Pair Correlation of the Fractional Parts of αn^θ

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Abstract

Fix $\alpha, \theta > 0$, and consider the sequence $(\alpha n^\theta \bmod 1)_{n\geq 1}$. Since the seminal work of Rudnick–Sarnak (1998), and due to the Berry–Tabor conjecture in quantum chaos, the fine-scale properties of these dilated mononomial sequences have been intensively studied. In this paper we show that for $\theta \leq 1/3$, and $\alpha > 0$, the pair correlation function is Poissonian. While (for a given $\theta \neq 1$) this strong pseudo-randomness property has been proven for almost all values of α, there are next-to-no instances where this has been proven for explicit α. Our result holds for all $\alpha > 0$ and relies solely on classical Fourier analytic techniques. This addresses (in the sharpest possible way) a problem posed by Aistleitner–El-Baz–Munsch (2021).

1 Introduction

Let $x = (x_n)_{n\geq 1}$ be a sequence on the unit interval $[0, 1)$. The following function measures the correlation between points in the initial segment $\{x_n : n \leq N\}$ on the scale of the mean spacing, $1/N$. That is, define the *pair correlation function* of the sequence x by

$$R(x, N, f) := \frac{1}{N} \sum_{i\neq j\leq N} \sum_{k\in\mathbb{Z}} f(N(x_i - x_j + k)), \tag{1.1}$$

where $f \in \mathcal{C}_c^\infty(\mathbb{R})$ is a compactly supported, C^∞-function. Here f plays the role of a smooth alternative to the indicator function of $[-s/N, s/N]$ for some $s > 0$. The sequence x is said to have *Poissonian pair correlation* if the pair correlation function converges to the expectation of f as $N \to \infty$, just as one would expect for uniformly distributed and independent random variables. That is, if for all $f \in \mathcal{C}_c^\infty(\mathbb{R})$

$$\lim_{N\to\infty} R(x, N, f) = \int_{\mathbb{R}} f(t) \, \mathrm{d}t. \tag{1.2}$$

The notion of Poissonian pair correlation defines a strong measure of pseudo-randomness and is a basic concept in quantum chaos. Unsurprisingly, various efforts have been made [RS98, BZ00, RZ02, MS03, HB10, ALL17] to study the pair correlation function of monomial sequences

$$(\alpha n^\theta \bmod 1)_{n\geq 1}, \tag{1.3}$$

where $\theta > 0$ and $\alpha > 0$. However, little progress has been made to verify that the pair correlation of such monomial sequences is Poissonian (under explicit conditions on α, θ). We present the state of the art for (1.3) in Section 1.1. In this paper we prove the first general and explicit result showing that such monomial sequences exhibit Poissonian pair correlation. Namely,

Theorem 1. *If $\theta \in (0, 14/41)$ and $\alpha > 0$, then (1.3) has Poissonian pair correlation.*

Remark.

1. Our method applies to higher level correlations, although this generalisation is not straightforward as it requires a genuinely multidimensional approach (see Section 1.3). Moreover, the only arithmetic input of our method are exponential sum bounds. Thus, with some modification, the method can be extended to more general sequences satisfying certain growth conditions. We intend to address these in a forthcoming paper.
2. The method of proof allows one to show that the pair correlation function converges to $\int_{\mathbb{R}} f(x)\,\mathrm{d}x$ with a polynomially decaying error in N which is uniform for all α in a fixed compact interval.
3. When $\theta = 1/3$ and $\alpha^3 \in \mathbb{Q}$, the triple correlation is not Poissonian (because of the cubes n^3). Thus, Theorem 1 gives an example of a sequence whose pair correlation is Poissonian, but whose triple correlation is not.

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Organization of the Paper: Subsection 1.1 presents a brief history of these monomial sequences. Subsection 1.2 sketches the proof of Theorem 1, and Subsection 1.3 provides a heuristic argument which indicates the limitations of our method. In Section 2 we collect lemmata, reducing matters to bounding certain exponential sums. Finally, in Section 3 we prove Theorem 1.

1.1 Background

The study of monomial sequences dates back to Weyl [Wey16] who used them in his study of uniform distribution (see [KN74] or [DT06]). More recently, there has been renewed interest in these sequences. In part, this is due to the well-known Berry–Tabor conjecture [BT77] which hypothesizes a link between the pseudo-randomness properties of energy levels, and dynamics of quantum systems. For more details, see either of the following review papers [Mar00, Rud08].

