mathbb{T}} \int_0^\infty \tilde{f}_1(n_-(w)a_{-r}) e^{\delta_\Gamma r} dr d\mu^{PS}(w) \right|$$
$$+ \left| e^{-\delta_\Gamma t} \sum_{r \in \mathcal{F}_Q} f_2(n_-(r)a_{-t})) - \frac{1}{|m^{BMS}|} \int_{\mathbb{T}} \int_0^\infty \tilde{f}_2(n_-(w)a_{-r}) e^{\delta_\Gamma r} dr d\mu^{PS}(w) \right|. \tag{7.5.3}$$

Applying Theorem 7.5.1 for compact functions implies we can take t large enough that the first term is less than $\epsilon/2$.

We may assume that f_2 is supported on the cusp at infinity, i.e $\text{supp}(f_2) = \{z \in \mathbb{H} : \Im(z) > \varrho^{-1}\}$. With that, using the bounded property of f, there exists a $C < \infty$ such that

$$\left| |\mathcal{F}_Q|^{-1} \sum_{r \in \mathcal{F}_Q} f_2(n_-(r)a_{-t}) \right| \le \frac{C \#\{r \in \mathcal{F}_Q : \Im(\pi_1(n_-(r)a_{-t})) > \varrho^{-1}\}}{|\mathcal{F}_Q|} \tag{7.5.4}$$

where π_1 denotes the projection to the fundamental domain above i extending to infinity. This proportion can be upper bounded by $\frac{C|\mathcal{F}_{\varrho Q}|}{|\mathcal{F}_Q|} = C\varrho^{2\delta_\Gamma}$ for some constant $C < \infty$. Thus by choosing ϱ large enough the summation in the right hand term in (7.5.3) can be bounded by $\epsilon/4$.

Lastly, consider the term

$$\left| \int_{\mathbb{T}} \int_0^\infty \tilde{f}_2(n_-(w)a_{-r}) e^{\delta_\Gamma r} dr d\mu^{PS}(w)) \right| < \infty. \tag{7.5.5}$$

As Γ has a cusp, $\delta_\Gamma > 1/2$. Thus the Patterson-Sullivan measure of $\text{supp}(\tilde{f}_2) \cap \mathcal{L}(\Gamma)$ goes to 0 as $\text{vol}(\text{supp}(\tilde{f}_2))$ goes to 0. Hence we can choose ϱ such that (7.5.2) is bounded by ϵ. Thus Theorem 7.5.1 for compactly supported f implies the theorem for f with finite volume support.

Henceforth take f to be compactly supported.

Step 2 Note that because f is continuous and has compact support it is uniformly continuous. Hence for every $\varrho > 0$ there exists a $\epsilon > 0$ such that for all $(x, \alpha), (x'\alpha') \in \mathbb{R} \times G$

$$|x - x'| < \epsilon \qquad d(\alpha, \alpha') < \epsilon \tag{7.5.6}$$

imply $|f(x, \alpha) - f(x', \alpha')| < \varrho$.

Step 3 For $0 \le \theta < 1$ and $\epsilon > 0$ define

$$\begin{aligned} \mathcal{F}_{Q,\theta} &:= \left\{ \frac{p}{q} \in [0,1) : (p,q) \in \mathcal{Z},\ \theta Q < q < Q \right\} \\ \mathcal{F}_Q^\epsilon &:= \bigcup_{r \in \mathcal{F}_{Q,\theta} + \mathbb{Z}} \left\{ x \in \mathbb{R} : \|x - r\| < \epsilon e^{-t} \right\}. \end{aligned}$$

The latter we can write as

$$\mathcal{F}_Q^\epsilon = \bigcup_{\mathbf{a}\in\mathcal{Z}} \{x \in \mathbb{R} : (a_1, a_2)n_+(x)a_t \in \mathfrak{C}_\epsilon\},$$

where

$$\mathfrak{C}_\epsilon := \{(y_1, y_2) \in \mathbb{R}^2 : |y_1| < \epsilon y_2, \quad \theta < y_2 \leq 1\}.$$

Our goal is to write the characteristic function for $\mathcal{F}_Q^\epsilon$ as a sum over simpler characteristic functions. Thus, let

$$\mathcal{H}_\epsilon := \bigcup_{\mathbf{a}\in\mathcal{Z}} \mathcal{H}_\epsilon(\mathbf{a}), \qquad \mathcal{H}_\epsilon(\mathbf{a}) := \{\alpha \in G : (a_1, a_2)\alpha \in \mathfrak{C}_\epsilon\}.$$

By considering the bijection

$$\Gamma_{N_-}\backslash\Gamma \to \mathcal{Z}, \qquad \Gamma_{N_-}\gamma \mapsto (0,1)\gamma$$

we can write

$$\begin{aligned} \mathcal{H}_\epsilon &= \bigcup_{\gamma\in\Gamma_{N_-}\backslash\Gamma} \mathcal{H}_\epsilon((0,1)\gamma) \\ &= \bigcup_{\gamma\in\Gamma_{N_-}\backslash\Gamma} \gamma\mathcal{H}_\epsilon^1, \end{aligned} \tag{7.5.7}$$

where

$$\mathcal{H}_\epsilon^1 := \mathcal{H}_\epsilon((0,1)) = H\{M_{\mathbf{y}} : \mathbf{y} \in \mathfrak{C}_\epsilon\}$$

with $M_{\mathbf{y}} := \begin{pmatrix} y_2^{-1} & 0 \\ y_1 & y_2 \end{pmatrix}$.

Step 4

Claim: Given $\mathcal{C} \subset G$ compact there exists an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$

$$\gamma\mathcal{H}_\epsilon^1 \cap \mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C} = \emptyset, \tag{7.5.8}$$

for all $\gamma \in \Gamma/\Gamma_{N_-} \neq 1$

Proof of Claim.(7.5.8) is equivalent to

$$\mathcal{H}_\epsilon((p,q)) \cap \mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C} = \emptyset, \qquad \forall (p,q) \neq (0,1) \in \mathcal{Z}$$

Consider an $\alpha \in G$ such that $(p,q)\alpha \in \mathfrak{C}_\epsilon$. We can write any such α as

$$\alpha = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2^{-1} & 0 \\ y_1 & y_2 \end{pmatrix}$$

for $b \in \mathbb{R}$ and $y_1 \in \mathbb{R}$.

Therefore if we assume for the sake of contradiction that $(p,q)\alpha \in \mathfrak{C}_\epsilon$ and $(0,1)\alpha \in \mathfrak{C}_\epsilon$ we have the following 4 inequalities

$$|y_2^{-1}p + (pb+q)y_1| < \epsilon y_2(pb+q) \tag{7.5.9}$$

$$\theta < y_2(pb+q) \leq 1 \tag{7.5.10}$$

$$|y_1| < \epsilon y_2 \tag{7.5.11}$$

$$\theta < y_2 \leq 1. \tag{7.5.12}$$

Using (7.5.10) and (7.5.11) gives

$$|(pb+q)y_1| < \epsilon$$

which, plugging into (7.5.9) gives

$$|y_2^{-1}p| < 2\epsilon.$$

Hence

$$|p| < 2\epsilon.$$

Thus $p = 0$. Therefore $(0, q) = (0, 1)\gamma$ for some $\gamma \in \Gamma$. However since $\Gamma_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$, $q = 1$. Which is a contradiction proving the statement. □

Step 5 The claim implies that for $\mathcal{C} \subset G$ compact there is an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ such that

$$\mathcal{H}_\epsilon \cap \Gamma\mathcal{C} = \bigcup_{\gamma \in \Gamma/\Gamma_{N_-}} (\gamma \mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C}) \tag{7.5.13}$$

is a disjoint union. Thus let χ_ϵ and χ_ϵ^1 denote the characteristic functions of $\mathcal{H}_\epsilon$ and $\mathcal{H}_\epsilon^1$ respectively, then

$$\chi_\epsilon(\alpha) = \sum_{\gamma \in \Gamma_{N_-} \backslash \Gamma} \chi_\epsilon^1(\gamma\alpha)$$

for all $\alpha \in \Gamma\mathcal{C}$. Moreover all of the sets we consider have boundary of BR-measure 0. Set $\tilde{\chi}_\epsilon(\alpha) := \chi_\epsilon({}^t\alpha^{-1})$ and note that $\chi_\epsilon(n_+(x)a_t) = \tilde{\chi}_\epsilon(n_-(-x)a_{-t})$ is the characteristic function for $\mathcal{F}_Q^\epsilon$.
Therefore we write

$$\begin{aligned} \int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}} f(x, n_-(x)a_{-t})dx &= \int_{\mathbb{T}} f(x, n_-(x)a_{-t})\chi_\epsilon(n_+(-x)a_t)dx \\ &= \int_{\mathbb{T}} \tilde{f}(x, n_+(-x)a_t)\chi_\epsilon(n_+(-x)a_t)dx, \end{aligned} \tag{7.5.14}$$

to which we can apply Theorem 7.3.1 giving:

$$\lim_{t\to\infty} e^{(1-\delta_\Gamma)t} \int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}} f(x, n_-(x)a_{-t})dx = \frac{1}{|m^{BMS}|} \int_{\Gamma\backslash G \times \mathbb{T}} \tilde{f}(x, \alpha)\chi_\epsilon(\alpha)dm^{BR}(\alpha)d\mu_{N_+}^{PS}(x). \tag{7.5.15}$$

Which we write as

$$= \frac{1}{|m^{BMS}|} \int_{\Gamma_{N_-}\backslash G \times \mathbb{T}} \tilde{f}(x,\alpha)\chi_\epsilon^1(\alpha) dm^{BR}(\alpha) d\mu_{N_+}^{PS}(x),$$
$$= \frac{1}{|m^{BMS}|} \int_{\Gamma_{N_-}\backslash N_- \{M_{\mathbf{y}} : \mathbf{y}\in\mathfrak{C}_\epsilon\}\times\mathbb{T}} \tilde{f}(x,\alpha) dm^{BR}(\alpha) d\mu_{N_+}^{PS}(x). \tag{7.5.16}$$

Step 6

Using Proposition 7.2.1 we write (7.5.16) as (noting that $(0,1)n_- = (0,1)$)

$$\lim_{t\to\infty} e^{(1-\delta_\Gamma)t} \int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}} f(x, n_-(x)a_{-t})dx =$$
$$\frac{2}{|m^{BMS}|} \int_{\mathbb{T}\times\{\mathbf{y}\in\mathfrak{C}_\epsilon\}\times\mathbb{T}} y_2^{2\delta_\Gamma - 2} \tilde{f}(x, n_-(w)M_{\mathbf{y}}) dy_2 dy_1 d\mu^{PS}(w) d\mu_{N_+}^{PS}(x). \tag{7.5.17}$$

Which we can write

$$= \frac{2}{|m^{BMS}|} \int_{\mathbb{T}\times\mathbb{T}} \int_\theta^1 \int_{\mathcal{B}_{\epsilon y_2}(0)} y_2^{2\delta_\Gamma - 2} \tilde{f}(x, n_-(w)M_{\mathbf{y}}) dy_2 dy_1 d\mu^{PS}(w) d\mu_{N_+}^{PS}(x). \tag{7.5.18}$$

Next we write $D(y_2) := \begin{pmatrix} y_2^{-1} & 0 \\ 0 & y_2 \end{pmatrix}$ and note

$$d(M_{\mathbf{y}}, D(y_2)) = d(n_+(y_2^{-1}y_1), Id) \leq \epsilon \tag{7.5.19}$$

for $\mathbf{y} \in \mathfrak{C}_\epsilon$ (this is the same calculation as [Mar10, (3.42)]). Therefore, using uniform continuity

$$\begin{aligned} &\left| (7.5.16) - \frac{2}{|m^{BMS}|} \int_{\mathbb{T}\times\mathbb{T}} \int_\theta^1 \int_{\mathcal{B}(\epsilon y_2)} \tilde{f}(x, n_-(w)D(y_2)) y_2^{2\delta_\Gamma - 2} dy_2 dy_1 d\mu^{PS}(w) d\mu_{N_+}^{PS}(x) \right| \\ &= \left| (7.5.16) - \frac{4\epsilon}{|m^{BMS}|} \int_{\mathbb{T}\times\mathbb{T}} \int_\theta^1 \tilde{f}(x, n_-(w)D(y_2)) y_2^{2\delta_\Gamma - 1} dy_2 d\mu^{PS}(w) d\mu_{N_+}^{PS}(x) \right| \\ &\leq \frac{4\varrho\epsilon|\mu^{PS}|^2}{|m^{BMS}|} \int_\theta^1 y_2^{2\delta_\Gamma - 1} dy_2. \end{aligned} \tag{7.5.20}$$

Evaluating this integral then gives that the right hand side of the inequality in (7.5.20) is equal to

$$\frac{2\epsilon\varrho|\mu^{PS}|^2}{|m^{BMS}|\delta_\Gamma}(1 - \theta^{2\delta}). \tag{7.5.21}$$

Finally reinserting the right hand side of (7.5.15) and applying the change of variables $y_2 = e^{r/2}$, we conclude that

$$\left| \lim_{t\to\infty} e^{(1-\delta_\Gamma)t} \int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}} f(x, n_-(x)a_{-t})dx \right.$$
$$\left. - \frac{2\epsilon}{|m^{BMS}|} \int_{\mathbb{T}\times\mathbb{T}} \int_0^{2|\ln\theta|} \tilde{f}(x, n_-(w)a_{-t})e^{\delta_\Gamma r} dr d\mu^{PS}(w) d\mu_{N_+}^{PS}(x) \right|$$
$$< \frac{2\varrho\epsilon|\mu^{PS}|^2}{|m^{BMS}|\delta_\Gamma}(1-\theta^{2\delta_\Gamma}). \quad (7.5.22)$$

Step 7

To conclude consider

$$\lim_{t\to\infty} e^{-\delta_\Gamma t} \sum_{r\in\mathcal{F}_{Q,\theta}} f(r, n_-(r)a_{-t}) \tag{7.5.23}$$

taking the asymptotic formula (7.1.10) and using a volume estimate together with uniform continuity we can write this as

$$= \lim_{\epsilon\to 0} \lim_{t\to\infty} \frac{e^{(1-\delta_\Gamma)t}}{2\epsilon} \sum_{r\in\mathcal{F}_{\theta,Q}} \int_{|x-r|<\epsilon e^{-t}} f(x, n_-(x)a_{-t})dx. \tag{7.5.24}$$

Now using the disjoint union in (7.5.13) we can say

$$= \lim_{\epsilon\to 0} \lim_{t\to\infty} \frac{e^{(1-\delta_\Gamma)t}}{2\epsilon} \int_{\mathcal{F}_Q^\epsilon\backslash\mathbb{Z}} f(x, n_-(x)a_{-t})dx \tag{7.5.25}$$

and using (7.5.22) we thus conclude after taking $\epsilon \to 0$ (and therefore $\varrho \to 0$) this is equal

$$= \frac{1}{|m^{BMS}|} \int_{\mathbb{T}\times\mathbb{T}} \int_0^{2|\ln\theta|} \tilde{f}(x, n_-(w)a_{-r})e^{\delta_\Gamma r} dr d\mu^{PS}(w) d\mu_{N_+}^{PS}(x) \tag{7.5.26}$$

Taking the limit as $\theta \to 0$ is then possible as

$$\limsup_{t\to\infty} \frac{|\mathcal{F}_Q \backslash \mathcal{F}_{Q\theta}|}{e^{\delta_\Gamma t}} = \theta c_\Gamma. \tag{7.5.27}$$

□

7.6 Local Statistics

Theorem 7.3.3 and Theorem 7.5.1 can also be used to study the local statistics of $\mathcal{F}_Q$ when viewed as a point process on $[0,1]$ (note once more we are assuming for notation, that Γ^∞ is periodic on $[0,1]$).

7.6.1 Statement

For $Q = e^{t/2}$. Let $\mathscr{A} \subset \mathbb{R}$ be bounded interval and set $\mathscr{A}_t = \mathscr{A}e^{-t}$. For a bounded $\mathcal{D} \subset \mathbb{T}$, define

$$P_Q(\mathcal{D}, \mathscr{A}, k) = \frac{e^t \operatorname{vol}(\{x \in \mathcal{D} : |x + \mathscr{A}_t + \mathbb{Z} \cap \mathcal{F}_Q| = k\})}{\mu_{N_+}^{PS}(\mathcal{D})e^{\delta_\Gamma t}} \tag{7.6.1}$$

and

$$P_{0,Q}(\mathcal{D},\mathscr{A},k) = \frac{|\{r \in \mathcal{F}_Q \cap \mathcal{D} : |r + \mathscr{A}_t + \mathbb{Z} \cap \mathcal{F}_Q| = k\})}{\mu^{PS}_{N_+}(\mathcal{D})e^{\delta_\Gamma t}} \tag{7.6.2}$$

Theorem 7.6.1. *Given an interval $\mathscr{A} \subset \mathbb{R}$ and $\mathcal{D} \subset \mathbb{T}$ then for all $k > 0$*

$$\lim_{Q\to\infty} P_Q(k,\mathcal{D},\mathscr{A}) = P(k,\mathscr{A}) \tag{7.6.3}$$

$$\lim_{Q\to\infty} P_{0,Q}(\mathcal{D},\mathscr{A},k) = P_0(k,\mathscr{A}) \tag{7.6.4}$$

where $P(k,\mathscr{A})$ and $P_0(k,\mathscr{A})$ are given explicitly.

Remark. In particular (7.6.4) implies that the limiting gap distribution exists everywhere.

Remark. Note that the above theorem is restricted to $k > 0$. The reason for this is that the scaling in P_Q and $P_{0,Q}$ is incorrect for the case $k = 0$. For geometrically finite subgroups the boundary points cluster close together in far apart cluster. This phenomenon was noticed by Zhang [Zha17] and again in [Lut18] (see Chapter 6, remark below Theorem 6.2.2).

To give a qualitative example, we have graphed the gap distribution for $\widehat{\Gamma}^\infty$ in Figure 7.3.

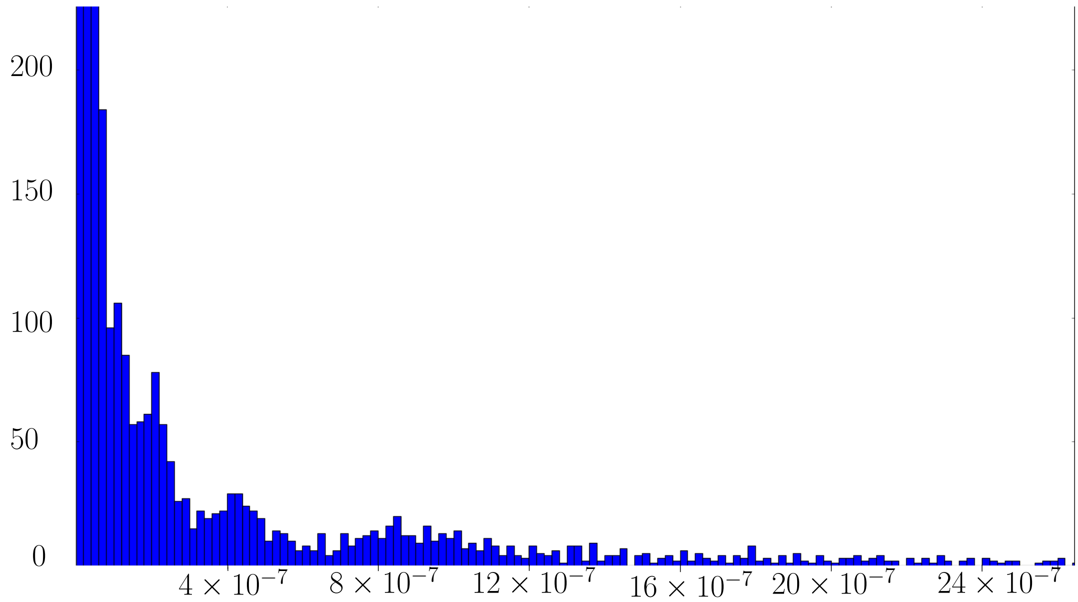

Figure 7.3: Above we have shown the gaps in the point set $\widehat{\Gamma}^\infty$. The point set is exactly the one shown in Figure 7.1 on page 146. We have cut off the image at 240 (thus the first three bars do not have the same height) and the bin size here is 4×10^{-8}. Hence the bars represent the number of gaps lying in a particular bin.

7.6.2 Proof

Proof of Theorem 7.6.1. Theorem 7.6.1 is a straightforward consequence of Theorem 7.3.3 and Theorem 7.5.1. We begin by addressing (7.6.3), define

$$\mathfrak{C}(\mathscr{A}) := \{(x,y) \in \mathbb{R} \times (0,1] : x \in \mathscr{A}y\} \subset \mathbb{R}^2 \tag{7.6.5}$$

and note that

$$\frac{p}{q} \in x + \mathscr{A}_t \quad , \quad 0 < q \leq Q \tag{7.6.6}$$

is equivalent to

$$\iff (p,q)n_+(\mathbf{x})a_t \in \mathfrak{C}(\mathscr{A}). \tag{7.6.7}$$

Therefore for a given $x \in \mathcal{D}$

$$P_Q(\mathcal{D},\mathscr{A},k) = \frac{e^{(1-\delta_\Gamma)t}}{\mu^{PS}_{N_+}(\mathcal{D})} \operatorname{vol}(\{x \in \mathcal{D} : |\mathcal{Z}n_+(x)a_t \cap \mathfrak{C}(\mathscr{A})| = k\}). \tag{7.6.8}$$

Applying Theorem 7.3.3 then implies

$$P(k,\mathscr{A}) = \frac{1}{|m^{BMS}|} m^{BR}(\mathcal{S}_k). \tag{7.6.9}$$

where $\mathcal{S}_k = \{\alpha \in \Gamma\backslash G : |\mathcal{Z}\alpha \cap \mathfrak{C}(\mathscr{A})| = k\}$.