The holy grail of this field is to find circumstances for which a sequence has Poissonian gap statistics. That is, consider the distribution of gaps between neighboring first N elements of sequence – scaled to have average 1 – then we say the sequence exhibits Poissonian gap statistics if this (finite) distribution converges to the exponential distribution, as one would expect for independent random variables. While the aforementioned behavior is conjectured in many instances, it is truly challenging to prove. Thus, mathematicians have turned to weaker measures of pseudo-randomness. In particular, there has been a lot of recent work on the pair correlation. Indeed, if one could show that the m-level correlation converges to the expected value for independent random variables (for every $m \geq 2$) then, by the method of moments, one can infer that the sequence has Poissonian gap statistics.

If we consider the random variable counting the number of sequence elements in a randomly shifted set of size comparable to $1/N$, then the m-level correlations arise from the moments of this variable. Thus, the m-level correlations are natural measures of pseudo-randomness in their own right. We refer to [Mar07] for further discussion.

1.1.1 Pair correlation of deterministic sequences

The few deterministic sequences whose pair correlation functions are known to be Poissonian either require the presence of particularly strong arithmetic structure, or tools from homogeneous dynamics to apply. An example of the former is the work of Kurlberg and Rudnick [KR99] on the (appropriately normalised) spacing of the quadratic residues of a highly composite modulus. In fact, they show that the gap statistics are Poissonian. However this setting requires the use of arithmetic tools which cannot be relied on in our situation.

On the homogeneous dynamics side, Elkies and McMullen [EM04] established a remarkable link between (1.3), for $(\theta, \alpha) = (1/2, 1)$, and flows on the modular surface $\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SL}_2(\mathbb{Z})$. They used this connection, and tools from homogeneous dynamics, to establish that the corresponding gap distribution is *not* Poissonian. Surprisingly, El-Baz, Marklof, and Vinogradov [EBMV15] then exploited said relationship further to show that, if one removed the squares, the pair correlation *is Poissonian*. However, the connection to homogeneous dynamics requires a particular scaling property which only holds when $\theta = 1/2$ and $\alpha^2 \in \mathbb{Q}$. Indeed for $\alpha^2 \notin \mathbb{Q}$ it is conjectured [EM04] the gap statistics are Poissonian.

For the sequence $(\alpha n \mod 1)_{n\geq 1}$, the three gap theorem (also known as the Steinhaus conjecture) states that the size of the gaps between neighboring points, at any time N, form a set of cardinality at most 3. Hence, the local statistics are certainly not Poissonian. For background see [MS17, MK98].

1.1.2 Metric Poisson Pair Correlation

Generally speaking, it is believed that, given a $\theta > 0$, the pseudo-random properties of (1.3) are determined by the Diophantine properties of α (e.g see [RS98, Remark 1.2]). However, in the absence of methods to prove Poissonian pair correlation for explicit values of α, Rudnick and Sarnak [RS98] introduced the concept of *metric Poisson pair correlation*. Namely a general sequence $(x_n)_{n\geq 1}$ has metric Poisson pair correlation, if the dilated sequence $(\alpha x_n \bmod 1)_{n\geq 1}$ has Poissonian pair correlation for all $\alpha > 0$ outside of a (Lebesgue) null set.

For $\theta \in \mathbb{N}_{>1}$, Rudnick and Sarnak [RS98] proved the metric Poissonian pair correlation of (1.3) in the late 90s. The case of non-integer $\theta > 1$ was only recently settled by Aistleitner, El-Baz, and Munsch [AEBM21]. The regime $\theta < 1$ will be addressed in a forthcoming work of Rudnick and the second named author, see [RT].

Special attention has been given to the quadratic case, $\theta = 2$, due to its connection with quantum chaos and the boxed harmonic oscillator. Here, Heath-Brown [HB10] gave an algorithmic construction of a dense set of α for which the pair correlation is Poissonian. Moreover, there have been some results for longer-range correlations [TW20, Lut20], convergence along sparse subsequences [RSZ01, FKZ21], and minimal gaps [Zah95, Reg21, Rud18]. However, finding explicit α for which the pair correlation is Poissonian remains out of reach.

Finally, it is worth noting that the metric Poisson pair correlation theory has been generalized beyond monomial sequences and exploits some deep connections to additive combinatorics [ALL17, BW20]. However, this connection is beyond the scope of this paper.

Notation: Throughout, we use the usual Bachmann–Landau notation: for functions $f, g : X \to \mathbb{R}$, defined on some set X, we write $f \ll g$ (or $f = O(g)$) to denote that there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in X$. Moreover let $f \asymp g$ denote $f \ll g$ and $g \ll f$. Furthermore, let $f = o(g)$ denote that $\frac{f(x)}{g(x)} \to 0$.

Throughout we denote $e(x) = e^{2\pi i x}$ and $\widehat{f}$ is the Fourier transform (on $\mathbb{R}$) of f. All of the sums which appear range over integers, in the indicated interval. As $\alpha, \varepsilon, \theta$, and f are considered fixed, we suppress any dependence in the implied constants. Moreover, for ease of notation, $\varepsilon > 0$ may vary from line to line by a bounded constant. Further, we will frequently encounter the exponent

$$\Theta := \frac{1}{1-\theta}.$$

1.2 Idea of the Proof

The proof relies on a well-known Fourier decomposition. First, we include the diagonal term in the pair correlation function, to simplify technicalities. Thus, define

$$\widetilde{R}(N, f) := \frac{1}{N} \sum_{\mathbf{y} \in [1,N]^2} \sum_{k \in \mathbb{Z}} f(N(\alpha y_1^\theta - \alpha y_2^\theta + k)). \tag{1.4}$$

Note that Theorem 1 is equivalent to showing that (as $N \to \infty$)

$$\widetilde{R}(N, f) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x + f(0) + o(1). \tag{1.5}$$

By the Poisson summation formula,

$$\widetilde{R}(N, f) = \widehat{f}(0) + \frac{1}{N^2} \sum_{|k| \in [1, N^{1+\varepsilon}]} \widehat{f}\Big(\frac{k}{N}\Big) \Bigg| \sum_{y \in [1,N]} e(\alpha k y^\theta) \Bigg|^2 + o(1) \tag{1.6}$$

for $\varepsilon > 0$ where the $o(1)$-error comes from the fast decay of $\widehat{f}$. Note that f can be decomposed into a sum of an even and an odd function. Further, the Fourier coefficients of the odd part cancel out, and the Fourier coefficients of the even part are even functions themselves. Thus, without loss of generality we may assume f is even. Hence, it suffices to show that

$$\mathcal{E}(N) := \frac{2}{N^2} \sum_{k \in [1, N^{1+\varepsilon}]} \widehat{f}\Big(\frac{k}{N}\Big) \Bigg| \sum_{y \in [1,N]} e(\alpha k y^\theta) \Bigg|^2 = f(0) - \widehat{f}(0) + o(1). \tag{1.7}$$

To achieve the desired bound requires a detailed analysis of the exponential sums in (1.7). We argue in, roughly, two steps: first we decompose the innermost summation, and apply van der Corput's B-process to obtain a saving in the y-summation. Second, we expand the square and use some analytic tricks to reduce the estimates to exponential sums over k. Now to obtain a saving in the k-summation, we again use the B-process coupled with other estimates (such as Weyl differencing).

1.3 Heuristic

After applying the B-process, interchanging the order of summation, extracting the main terms and dealing with the error terms, our task is the following. We need to show that

$$\mathrm{Err} := \frac{2}{N^2} \sum_{\substack{r_1, r_2 > cN^\theta \\ r_1 \neq r_2}}^{N} \frac{1}{(r_1 r_2)^{\frac{\Theta+1}{2}}} \sum_{k \in [1, N^{1+\varepsilon}]} \widehat{f}\Big(\frac{k}{N}\Big) k^\Theta e(\gamma(\mathbf{r}) k^\Theta)$$

is $o(1)$, as $N \to \infty$, where $\gamma(\mathbf{r}) = \beta(r_1^{1-\Theta} - r_2^{1-\Theta})$, and β and c depend only on θ and α. Now we apply partial summation to reduce matters to estimating

$$\left| \sum_{k \in [1, N^{1+\varepsilon}]} e(\gamma(\mathbf{r}) k^{\Theta}) \right|.$$

If we had square root cancellation for this sum − uniformly in $\gamma(\mathbf{r})$ − then our method yields Err $\approx N^{\theta - 1/2 + \varepsilon}$. In other words, even with optimal bounds, we cannot hope to go past the barrier $\theta = 1/2 - \varepsilon$. To move past this barrier, our analytic method would require taking advantage of the cancellation between exponential sums for different values of $\mathbf{r}$. This seems to be well beyond currently technology.