Turning now to (7.6.4). Write

$$\begin{aligned} P_0(\mathscr{A},k) &= \lim_{t\to\infty} \frac{|\{r \in \mathcal{F}_Q \cap \mathcal{D} : |\mathcal{Z}n_+(r)a_t \cap \mathfrak{C}(\mathscr{A})| = k\}|}{e^{\delta_\Gamma t}\mu^{PS}_{N_+}(\mathcal{D})} \\ &= \lim_{t\to\infty} \frac{\sum_{r\in\mathcal{F}_Q} \chi_{\mathcal{S}_k}(r, n_+(r)a_t)}{\mu^{PS}_{N_+}(\mathcal{D})e^{\delta_\Gamma t}}. \end{aligned} \tag{7.6.10}$$

Applying Theorem 7.5.1 (after extending it to characteristic functions using the methodology of Chapter 6, Section 5.6) gives

$$P_0(\mathscr{A},k) = \frac{1}{|m^{BMS}|} \int_{\mathbb{T}\times[0,\infty)} \tilde{\chi}_{\mathcal{S}_k}(n_-(w)a_{-r})e^{\delta_\Gamma r} dr d\mu^{PS}(w). \tag{7.6.11}$$

Note that the quantity in (7.6.9) is finite for $k > 0$. This follows from Chapter 6, Proposition 6.3.3. However finiteness does not hold for $k = 0$, which is the reason for that restriction in the Theorem. The integral on the right hand side of (7.6.11) is finite whenever the Burger-Roblin measure is finite. Hence the same Chapter 6, Proposition 6.3.3 also implies finiteness of (7.6.11).

□

7.7 Explicit Gap Distribution for $\widehat{\Gamma}$

We now return to the example, $\widehat{\Gamma}$, discussed in Section 7.1. First note that Theorem 7.6.1 implies that, in the limit $T \to \infty$, the gap distribution in (7.1.13) exists for all $s > 0$. Our goal is to prove the following theorem which gives a far more explicit formula for the limiting gap distribution:

Theorem 7.7.1. *For $s < s_0 = 7$, and $\mathcal{I}$ a closed interval in $[0,1]$, the limiting gap distribution can be written*

$$\lim_{T\to\infty} \widehat{F}_{T,\mathcal{I}}(s) =: \widehat{F}_{\mathcal{I}}(s) = F^{1,2}_{\mathcal{I}}(s) + F^{2,3}_{\mathcal{I}}(s) \tag{7.7.1}$$

where $F^{1,2}_{\mathcal{I}}(s)$ and $F^{2,3}_{\mathcal{I}}(s)$ are explicit integrals (see (7.7.35)) over compact regions involving the Patterson-Sullivan density ν_i (defined below (5.5.5)).

The proof follows the methodology of [RZ17], however there are significant differences. In [RZ17] Rudnick and Zhang looked at Ford configurations associated to lattices. Thus our analysis represents one example of the infinite covolume analogue of their chapter. The plan is to break up the gap distribution into a sum, with each term coming from a pair of circles in the initial configuration $\mathcal{K}_0$. Then, using the following elementary lemma (proved in [RZ17]) we can express each term in this sum as an integral over a compact area.

Lemma 7.7.2 ([RZ17, Lemma 3.5]). *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$.*

(i) *If $c \neq 0$ then under the Möbius transform M, a circle $C(x+yi, y)$ is mapped to*

$$C\left(\frac{ax+b}{cx+d} + \frac{yi}{(cx+d)^2}, \frac{y}{(cx+d)^2}\right) \tag{7.7.2}$$

if $cx+d \neq 0$, and to the line $\Im \mathbf{z} = 1/2c^2y$ if $cx+d=0$. When $c=0$, the image circle is

$$C\left(\frac{ax+b}{d}, \frac{y}{d^2}\right). \tag{7.7.3}$$

(ii) *If $c \neq 0$ then the line $C = \mathbb{R} + yi$ is mapped to*

$$C\left(\frac{a}{c} + \frac{1}{2c^2y}i, \frac{1}{2c^2y}\right), \tag{7.7.4}$$

and to the line $\mathbb{R} + a^2yi$ if $c = 0$.

7.7.1 Breaking the Gap Distribution Up

In [RZ17] a fundamental observation is that a pair of neighbouring tangencies at a given height T, are the image of a pair of circles in the initial configuration by exactly one or two group elements in Γ. That is not true here, however the following proposition states that this is the case in the interval $[0, s_0)$.

Proposition 7.7.3. *For any T and $\mathcal{I}$, suppose $\mathcal{C}$ and $\mathcal{C}'$ are the circles tangent to $\mathcal{C}_0$ at $x^j_{T,\mathcal{I}}$ and $x^{j+1}_{T,\mathcal{I}}$. If $T(x^{j+1}_{T,\mathcal{I}} - x^j_{T,\mathcal{I}}) \leq s$ for $s < s_0$ then there exists a $\gamma \in \Gamma$ such that $\mathcal{C} = \gamma\mathcal{C}_l$ and $\mathcal{C}' = \gamma\mathcal{C}_m$ for $\mathcal{C}_l \neq \mathcal{C}_m \in \mathcal{K}_0$ and neither equal $\mathcal{C}_0$. Moreover if $\mathcal{C}$ and $\mathcal{C}'$ are not tangent then γ is unique and if they are tangent then there exist exactly two such γ.*

Remark. The reason we consider $s < s_0$ in Theorem 7.7.1 is that Proposition 7.7.3 fails for larger s. In words, for larger s some of the gaps considered are not the image of a pair in the initial configuration. To get around this, one could consider a larger initial configuration (i.e consider $\mathcal{K}$ together with the circles tangent at $1/4$ and $4 - 1/4$). This would allow Proposition 7.7.3 to hold for slightly larger s_0. Therefore as one considered larger and larger gaps, one would need to consider larger and larger initial configurations and more and more terms in the decomposition below. In this chapter we will stick to the case $s_0 = 7$ as it will simplify the following proofs.

For ease of notation, we restrict our attention to circles tangent to $\mathcal{C}_0$ in $[0, 2]$ (i.e beneath $\mathcal{C}_2$) and adopt the following notation shown in Figure 7.4: first label $\mathcal{C}_2 = \mathcal{C}^0$ and

- The tangencies are labelled by their continued fraction expansions $\alpha^{(i)}_{k_1,\ldots,k_i} = [0; 4k_1, \ldots 4k_i]$.
- The associated circles are labelled $\mathcal{C}^{(i)}_{k_1,\ldots,k_i}$.
- The diameter of each circle is similarly labelled $h^{(i)}_{k_1,\ldots,k_i}$.

Thus, each circle $\mathcal{C}^{(i)}_{k_1,\ldots,k_i}$ is the *child* of the circle $\mathcal{C}^{(i-1)}_{k_1,\ldots,k_{i-1}}$ (to which it is tangent) and the *parent* of $\mathbb{Z}_{\neq 0}$ children - $\mathcal{C}^{(i+1)}_{k_1,\ldots,k_i,k_{i+1}}$ (to which it is also tangent).

Define a rectangle to be any collection of circles

$$\mathcal{R} = (\mathcal{C}^{(i)}_{k_1,\ldots,k_{i-1},k_i}, \mathcal{C}^{(i)}_{k_1,\ldots,k_{i-1},k_i\pm 1}, \mathcal{C}^{(i-1)}_{k_1,\ldots,k_{i-1}}, \mathcal{C}_0) \quad , \quad (k_i \neq 0) \tag{7.7.5}$$

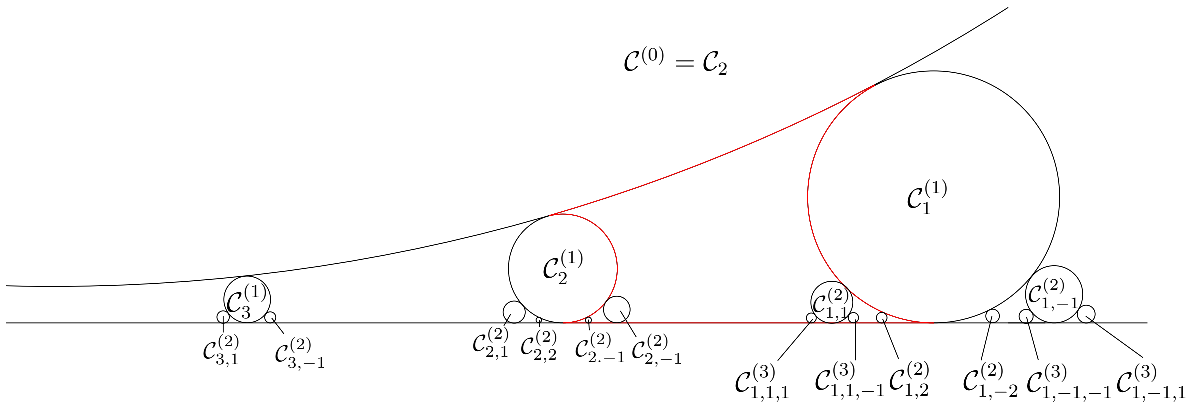

Figure 7.4: The labelling used in this section. For clarity, we only show a portion of the interval and a few circles in $\mathcal{K}$. The red section is what we call the rectangle $(\mathcal{C}_1^{(1)}, \mathcal{C}_2^{(1)}, \mathcal{C}^{(0)}, \mathcal{C}_0)$.

where $k_i \pm 1 \neq 0$ (see for example the rectangle in Figure 7.4). A rectangle is thus a pair of neighbours in a generation, the shared parent, and the real line. Let $\mathcal{R}_0$ denote the rectangle $(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ of the initial configuration. The following simple observation is the basis of the proof of Proposition 7.7.3.

Fact 7.7.1. *For any rectangle $\mathcal{R}$ there exists a unique $\gamma \in \widehat{\Gamma}$*

$$\mathcal{R} = \gamma \mathcal{R}_0. \tag{7.7.6}$$

The configuration $\mathcal{K} = \Gamma \mathcal{K}_0$ where $\mathcal{K}_0$ is the initial configuration. Since circle inversions send circles to circles preserving tangencies there must be a $\gamma \in \widehat{\Gamma}$ sending $\mathcal{R}_0$ to $\mathcal{R}$. Moreover the uniqueness follows as we are working in $\mathrm{PSL}(2, \mathbb{Z})$.

Proof of Proposition 7.7.3. In this proof, given two circles with tangencies α_1 and α_2 and diameters h_1 and h_2 we refer to $|\alpha_1 - \alpha_2|$ as the gap associated to them and to $\min\{h_1, h_2\}^{-1} |\alpha_1 - \alpha_2|$ as the *scaled gap* associated to them. Note that if a scaled gap is larger than s_0, then the gap *will never* contribute to $\widehat{F}_{T,\mathcal{I}}(s)$ for any T. Thus that gap can be ignored. Fact 7.7.1 implies that Proposition 7.7.3 follows if we show that all scaled gaps associated to pairs of circles *not in* rectangles are larger than s_0.

Step 1 The scaled gap associated to a pair of *non-tangent* circles *in a rectangle* has the form

$$\min\{h_{k_1,\ldots,k_i}^{(i)}, h_{k_1,\ldots,k_i \pm 1}^{(i)}\}^{-1} \left| \alpha_{k_1,\ldots,k_i}^{(i)} - \alpha_{k_1,\ldots,k_i \pm 1}^{(i)} \right| \tag{7.7.7}$$

(again $k_i \pm 1 \neq 0$).