Interestingly, if we consider instead the triple correlation function, the natural barrier to our methods turns out to be $\theta < 1/3$. In fact as we consider higher and higher correlations, that barrier goes to 0.

1.4 Preliminaries

The following two results are fundamental in the modern study of exponential sums. First, we recall an application of Weyl's differencing method (called the A-Process, see [GK91, Theorem 2.9]):

Theorem 2 (A-Process)**.** *Let $l \geq 0$ be an integer and let $M > 2$. Suppose $\phi : [a, b) \to \mathbb{R}$ has $l + 2$ continuous derivatives on $[a, b) \subseteq [M, CM)$, where $C > 1$ is some fixed constant, and assume there exists a constant $F > 0$ such that*

$$\phi^{(r)}(x) \asymp FM^{-r} \tag{1.8}$$

for $r = 1, \ldots, l + 2$. Then

$$\sum_{x \in [a,b)} e(\phi(x)) \ll F^{1/(4L-2)} M^{1-(l+2)/(4L-2)} + F^{-1} M, \tag{1.9}$$

where $L := 2^l$. The implicit constant in (1.9) depends on the choice of l and (1.8).

Further, we will use van der Corput's B-process, which follows from Poisson summation and a stationary phase argument (see [IK04, Theorem 8.16]):

Theorem 3 (B-Process)**.** *Let $\phi : [A, B) \to \mathbb{R}$ be a C^4-function so that there are $\Lambda > 0$ and $\eta \geq 1$ with*

$$\Lambda \leq \phi^{(2)}(x) < \eta\Lambda, \qquad \left|\phi^{(3)}(x)\right| < \frac{\eta\Lambda}{B - A}, \qquad \left|\phi^{(4)}(x)\right| < \frac{\eta\Lambda}{(B - A)^2} \tag{1.10}$$

for all $x \in [A, B)$. Let $a = \phi'(A)$, and $b = \phi'(B)$. Then

$$\sum_{n \in [A,B)} e(\phi(n)) = e(1/8) \sum_{m \in [a,b)} \frac{e(\phi(x_m) - m x_m)}{\sqrt{\phi^{(2)}(x_m)}} + \omega_\phi(A, B)$$

where x_m denotes the unique solution to $\phi'(x) = m$. Furthermore,

$$\omega_\phi(A, B) \ll \Lambda^{-\frac{1}{2}} + \eta^2 \log(b - a + 1), \tag{1.11}$$

where the implied constant is absolute.

We will often need to bound weighted exponential sums. To reduce these estimates to bounding unweighted sums, we use partial summation in the form of:

Lemma 4. *Let $(a_s)_s$ and $(b_s)_s$ be sequences of complex numbers. Fix a constant $c > 0$. If $T > 0$ is such that $|b_s - b_{s+1}| \leq T/s$, then*

$$\sum_{S < s < \tilde{S}} a_s b_s \leq \left(\max_{S \leq s \leq cS} |b_s| + O(T) \right) \max_{S \leq \tilde{S} \leq cS} \left| \sum_{S \leq s < \tilde{S}} a_s \right|$$

for any positive integers S and $\tilde{S}$ satisfying $S \leq \tilde{S} \leq cS$.

2 Reducing to Exponential Sum Bounds

2.1 Decomposing the sums and applying the B-Process

Now consider the term $\mathcal{E}(N)$, defined in (1.7). We shall apply the B-process (Theorem 3) to the exponential sum in $\mathcal{E}(N)$, but presently we do not have sufficient control on the derivative of $y \mapsto \alpha k y^{\theta}$.