Step 2 We now use some theory of continued fractions to show that (7.7.7) is bounded below by 4. I.e the gap associated to *non-tangent* pairs *in* a rectangle is bounded below by 4. Given a tangency $\alpha_{k_1,\ldots,k_i}^{(i)} = [0; a_1, \ldots a_i]$, let

$$\frac{b_n}{d_n} := [0; a_1, \ldots, a_n] \tag{7.7.8}$$

for $n < i$ where b_n and d_n share no common factors. It is a classical exercise to show (see [Khi03]):

$$b_n = a_n b_{n-1} + b_{n-2}, \qquad b_{-2} = 0, \qquad b_{-1} = 1 \tag{7.7.9}$$

$$d_n = a_n d_{n-1} + d_{n-2}, \qquad d_{-2} = 1, \qquad d_{-1} = 0 \tag{7.7.10}$$

and

$$d_n b_{n-1} - d_{n-1} b_n = (-1)^n. \tag{7.7.11}$$

Hence, if we let b_i and d_i be respectively the numerator and denominator of $\alpha^{(i)}_{k_1,\dots,k_i}$ and b'_i and d'_i, the numerator and denominator of $\alpha^{(i)}_{k_1,\dots,k_i\pm1}$, then:

$$\begin{aligned}
\min\{h^{(i)}_{k_1,\dots,k_i}, h^{(i)}_{k_1,\dots,k_i\pm1}\}^{-1} &\left|\alpha^{(i)}_{k_1,\dots,k_i} - \alpha^{(i)}_{k_1,\dots,k_i\pm1}\right| \\
&= \max\{d'_i, d_i\}^2 \left|[1; a_1, \dots a_i] - [1; a_1, \dots a_i \pm 4]\right| \\
&= \max\{d'_i, d_i\}^2 \left|\frac{a_i b_{i-1} + b_{i-2}}{a_i d_{i-1} + d_{i-2}} - \frac{(a_i \pm 4) b_{i-1} + b_{i-2}}{(a_i \pm 4) d_{i-1} + d_{i-2}}\right| \\
&= \max\{d'_i, d_i\}^2 \frac{4}{d_i d'_i} \geq 4,
\end{aligned} \tag{7.7.12}$$

Step 3 Suppose $\mathcal{C}^{(i)}_{m_1,\dots,m_i} = \mathcal{D}_1$ and $\mathcal{C}^{(j)}_{n_1,\dots,n_j} = \mathcal{D}_2$ are adjacent at time T (i.e there is no circle of height larger than T between $\mathcal{D}_1$ and $\mathcal{D}_2$) and *do not* belong to the same rectangle. For notation we assume $\alpha^{(i)}_{m_1,\dots m_i} < \alpha^{(j)}_{n_1,\dots n_j}$.

- By construction there is a 'youngest' shared ancestor of $\mathcal{D}_1$ and $\mathcal{D}_2$, $\mathcal{C}^{(k)}_{m_1,\dots,m_k} = \mathcal{B}_1$
- At the $k+1$-st generation $\mathcal{D}_1$ is the descendent of $\mathcal{C}_{m_1,\dots,m_{k+1}} = \mathcal{B}_3$ and $\mathcal{D}_2$ is the descendent of $\mathcal{C}^{(k+1)}_{n_1,\dots,n_{k+1}} = \mathcal{B}_2$ (see Figure 7.5) and $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{C}_0)$ must form a rectangle (otherwise $\mathcal{D}_1$ and $\mathcal{D}_2$ are clearly not adjacent at any times).
- Lastly it is evident that $\mathcal{D}_1$ must be the *right-most* descendent of $\mathcal{B}_3$ of its generation. Thus $|m_l| = 1$ for all $l > k+1$. Moreover $\mathcal{D}_2$ must be the *left-most* descendent of $\mathcal{B}_2$ in its generation.

Motivated by these three geometric facts we adopt the following notation (see Figure 7.5). In each generation l, we label the left-most descendent of $\mathcal{B}_2$ by $\mathcal{B}_{2(l-k)}$. Moreover we label the right-most descendent of $\mathcal{B}_3$ by $\mathcal{B}_{2(l-k)+1}$. With that notation, all non-tangent adjacent pairs of circles at a given time are of the form $\mathcal{B}_x$, $\mathcal{B}_{x+1}$ for some x.

Label the tangency associated to $\mathcal{B}_i$, α_i. Label the diameter of $\mathcal{B}_i$, h_i. We assume (w.l.o.g) $h_1 > h_2 \geq h_3$. Label the gap between $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$, $g_i = |\alpha_i - \alpha_{i+1}|$.

We show that $h^{-1}_{i+1} g_i$ (the scaled gap) is larger than 7 for all $i > 2$. This will prove the proposition as all gaps associated to non-tangent pairs are of this form. We assume $h_3 = 1$ (this is w.l.o.g by a simple scaling argument).

Now we collect two facts:

- By (7.7.10) $h_{n+2} \leq \frac{h_n}{3^2}$
- By (7.7.9)(7.7.10)(7.7.11) $g_{i+1} \geq g_i - h^{1/2}_{i+1} h^{1/2}_{i+2}$

First by (7.7.10) it is fairly easy to see that $h_2 < 9$. Suppose $4 < h_2 < 9$, then by (7.7.12) we know that $h_3^{-1} g_2 \geq 8$, thus

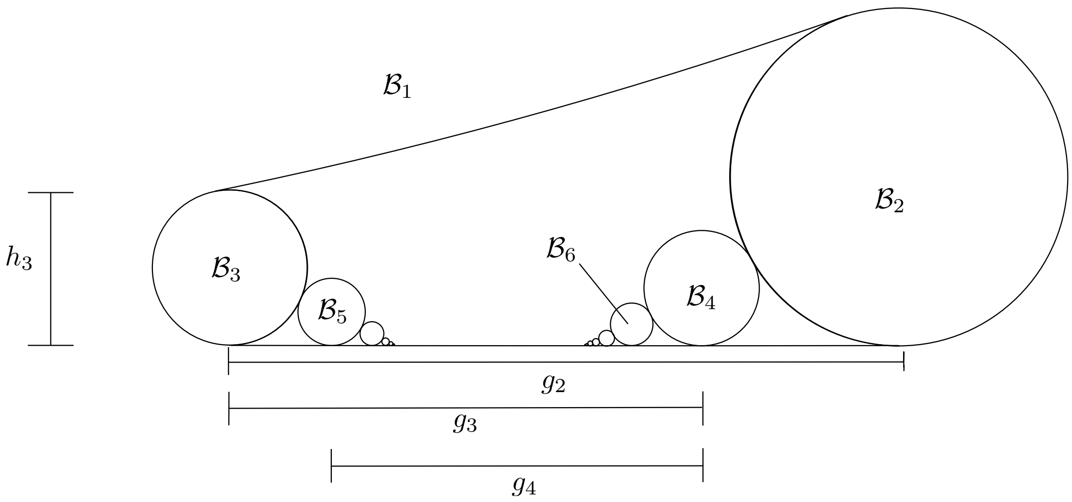

Figure 7.5: Above we show the relevant rectangle, circles and labelling for Step 3. We are only concerned with the 'innermost circles' in the rectangle. The circles are labelled in decreasing order of size.

$$\begin{aligned} h_3^{-1}g_2 &\geq 8 \\ h_4^{-1}g_3 &\geq \frac{9}{h_2}(8-1) > 7 \\ h_5^{-1}g_4 &\geq \frac{9}{h_3}(8-1-\frac{1}{3}) = 60 \end{aligned} \tag{7.7.13}$$

and so forth (a messy recursive inequality shows that this quantity is bounded by 7). Now assume $h_2 < 4$. Hence, using the two facts listed above we may conclude:

$$\begin{aligned} h_3^{-1}g_2 &\geq 4 \\ h_4^{-1}g_3 &\geq \left(4-\frac{2}{3}\right)\left(\frac{3}{2}\right)^2 > 7 \\ h_5^{-1}g_4 &\geq \left(4-\frac{2}{3}-\frac{2}{9}\right)3^2 \\ h_6^{-1}g_5 &\geq \left(4-\frac{2}{3}-\frac{2}{9}-\frac{2}{81}\right)\left(\frac{9}{2}\right)^2 \end{aligned} \tag{7.7.14}$$

and so forth. Hence the gap arising from circles which do not form the boundary of a rectangle is at least 7.

This proves the proposition with $s_0 = 7$ (this may not be sharp).

□

Now that we have established this proposition, the argument to prove Theorem 7.7.1 follows similar lines to Rudnick and Zhang. Note that Proposition 7.7.3, implies we can write the gap distribution for $s < s_0$ as

$$\widehat{F}_{T,\mathcal{I}}(s) = F_{T,\mathcal{I}}^{1,2}(s) + F_{T,\mathcal{I}}^{2,3}(s) \tag{7.7.15}$$

$$F_{T,\mathcal{I}}^{i,j}(s) := \frac{\#\left\{ (x_{T,\mathcal{I}}^l, x_{T,\mathcal{I}}^{l+1}) \in \Gamma(\alpha_i,\alpha_j) \middle| T(x_{T,\mathcal{I}}^{l+1} - x_{T,\mathcal{I}}^l) \leq s \right\}}{T^{\delta_{\widehat{\Gamma}}}}, \tag{7.7.16}$$

where α_i are the tangencies associated to $\mathcal{C}_i$ in the initial configuration (the contribution from the tangent pair $(1,3)$ has already been counted from the $(1,2)$ pair because of the overcounting in Proposition 7.7.3 for gaps associated with tangent pairs).

7.7.2 Geometric Description of the Gap Distribution

Lemma 7.7.2 and the Proposition 7.7.4 play a crucial role in what follows. As these theorems are taken from [RZ17] and are not specific to the subgroup considered, in what follows we will omit some of the technical details which are the same.

We use Lemma 7.7.2 to provide conditions under which the image of $\mathcal{C}_i$ and $\mathcal{C}_j$ are adjacent at time T. Indeed it follows from [RZ17, Proposition 4.6] that there exist two regions $\Omega_T^{1,2}$ and $\Omega_T^{2,3}$ such that, for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the image $M(\alpha_i, \alpha_j)$ is an adjacent pair at time T if and only if $(c,d) \in \Omega_T^{i,j}$ (where $(i,j) = (1,2)$ or $(2,3)$).

We define these two regions as subsets of the cd-plane $\{(c,d)|c \geq 0\}$:

(a) We define $\Omega_T^{1,2}$ to be those $\{(c,d)|c \geq 0\}$ such that

$$c^2 \leq \frac{T}{2} \quad , \quad d^2 \leq \frac{T}{2} \tag{7.7.17}$$

$$(4c + |d|)^2 > \frac{T}{2} \tag{7.7.18}$$

(b) We define $\Omega_T^{2,3}$ to be those $\{(c,d)|c \geq 0\}$ such that

$$d^2 \leq \frac{T}{2} \quad , \quad (4c+d)^2 \leq \frac{T}{2}. \tag{7.7.19}$$

$$\text{If } d(4c+d) < 0 \text{ then } c^2 > \frac{T}{2}. \tag{7.7.20}$$

Note that $\Omega_T^{i,j}$ is in both cases a union of convex sets and

$$\Omega_T^{i,j} = \sqrt{T}\Omega_1^{i,j} \tag{7.7.21}$$

Hence we have the following restatement of [RZ17, Proposition 4.6] restricted to our context

Proposition 7.7.4 ([RZ17, Proposition 4.6]). *For* $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma$*:*

(a) the circles $\gamma(\mathcal{C}_1)$ and $\gamma(\mathcal{C}_2)$ are neighbours in $\mathcal{A}_T$ if and only if $(c_\gamma, d_\gamma) \in \sqrt{T}\Omega_1^{1,2}$.

(b) the circles $\gamma(\mathcal{C}_2)$ and $\gamma(\mathcal{C}_3)$ are neighbours in $\mathcal{A}_T$ if and only if $(c_\gamma, d_\gamma) \in \sqrt{T}\Omega_1^{2,3}$.

The relative gap condition in (7.7.16) can now be written (again following [RZ17, (18) - (20)]):

(a) For $i = 1$ and $j = 2$

$$c\,|d| \geq \frac{T}{s} \tag{7.7.22}$$

(b) For $i = 2$ and $j = 3$

$$|d(4c+d)| \geq \frac{4T}{s} \tag{7.7.23}$$

Thus we come to the same conclusion as Rudnick and Zhang that

$$F^{i,j}_{T,\mathcal{I}}(s) = \frac{1}{T^{\delta_{\widehat{\Gamma}}}}\#\left\{\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \Gamma \mid \gamma\alpha_i, \gamma\alpha_j \in \mathcal{I}, (c_\gamma, d_\gamma) \in \Omega^{i,j}_T(s)\right\} \tag{7.7.24}$$

for $(i,j) = (1,2),(2,3)$, where $\Omega^{i,j}_T(s)$ is defined to be those elements $(c,d) \in \Omega^{i,j}_T$ satisfying (7.7.22) for $(1,2)$ and (7.7.23) for $(2,3)$.

Note that $\Omega^{i,j}_T(s)$ are unions of convex, compact sets, and

$$\Omega^{i,j}_T(s) = \sqrt{T}\Omega^{i,j}_1(s) \tag{7.7.25}$$

7.7.3 Limiting Behaviour

To ease notation and remain consistent with [RZ17] we reparameterise the geodesic flow

$$A := \left\{\begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ 0 & y^{\frac{1}{2}} \end{pmatrix} | y > 0\right\} \tag{7.7.26}$$

and set

$$A_T := \left\{\begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ 0 & y^{\frac{1}{2}} \end{pmatrix} | 0 < y < T\right\}. \tag{7.7.27}$$

Note that this is the *backwards geodesic flow.* The following theorem of Bourgain, Kontorovich and Sarnak concerns counting points in the orbits of general discrete subgroups (i.e as considered in Section 7.1), Γ, in bisectors.

Theorem 7.7.5 ([BKS10])**.** *Consider bounded Borel subsets $\Omega_1 \subset N_-$ and $\Omega_2 \subset K$ such that $\mu^{PS}(\partial(\Omega_1(X_i)) = \nu_i(\partial(\Omega^{-1}(X_i^-))) = 0$, then*

$$\lim_{T\to\infty} \frac{\#(\Gamma \cap \Omega_1 A_T \Omega_2)}{T^{\delta_\Gamma}} = \frac{1}{\delta_\Gamma \cdot |\mathrm{m}^{BMS}|}\mu^{PS}(\Omega_1(X_i))\nu_i(\Omega_2^{-1}(X_i^-)). \tag{7.7.28}$$

Now for a given $\gamma \in \Gamma$ use the Iwasawa decomposition to write

$$\gamma = n_-(x(\gamma))\begin{pmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{7.7.29}$$

Theorem 7.7.5 then allows us to prove

Proposition 7.7.6. *Let $\mathcal{I}$ be an interval, and let $\Omega \subset \{(c,d) \mid c \geq 0\}$ be a bounded, convex, compact subset with piecewise smooth boundary. Moreover suppose that in polar coordinates the region Ω is bounded by two piecewise smooth curves $r_1(\theta) \leq r_2(\theta)$ for $\theta \in [\theta_1, \theta_2]$. Then*

$$\begin{aligned} &\#\left\{\gamma = \left(\begin{smallmatrix} * & * \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \Gamma_\infty\backslash\Gamma \;\middle|\; x(\gamma) \in \mathcal{I},\ (c_\gamma, d_\gamma) \in \sqrt{T}\Omega\right\} \\ &\qquad\sim \frac{T^{\delta_\Gamma}}{\delta_\Gamma\,|\mathrm{m}^{BMS}|}\mu^{PS}(\mathcal{I}(X_i))\int_{\theta_1}^{\theta_2}\left(r_2^{2\delta_\Gamma}(\theta) - r_1^{2\delta_\Gamma}(\theta)\right)d\nu_i(\theta) \end{aligned} \tag{7.7.30}$$

as $T \to \infty$, where $d\nu_i(\theta) = d\nu_i(k(\theta)X_i)$ and we have written γ in N_-AK coordinates as $x(\gamma)a(\gamma)k(\gamma)$.

Proof. The proof is the same as [RZ17, Proof of Proposition 5.3], with the exception that we use Theorem 7.7.5 rather than a more classical counting theorem (due to Good).

First note that using the Iwasawa decomposition of γ, we have $d_\gamma = y^{1/2}\cos\theta$, $c_\gamma = y^{1/2}\sin\theta$. Therefore $(y^{1/2}, \theta)$ give a polar coordinate decomposition of the plane. The rest of the argument follows from a Riemann sum approximation which works equally well when working with ν_i.

Split the interval $I = [\theta_1, \theta_2]$ into separate equally spaced intervals $\{I_i\}_{i=1}^{n}$. Take $\theta_{1,i}^{+}$, and $\theta_{1,i}^{-}$ to be the points in I_i where r_1 is maximised (resp. minimised) and $\theta_{2,i}^{+}$, and $\theta_{2,i}^{-}$ to be the points at which r_2 is maximised (resp. minimised). Now define

$$\begin{aligned} \Omega_n^- &= \bigcup_{i=1}^{n} I_i \times [r_1(\theta_{1,i}^-), r_2(\theta_{2,i}^+)] \\ \Omega_n^+ &= \bigcup_{i=1}^{n} I_i \times [r_1(\theta_{1,i}^+), r_2(\theta_{2,i}^-)]. \end{aligned} \tag{7.7.31}$$

Thus $\Omega_n^- \subseteq \Omega \subseteq \Omega_n^+$ and

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \int_{I_i} \left(r_2^{2\delta_\Gamma}(\theta_{2,i}^+) - r_1^{2\delta_\Gamma}(\theta_{1,i}^-) \right) d\nu_i(\theta) \\ = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{I_i} \left(r_2^{2\delta_\Gamma}(\theta_{2,i}^-) - r_1^{2\delta_\Gamma}(\theta_{1,i}^+) \right) d\nu_i(\theta) \\ = \int_{\theta_1}^{\theta_2} \left(r_2^{2\delta_\Gamma}(\theta) - r_1^{2\delta_\Gamma}(\theta) \right) d\nu_i(\theta). \end{aligned} \tag{7.7.32}$$

For the truncated regions Ω_n^+ and Ω_n^- the proposition follows readily with the observation that in (7.7.28), the fact that the conformal density is evaluated at Ω_2^{-1} simply means that the bounds of integration would be $[-\theta_2, -\theta_1]$. However since our group is symmetric this is equal the integral over $[\theta_1, \theta_2]$. From, since (7.7.30) satisfies finite additivity, the proposition follows.

□

Summarising: provided $s \leq s_0 = 7$ the gap distribution at time T can be written

$$\widehat{F}_{T,\mathcal{I}}(s) = F_{T,\mathcal{I}}^{1,2}(s) + F_{T,\mathcal{I}}^{2,3}(s). \tag{7.7.33}$$

Moreover we can take the limit as $T \to \infty$ and (7.7.16) becomes

$$\widehat{F}_{\mathcal{I}}(s) = F_{\mathcal{I}}^{1,2}(s) + F_{\mathcal{I}}^{2,3}(s) \tag{7.7.34}$$

where, for $(i,j) = (1,2), (2,3)$

$$F_{\mathcal{I}}^{i,j}(s) = \frac{1}{\delta_{\widehat{\Gamma}} |\mathrm{m}^{BMS}|} \mu^{PS}(\mathcal{I}(X_i)) \int_{\theta_1^{i,j}(s)}^{\theta_2^{i,j}(s)} \left(r_2^{i,j}(\theta, s)^{2\delta_{\widehat{\Gamma}}} - r_1^{i,j}(\theta, s)^{2\delta_{\widehat{\Gamma}}} \right) d\nu_i(\theta), \tag{7.7.35}$$

where $r_2^{i,j}(\theta, s)\Big|_{\theta \in [\theta_1^{i,j}(s), \theta_2^{i,j}(s)]}$ and $r_1^{i,j}(\theta, s)\Big|_{\theta \in [\theta_1^{i,j}(s), \theta_2^{i,j}(s)]}$ are the curves in polar coordinates forming the boundary of $\Omega^{i,j}(s)$.

For convenience define the constant

$$\kappa := \frac{1}{\delta_{\widehat{\Gamma}} |\mathrm{m}^{BMS}|} \mu^{PS}(\mathcal{I}(X_i)) \tag{7.7.36}$$

7.7.4 Properties of the Limiting Gap Distribution

In order to extract some properties of the limiting gap distribution we first consider $\Omega_1^{1,2}$ defined by (7.7.17), (7.7.18) and (7.7.22), however since $s < s_0 = 7$, (7.7.18) can be ignored. Hence we have the region (in (c,d)-coordinates):

$$\Omega_1^{1,2}(s) = \left([0, \frac{1}{\sqrt{2}}] \times [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]\right) \cap \left\{(c,d) : c \geq \frac{1}{s|d|}\right\}. \tag{7.7.37}$$

This region is symmetric under reflection across the y axis and since the conformal density in (7.7.35) is invariant under this reflection we can consider

$$\tilde{\Omega}_1^{1,2}(s) = \left([0, \frac{1}{\sqrt{2}}] \times [0, \frac{1}{\sqrt{2}}]\right) \cap \left\{(c,d) : c \geq \frac{1}{s|d|}\right\} \tag{7.7.38}$$

instead, and the only difference will be a factor of 2.

Regarding $\Omega_1^{2,3}(s)$, from (7.7.19) we know that $\Omega_1^{2,3}$ is a subset of the triangle

$$-\frac{1}{\sqrt{2}} \leq d \leq \frac{1}{\sqrt{2}} \quad , \quad 0 \leq c < \frac{1}{4\sqrt{2}} - d \tag{7.7.39}$$

Moreover (7.7.20) implies that when $d < 0$, if $c > -\frac{d}{4}$ then $c > \frac{1}{\sqrt{2}}$, thus $\Omega_1^{2,3} = T_1 \cup T_2$ where

$$T_1 := \left\{(c,d) : c, d \geq 0 \;,\; c < \frac{1}{4\sqrt{2}} - d\right\} \tag{7.7.40}$$

$$T_2 := \left\{(c,d) : c \geq 0 \;,\; -\frac{1}{\sqrt{2}} \leq d \leq 0 \;,\; c \leq -\frac{d}{4}\right\}. \tag{7.7.41}$$

Now looking at the condition imposed by (7.7.23), it is straightforawd to see that, for $s < 7$, $\Omega_1^{2,3}(s)$ does not intersect T_2. Hence, for $s < s_0 < 7$:

$$\Omega_1^{2,3}(s) = \left\{(c,d) \in T_1 \;:\; c \leq \frac{1}{sd} - \frac{d}{4}\right\}. \tag{7.7.42}$$

So far we have established that

$$\widehat{F}(s) = \kappa\mu(\Omega_1^{2,3}(s)) + 2\kappa\mu(\tilde{\Omega}_1^{1,2}(s)) \tag{7.7.43}$$

where, for a general set $A = \{(r\cos\theta, r\sin\theta) : r \in [r_1^A(\theta), r_2^A(\theta)], \theta \in [\theta_1^A, \theta_2^A]\}$,

$$\mu(A) := \int_{\theta_1^A}^{\theta_2^A} \left(r_2^A(\theta)^{2\delta_\Gamma} - r_1^A(\theta)^{2\delta_\Gamma}\right) d\nu_i(\theta). \tag{7.7.44}$$

Thus $\widehat{F}(s)$ is explicitly calculated in terms of the Patterson Sullivan density ν_i (5.5.5). Unfortunately this measure is not itself explicit (in that it is defined as the weak limit of a sequence of measures). However it does lend itself to simulations (which we will not do here) and analysis:

Proposition 7.7.7. *$\widehat{F}_{\mathcal{I}}(s) = 0$ for all $s < 2$ for any $\mathcal{I}$. Moreover,* all *gaps are larger than* 2.

This is a form of level repulsion and follows from the definitions of $\tilde{\Omega}_1^{1,2}(s)$ and $\Omega_1^{2,3}(s)$ and (7.7.43). Indeed $\tilde{\Omega}_1^{1,2}(s)$ is empty for $s < 2$ and $\Omega_1^{2,3}(s)$ is empty for $s < 4$.

ν_i is a fractal measure supported on the limit set. Hence, looking at (7.7.44), if neither θ_1^A nor θ_2^A is in $\mathcal{L}(\Gamma)$ (the support of ν_i). Then the derivative of $\widehat{F}$ will be easy to calculate:

Proposition 7.7.8. *Suppose $\mathcal{S} \subset (2, s_0)$ is a connected subset such that for all $s \in \mathcal{S}$, $\theta_1^{i,j}(s)$ and $\theta_2^{i,j}(s) \notin \mathcal{L}(\Gamma)$ for $(i,j) = (1,2)$ or $(2,3)$, then*

$$P(s) = \widehat{F}'(s) = \frac{C_{\mathcal{S}}}{s^{\delta_\Gamma + 1}}, \tag{7.7.45}$$

where $0 \leq C_{\mathcal{S}} < \infty$ depends on the region $\mathcal{S}$ but not on $s \in \mathcal{S}$ and is explicit.

Proof. Let $s_1 = \inf\{s \in \mathcal{S}\}$, in which case, for $s \in \mathcal{S}$ we separate the integral in (7.7.43) and write

$$\begin{aligned}
\widehat{F}(s) &= \kappa \int_{\theta_1^{2,3}(s)}^{\theta_2^{2,3}(s)} \left(r_2^{2,3}(\theta,s)^{2\delta_{\widehat{\Gamma}}} - r_2^{2,3}(\theta,s)^{2\delta_{\widehat{\Gamma}}}\right) d\nu_i(\theta) + 2\kappa \int_{\theta_1^{1,2}(s)}^{\theta_2^{1,2}(s)} \left(r_2^{1,2}(\theta,s)^{2\delta_{\widehat{\Gamma}}} - r_2^{1,2}(\theta,s)^{2\delta_{\widehat{\Gamma}}}\right) d\nu_i(\theta) \\
&= \kappa \int_{\theta_1^{2,3}(s_1)}^{\theta_2^{2,3}(s_1)} \left(r_2^{2,3}(\theta)^{2\delta_{\widehat{\Gamma}}} - r_2^{2,3}(\theta,s)^{2\delta_{\widehat{\Gamma}}}\right) d\nu_i(\theta) + 2\kappa \int_{\theta_1^{1,2}(s_1)}^{\theta_2^{1,2}(s_1)} \left(r_2^{1,2}(\theta)^{2\delta_{\widehat{\Gamma}}} - r_2^{1,2}(\theta,s)^{2\delta_{\widehat{\Gamma}}}\right) d\nu_i(\theta) \\
&\quad + R(s,\mathcal{S})
\end{aligned}$$

where we have noted that (by (7.7.38) and (7.7.42)), r_2 is independent of s . In fact, since on $\mathcal{S}$, $\theta_1^{i,j}(s)$ and $\theta_2^{i,j}(s)$ are outside $\mathcal{L}(\Gamma)$, $R(s,\mathcal{S})$ is 0 (as the measure is supported away from the range of integration). Hence, taking a derivative:

$$P(s) = -\kappa \int_{\theta_1^{2,3}(s_1)}^{\theta_2^{2,3}(s_1)} \frac{dr_1^{2,3}(\theta,s)^{2\delta}}{ds} d\nu_i(\theta) - 2\kappa \int_{\theta_1^{1,2}(s_1)}^{\theta_2^{1,2}(s_1)} \frac{dr_1^{1,2}(\theta,s)^{2\delta}}{ds} d\nu_i(\theta). \tag{7.7.46}$$

Moreover, for $s < s_0$ we have that

$$r_1^{1,2}(\theta,s) = \frac{1}{\sqrt{s}}\sqrt{\frac{1}{\cos\theta\sin\theta}} \quad , \quad r_1^{2,3}(\theta,s) = \frac{1}{\sqrt{s}}\sqrt{\frac{1}{(\sin\theta\cos\theta + \frac{\cos^2\theta}{4})}}. \tag{7.7.47}$$

Therefore, for $s \in \mathcal{S}$

$$P(s) = \frac{\kappa}{s^{\delta_{\widehat{\Gamma}}+1}} \left(\int_{\theta_1^{2,3}(s_1)}^{\theta_2^{2,3}(s_1)} \left(\frac{1}{(\sin\theta\cos\theta + \frac{\cos^2\theta}{4})}\right)^{\delta_{\widehat{\Gamma}}} d\nu_i(\theta) + 2\int_{\theta_1^{1,2}(s_1)}^{\theta_2^{1,2}(s_1)} \left(\frac{1}{\cos\theta\sin\theta}\right)^{\delta_{\widehat{\Gamma}}} d\nu_i(\theta) \right). \tag{7.7.48}$$

□

The final analytic property we calculate for $\widehat{F}$ is the following Lipschitz condition:

Proposition 7.7.9. *$\widehat{F}$ is Lipschitz in a neighbourhood of s whenever $s \in [0,4)$*

$$\left|\widehat{F}(s) - \widehat{F}(s+x)\right| \leq C_s x \tag{7.7.49}$$

for some constant $C_s < \infty$.

Proof. $\widehat{F}$ is 0 on $[0,2)$. Moreover Proposition 7.7.8 implies the $\widehat{F}$ is differentiable when both $\theta_1^{1,2}$ and $\theta_2^{1,2}$ are outside $\mathcal{L}(\widehat{\Gamma})$. Hence we only need to worry about when $\theta_1^{1,2}(s)$ or $\theta_2^{1,2}(s)$ is a parabolic fixed point (since parabolic points are dense in the limit set).

For any $2 \leq s < 4$ such that $\theta_1^{1,2}(s)$ or $\theta_2^{1,2}(s)$ is a parabolic fixed point:

$$\begin{aligned}
\left|\widehat{F}(s) - \widehat{F}(s+x)\right| \leq C \Bigg| &\int_{\theta_2^{1,2}(s)}^{\theta_2^{1,2}(s+x)} \left(r_2^{1,2}(\theta)^{2\delta_{\widehat{\Gamma}}} - r_1^{1,2}(\theta,s)^{2\delta_{\widehat{\Gamma}}}\right) d\nu_i(\theta) \\
&+ \int_{\theta_1^{1,2}(s+x)}^{\theta_1^{1,2}(s)} \left(r_2^{1,2}(\theta)^{2\delta_{\widehat{\Gamma}}} - r_1^{1,2}(\theta,s)^{2\delta_{\widehat{\Gamma}}}\right) d\nu_i(\theta)\Bigg|
\end{aligned} \tag{7.7.50}$$

Plugging in the formula for $r_2^{1,2}$ and $r_1^{1,2}$ and using Corollary 7.2.3 gives that the first term on the right hand side of (7.7.50) is less than

$$\leq C_s \left| \int_{\theta_2^{1,2}(s)}^{\theta_2^{1,2}(s+x)} \theta^{2\delta_{\widehat{\Gamma}}-2} \left(\left(\frac{1/\sqrt{2}}{\sin\theta} \right)^{2\delta_{\widehat{\Gamma}}} - \left(\frac{1}{(s+x)\cos\theta\sin\theta} \right)^{\delta_{\widehat{\Gamma}}} \right) d\theta \right| \tag{7.7.51}$$

in the range with which we are concerned we can bound this integral (by adjusting the constant) by

$$\leq C_s \int_{\theta_2^{1,2}(s)}^{\theta_2^{1,2}(s+x)} \theta^{2\delta_{\widehat{\Gamma}}-2} d\theta. \tag{7.7.52}$$

Evaluating the integral and performing the same analysis on the other term in (7.7.50) gives

$$\left| \widehat{F}(s) - \widehat{F}(s+x) \right| \leq C_s \left(\theta_2^{1,2}(s+x)^{2\delta_{\widehat{\Gamma}}-1} - \theta_2^{1,2}(s)^{2\delta_{\widehat{\Gamma}}-1} \right) + C_s \left(\theta_1^{1,2}(s)^{2\delta_{\widehat{\Gamma}}-1} - \theta_1^{1,2}(s+x)^{2\delta_{\widehat{\Gamma}}-1} \right). \tag{7.7.53}$$

Inserting the definition of $\theta_2^{1,2}$ and $\theta_1^{1,2}$ then gives

$$\left| \widehat{F}(s) - \widehat{F}(s+x) \right| \leq C_s \left(\tan^{-1}(s+x)^{2\delta_{\widehat{\Gamma}}-1} - \tan^{-1}(s)^{2\delta_{\widehat{\Gamma}}-1} \right) + C_s \left(\cot^{-1}(s)^{2\delta_{\widehat{\Gamma}}-1} - \cot^{-1}(s+x)^{2\delta_{\widehat{\Gamma}}-1} \right). \tag{7.7.54}$$

From here, Taylor expanding gives

$$\left| \widehat{F}(s) - \widehat{F}(s+x) \right| \leq C \left| \left(\frac{\pi}{4} + \frac{x}{4} \right)^{2\delta_{\widehat{\Gamma}}-1} - \left(\frac{\pi}{4} \right)^{2\delta_{\widehat{\Gamma}}-1} \right| + C \left| \left(\frac{\pi}{4} \right)^{2\delta_{\widehat{\Gamma}}-1} - \left(\frac{\pi}{4} - \frac{x}{4} \right)^{2\delta_{\widehat{\Gamma}}-1} \right|. \tag{7.7.55}$$

Here, expanding again gives us that $\widehat{F}$ is Lipschitz.

□

7.8 Gauss-Like Measure

As in the previous section this section is restricted to the example $\widehat{\Gamma}$. The goal for this section is to derive and study the probability measure

$$m^0(E) = C_0 \int_E \int_{-2}^{2} \frac{d\mu^{PS}(x)}{|xy-1|^{2\delta_{\widehat{\Gamma}}}} d\mu^{PS}(y). \tag{7.8.1}$$

where E is a Borel set in $\mathcal{L}(\widehat{\Gamma}) \cap (-2,2)$, and C_0 is a normalising constant. In particular we show that this measure is invariant and ergodic for the Gauss map. Then, as a corollary of this ergodicity, we are able to show that the Gauss-Kuzmin statistics on $\mathcal{Q}_4$ converge to an explicit function.

Note that m^0 is equivalent to the Patterson-Sullivan measure (and thus the Hausdorff measure for the fractal) up to a bounded density. It should also be noted that the density in (7.8.1) is a normalised eigenfunction for the transfer operator associated to the Gauss map. We shall avoid this transfer operator approach here, however it is a promising avenue for later research.

7.8.1 Setup

In [Ser85] Series, for the modular group, shows that one can encode the endpoints of geodesics by a 'cutting sequence' which generates the continued fraction expansions of the endpoints. Moreover she identifies a cross-section of the unit tangent bundle such that the return map to this cross-section corresponds to the (classical) Gauss map on the end point. As an application of this, she shows that the Gauss measure is simply a projection of the Haar measure onto these end points. Thus, because the Haar measure is ergodic for the geodesic flow, the Gauss measure is ergodic for the Gauss map. The goal for this subsection is to construct the analogous measure in our context (for $\widehat{\Gamma}$). To do this we will project the BMS measure in the same way and show that the resulting measure is ergodic for the Gauss map (for $\widehat{\Gamma}$). In the end we will only be working with this measure, however for those interested, in the Appendix, we show how to construct the analogous cutting sequences and cross-section in our context (we omit the formal proofs concerning the commuting diagrams as we do not use them and the details are the same as [Ser85]).

Throughout this section let $(-2,2)^* = (-2,2) \setminus \{0\}$. Consider the restriction of Gauss map to the limit set, $\mathcal{L}(\widehat{\Gamma}) = \overline{\mathcal{Q}_4}$ (where $\overline{\mathcal{Q}_4}$ denotes the closure):

$$\begin{aligned} T : &\mathcal{L}(\widehat{\Gamma}) \to \mathcal{L}(\widehat{\Gamma}) \\ &[0; a_1, a_2, \dots] \mapsto [0; a_2, \dots] \end{aligned} \tag{7.8.2}$$

and its inverse

$$T^{-1}([0; a_1, \dots, a_{n-1}]) = \bigcup_{k \in 4\mathbb{Z}^*} [0; k, a_1, \dots, a_{n-1}]. \tag{7.8.3}$$

The σ-algebra associated to this Gauss map is now the Borel σ-algebra on $\mathbb{R}$ intersected with $\mathcal{L}(\widehat{\Gamma})$. The goal is now to take the Bowen-Margulis-Sullivan measure and project it to a measure on $(-2,2)$. We choose the BMS measure as it is invariant and ergodic under the geodesic flow. Thus after projecting we are left with a measure invariant and ergodic under the Gauss map. The following lemma gives a parameterisation of the BMS measure used in Sullivan's work [Sul79].

Lemma 7.8.1. *For $u \in T^1(\mathbb{H})$ let z denote the Euclidean midpoint of the geodesic containing u and $t := \beta_{u^-}(z,u)$ (thus t is the arclength from z to u). Then*

$$d\mathrm{m}^{BMS}(u) = \frac{1}{|u^+ - u^-|^{2\delta_\Gamma}} d\mu^{PS}(u^-) d\mu^{PS}(u^+) dt. \tag{7.8.4}$$

Remark. Note this Lemma is not specific to the subgroup $\widehat{\Gamma}$ and holds for any Bowen-Margulis-Sullivan measure associated to a subgroup considered in this paper.

Proof. First (recall s from the definition of m^{BMS} - Chapter 5, (5.5.6)) note

$$\begin{aligned} s &:= \beta_{u^-}(i,u) \\ &= \beta_{u^-}(i,z) + \beta_{u^-}(z,u) \\ &= \beta_{u^-}(i,z) + t \\ &= \beta_{u^-}(i, i+u^-) + \beta_{u^-}(i+u^-, z) + t \end{aligned} \tag{7.8.5}$$

Now using the definition of the Busemann function, we note that $\beta_{u^-}(i+u^-, z)$, is the hyperbolic distance (along the vertical geodesic at u^-) between the horoball of height 1 based at u^- and the horoball of height $|u^+ - u^-|$. Thus

$$s = t + \beta_{u^-}(i, i+u^-) + \ln\left|u^+ - u^-\right|. \tag{7.8.6}$$

Similarly

$$\beta_{u^+}(i,u) = -t + \beta_{u^+}(i, i+u^+) + \ln\left|u^+ - u^-\right|. \tag{7.8.7}$$

Therefore, writing out the definition of the Burger Roblin measure and inserting (7.8.6) and (7.8.7):

$$
\begin{aligned}
\mathrm{m}^{BMS}(u) &:= e^{\delta_\Gamma s} e^{\delta_\Gamma \beta_{u^+}(i,u)} d\nu_i(u^-) d\nu_i(u^+) ds \\
&= \frac{1}{|u^+ - u^-|^{2\delta_\Gamma}} (e^{\delta_\Gamma \beta_{u^-}(i,i+u^-)} d\nu_i(u^-))(e^{\delta_\Gamma \beta_{u^+}(i,i+u^+)}) d\nu_i(u^+)) dt \\
&= \frac{1}{|u^+ - u^-|^{2\delta_\Gamma}} d\mu^{PS}(u^-) d\mu^{PS}(u^+) dt
\end{aligned}
\tag{7.8.8}
$$

where in the last line we insert the definition of μ^{PS}.

□

To derive the Gauss-type measure (similarly to [Ser85] for the classical Gauss measure) we restrict the BMS measure to the u^- coordinate. Integrating over the u^+ coordinate in $(-2,2)$ gives

$$
\int_{-2}^{2} \frac{d\mu^{PS}(u^+)}{|u^+ - u^-|^{2\delta_{\hat\Gamma}}}. \tag{7.8.9}
$$

Thus, for a set $E \subset (-\infty,-2) \cup (\infty, 2)$

$$
\int_E \int_{-2}^{2} \frac{d\mu^{PS}(x)}{|x-y|^{2\delta_{\hat\Gamma}}} d\mu^{PS}(y) \tag{7.8.10}
$$

is a measure. Changing coordinates and using that $d\mu^{PS}(1/y) = y^{-2\delta_{\hat\Gamma}} d\mu^{PS}(y)$ (this follows from Chapter 5, (5.5.5) and a calculation using the Busemann function) gives, for any set $E \subset (-2,2)^*$

$$
m^0(E) := C_0 \int_E \int_{-2}^{2} \frac{d\mu^{PS}(x)}{|xy-1|^{2\delta_{\hat\Gamma}}} d\mu^{PS}(y), \tag{7.8.11}
$$

where C_0 is a normalising constant. In the next section we show that this measure is T-invariant and ergodic.

7.8.2 Invariance and Ergodicity

Theorem 7.8.2. *On $(-2,2)^*$, m^0 is T-invariant and ergodic.*

Proof. To prove invariance, let $E \subset (-2,2)^*$ and consider the measure of its preimage

$$
m^0(T^{-1}(E)) = C_0 \int_{T^{-1}(E)} \int_{-2}^{2} \frac{d\mu^{PS}(x)}{|xy-1|^{2\delta_{\hat\Gamma}}} d\mu^{PS}(y)
$$

Plugging in the definition of $T^{-1}(E)$ and changing variables ($d\mu^{PS}(1/y) = y^{-2\delta_{\hat\Gamma}} d\mu^{PS}(y)$) together with the fact that the Patterson-Sullivan measure is invariant under translation by $4n$ gives

$$
\begin{aligned}
&= C_0 \sum_{n\in\mathbb{Z}^*} \int_{E+4n} \left(\int_{-2}^{2} \frac{d\mu^{PS}(x)}{|y-x|^{2\delta_{\hat\Gamma}}} \right) d\mu^{PS}(y) \\
&= C_0 \int_E \sum_{n\in\mathbb{Z}^*} \int_{-2}^{2} \left(\frac{d\mu^{PS}(x)}{|y-x-4n|^{2\delta_{\hat\Gamma}}} \right) d\mu^{PS}(y).
\end{aligned}
\tag{7.8.12}
$$

If we now change the x variable to $x+4n$ this gives

$$
= C_0 \int_E \int_{(-\infty,-2)\cup(2,\infty)} \frac{d\mu^{PS}(x)}{|y-x|^{2\delta_{\hat\Gamma}}} d\mu^{PS}(y).
$$

Hence applying the change of variables $x \mapsto x^{-1}$ gives

$$= C_0 \int_E \int_{-2}^{2} \frac{d\mu^{PS}(x)}{|xy-1|^{2\delta_{\widehat{\Gamma}}}} d\mu^{PS}(y) = m^0(E).$$

This new measure is ergodic for the Gauss map because the BMS is ergodic for the geodesic flow. However to see this directly note first that the density

$$\rho(y) = \int_{-2}^{2} \frac{d\mu^{PS}(x)}{|xy-1|^{2\delta_{\widehat{\Gamma}}}}$$

is bounded on $\mathcal{L}(\widehat{\Gamma})$. Given $a_1, \ldots, a_n$ and writing $\frac{p_i}{q_i} = [0; a_1, \ldots, a_i]$, define the cylinder sets for $\frac{p_n}{q_n} \in \mathcal{Q}_4$:

$$\Delta_n := \left\{ \psi_n(t) := \frac{4p_n + p_{n-1}t}{4q_n + q_{n-1}t} : 0 \le t \le 1 \right\}. \tag{7.8.13}$$

Note that the sets $\Delta_n \cap \mathcal{L}(\widehat{\Gamma})$ generate the Borel σ-algebra on $\mathcal{L}(\widehat{\Gamma})$.

First we note that the measure μ^{PS} is 0 if and only if ν_i is 0, hence for what follows we can work with ν_i instead of μ^{PS}. Hence note that for any $n > 0$, for $s < t \in [0,1]$, there exists a $\gamma \in \widehat{\Gamma}$ such that

$$\begin{aligned} \mu^{PS}\left(T^{-n}([\frac{s}{4}, \frac{t}{4}))\middle|\, \Delta_n \right) &\asymp \nu_i \left(T^{-n}([\frac{s}{4}, \frac{t}{4}))\middle|\, \Delta_n \right) \\ &= \frac{\nu_i(\gamma[\frac{s}{4}, \frac{t}{4}))}{\nu_i(\gamma[0, \frac{1}{4}))} \\ &= \frac{\nu_i([\frac{s}{4}, \frac{t}{4}))}{\nu_i([0, \frac{1}{4}))} \end{aligned} \tag{7.8.14}$$

Therefore, as the density for m^0 with respect to the PS measure is bounded above and below, for any $A \subset \mathcal{L}(\widehat{\Gamma}) \cap (-2,2)^*$ measurable

$$\frac{1}{C} m^0(A) \le m^0(T^{-n}(A)|\, \Delta_n) \le C m^0(A). \tag{7.8.15}$$

To conclude, assume A is T-invariant, then $\frac{1}{C} m^0(A) \le m^0(A|\, \Delta_n)$. If $m^0(A) > 0$, then $\frac{1}{C} m^0(\Delta_n) \le m^0(\Delta_n | A)$. Therefore, since the cylinders Δ_n generate the Borel σ-algebra of measurable sets, we have that

$$\frac{1}{C} m^0(B) \le m^0(B|A)$$

for all B measurable. Setting $B = A^c$ implies that $m^0(A^c) = 0$ and $m^0(A) = 1$. Hence m^0 is ergodic.

□

7.8.3 Gauss-Kuzmin Statistics

Given a point $x = [0; a_1, a_2, \ldots] \in \mathbb{R}$ ($a_i \in \mathbb{N}$), Gauss considered the following problem (further studied by Kuzmin in 1928): let $\widetilde{P}_{n,k}(x) = \frac{\#(k,n)}{n}$ where $\#(k,n)$ is the number of $a_i = k$ with $i \le n$. Does there exist a limiting distribution for $\widetilde{P}_{n,k}(x)$? Using the ergodicity of the Gauss measure it is fairly simple to show that for Lebesgue-almost every x

$$\lim_{n \to \infty} \widetilde{P}_{n,k}(x) = \frac{1}{\ln(2)} \ln \left(1 + \frac{1}{k(k+2)} \right). \tag{7.8.16}$$

This distribution is now known as Gauss-Kuzmin statistics. For a detailed description of the original problem and history see [Khi03, Section 15]. The problem has an analogue in our setting.

For $[0; a_1, a_2, ...] = x \in \overline{\mathcal{Q}_4} \cap (-2,2)$ define $\widehat{P}_{n,k}(x) = \frac{\#(k,n)}{n}$ where $\#(k,n)$ is the number of a_i equal k for $i \leq n$. For simplicity of notation we assume $k > 0$. In that case, writing

$$\widehat{P}_{n,k}(x) = \frac{1}{n} \sum_{s=0}^{n-1} \chi_{(\frac{1}{k+4}, \frac{1}{k}]}(T^s x) \tag{7.8.17}$$

and applying the Birkhoff ergodic theorem for m^0 imply:

Theorem 7.8.3. *For every positive integer k and μ^{PS}-almost every $x = [0; a_1, \dots] \in \overline{\mathcal{Q}_4} \cap (-2,2)$*

$$\widehat{P}_k(x) = \lim_{n\to\infty} \widehat{P}_{n,k}(x) = m^0\left(\left(\frac{1}{k+4}, \frac{1}{k}\right]\right). \tag{7.8.18}$$

Once more, we note that, given a set $A \subset \mathcal{L}(\widehat{\Gamma})$, it can be shown that $m^0(A) \asymp \mu^{PS}(A) \asymp H^{\delta_{\widehat{\Gamma}}}(A)$ where $H^{\delta_{\widehat{\Gamma}}}$ denotes the Hausdorff measure on $\mathcal{L}(\widehat{\Gamma})$. Hence Theorem 7.8.3 gives a rather fundamental property of the fractal $\mathcal{L}(\widehat{\Gamma})$ in terms of the Hausdorff measure.

Appendix to Chapter 7- Cutting Sequences for $\widehat{\Gamma}$

Working with $\widehat{\Gamma}$ the goal of this section is to show that, given a geodesic with right end point in $(-2,2) \cap \mathcal{L}(\widehat{\Gamma})$ (and left end point in $(-\infty,-2)$) there is a correspondence between the way this geodesic cuts the boundaries of fundamental domains and the continued fraction expansion of the end point. This section is analogous to the Bowen-Series coding for geodesics in $\mathrm{PSL}(2,\mathbb{R})/\mathrm{PSL}(2,\mathbb{Z})$.

Let $\xi \in (-2,2) \cap \mathcal{L}(\widehat{\Gamma})$ and let γ be any geodesic whose right endpoint is ξ and which intersects the line $x = -2$. As this geodesic moves from left to right, it will cut (bisect) each fundamental domain. Each fundamental domain has two funnels and a cusp. Thus the geodesic will separate one of the three from the others. If the geodesic separates a cusp we write a c. If it separates a funnel we write an l or an r depending on whether the funnel is to the left or right of the geodesic. See Figure 7.6.

It is easy to see that the first term in the sequence will always be r and the next term will be l/r after that there will be a sequence of c's followed by the same l/r. Thus we end up with a sequence of the form

$$\xi \mapsto r, q_0, c^{\alpha_0}, q_0, q_1, c^{\alpha_1}, q_1, q_2, c^{\alpha_2}, q_2 \dots \tag{7.8.19}$$

(the sequence is finite if the geodesic ends in a cusp) where $q_i = l$, or r and $\alpha_i \geq 0$. With that it is fairly easy to see that

$$\xi = [0; (-1)^{\eta_0} 4(\alpha_0 + 1), (-1)^{\eta_1} 4(\alpha_1 + 1), \dots] \tag{7.8.20}$$

where

$$\eta_i = \begin{cases} 0 & \text{if } q_i = l \\ 1 & \text{if } q_i = r \end{cases}. \tag{7.8.21}$$

Thus there is a correspondence between such sequences and geodesics with end points in $(-2,2)$.

To understand how the Gauss map acts on a point, we need to identify a particular cross-section in $T^1(\Gamma\backslash\mathbb{H})$. Consider the fundamental domain above i and the semi-circular arc centred at 0 of radius 1, which we call S - this arc forms part of the boundary of the fundamental domain. Given a geodesic

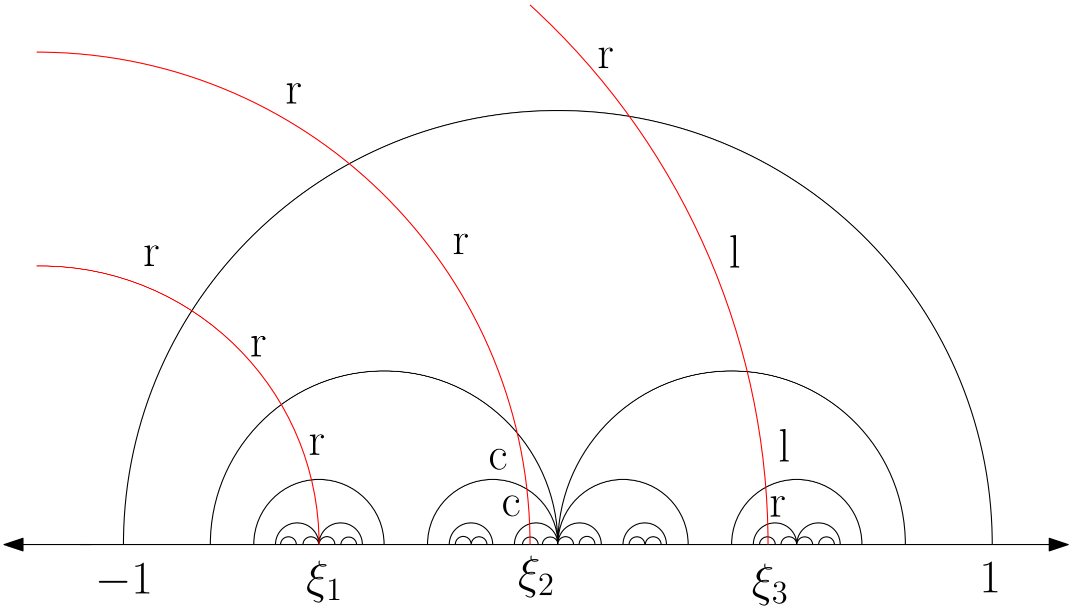

Figure 7.6: In this diagram we show the cutting sequence for 3 different points ξ_1, ξ_2, ξ_3. For ξ_2, first a funnel is cut off to the *right* of the geodesic, then again a funnel is cut off to the *right*, then a *cusp* is cut off and then another *cusp*. Thus the first 4 terms in the cutting sequence are r, r, c, c.

γ whose left end point is in $(-\infty, -2) \cap \mathcal{L}(\widehat{\Gamma})$ and whose right endpoint is in $(-2, 2) \cap \mathcal{L}(\widehat{\Gamma})$ and a point $x \in \gamma \cap S$, we insert x into the cutting sequence of γ, at its position in the sequence of fundamental domains, resulting in a sequence of the form for example:

$$r, l, c^{\alpha_0}, l, l, c^{\alpha_1} l, x, r, c^{\alpha_2}, \ldots \tag{7.8.22}$$

We say a cutting sequence *changes type* at x if x lies between a q_i and q_{i+1}. Note that x must lie before an l or an r

With that, the cross-section $\mathcal{C} \subset T^1(\Gamma \backslash \mathbb{H})$ are those points, based at $x \in S$ pointed along geodesics whose cutting sequence changes type at x. In that case, the return map for the geodesic flow to this cross-section corresponds to the Gauss map acting on the end point. For a more formal discussion for the modular group (however the same details apply here) see [Ser85].

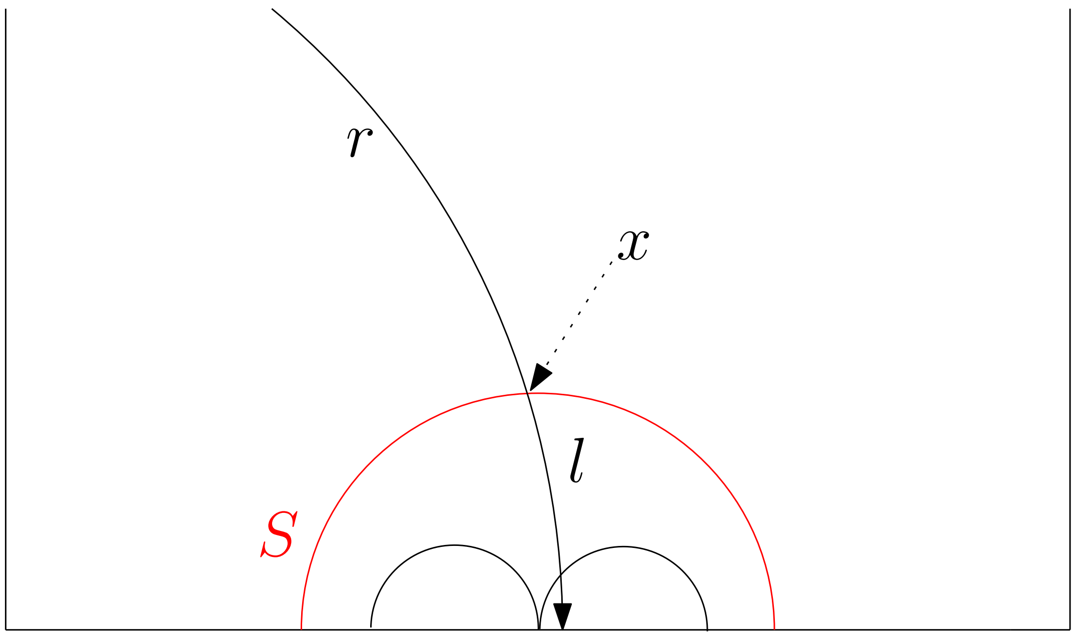

Figure 7.7: In this diagram we show a geodesic and a point $x \in S \cap \gamma$ such that the cutting sequence for γ changes type at x. This is because the cutting sequence with x inserted will read $..., r, x, l, ...$.

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