To gain control, we use several dyadic decompositions. First let $U \in \mathbb{N}$ be such that $e^U \le N^{1+\varepsilon} < e^{U+1}$. Further we assume (w.l.o.g.) that $N = N_Q := Q^\Gamma$ as it is enough to prove the correlations converge along such a subsequence, see [RT20, Lemma 3.1]. Thus, we decompose

$$\mathcal{E}(N) = \frac{2}{N^2} \sum_{u \le U} \sum_{k \in [e^u, e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) \Bigg| \sum_{y \in [1,N]} e(\alpha k y^\theta) \Bigg|^2,$$

and decompose the inner summation further into the pieces

$$E_q(k) := \sum_{y \in [N_q, N_{q+1})} e(\alpha k y^\theta)$$

where $N_q := q^\Gamma$. To catch the largest term, we set $N_{Q+1} = N_Q + 1$. Thus,

$$\mathcal{E}(N) = \frac{2}{N^2} \sum_{\substack{u \le U \\ q_1, q_2 \le Q}} \sum_{k \in [e^u, e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) E_{q_1}(k) \overline{E_{q_2}(k)}. \tag{2.1}$$

As Γ will be a large constant, the sum over $\mathbf{q} := (q_1, q_2)$ will not play an important role. Likewise the sum over u will contribute a logarithmic factor which will not play an important role. Thus, for the sake of notation, fix a pair $\mathbf{q} := (q_1, q_2)$, a variable $u \in [1, U]$, and define

$$\mathcal{E}_{\mathbf{q},u}(N) := \frac{2}{N^2} \sum_{k \in [e^u, e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) E_{q_1}(k) \overline{E_{q_2}(k)}. \tag{2.2}$$

Now, the next lemma shows that we can replace each $E_q(k)$ by

$$E_q^{(B)}(k) := c_1 \sum_{r \in \mathcal{R}_q(k)} \frac{k^{\Theta/2}}{r^{(\Theta+1)/2}} e(\beta k^\Theta r^{1-\Theta}), \quad \text{where } \mathcal{R}_q(k) := \alpha\theta k (N_{q+1}^{\theta-1}, N_q^{\theta-1}], \tag{2.3}$$

which is the main term after applying the B-Process to $E_q(k)$; the constants are defined

$$c_1 := e(-1/8)\sqrt{\Theta(\alpha\theta)^\Theta}, \qquad \text{and} \qquad \beta := \alpha^\Theta(\theta^{1-\Theta} - \theta^\Theta).$$

It is helpful to distinguish whether the length of the interval $\mathcal{R}_q(k)$ is at least a small power of N_q. To this end, we fix an exponent $w \in \Gamma[1-\theta+\varepsilon, 1]$ be a constant depending only on θ. We stress that all implied constants in the following are uniform in $\mathbf{q}$ and u.

Lemma 5. *If $e^u > \max(q_1^w, q_2^w)$, then there exists a constant $\nu > 1/\Gamma$, depending only on θ, such that*

$$\mathcal{E}_{\mathbf{q},u}(N) = \frac{2}{N^2} \sum_{k \in [e^u, e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) E_{q_1}^{(B)}(k) \overline{E_{q_2}^{(B)}(k)} + O(N^{-\nu}), \tag{2.4}$$

as $N \to \infty$. If $e^u \le \max(q_1^w, q_2^w)$ then $\mathcal{E}_{\mathbf{q},u}(N) = O(N^{-\nu})$ as $N \to \infty$.

Proof. First we apply Theorem 3 with $\Lambda = e^u \alpha\theta(1-\theta) N_{q+1}^{\theta-2}$ and $\eta = 10$. Hence,

$$\Big| E_q(k) - E_q^{(B)}(k) \Big| \ll \frac{N_q}{\sqrt{N_q^\theta e^u}} + \log N_q,$$

with the implied constant being uniform in $e^u \le k < e^{u+1}$. Thus

$$\frac{N^2}{2} \mathcal{E}_{\mathbf{q},u}(N) = \sum_{k \in [e^u, e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) E_{q_1}^{(B)}(k) \overline{E_{q_2}^{(B)}(k)} + O(e^u(\mathrm{Err}_1 + \mathrm{Err}_2)),$$

where, by symmetry of q_1 and q_2 we can take

$$\mathrm{Err}_1 := \max_{k \in [e^u, e^{u+1})} \frac{N_{q_1}}{\sqrt{N_{q_1}^\theta e^u}} \Big| E_{q_2}^{(B)}(k) \Big|, \qquad \mathrm{Err}_2 := \max_{k \in [e^u, e^{u+1})} \frac{N_{q_1}}{\sqrt{N_{q_1}^\theta e^u}} \frac{N_{q_2}}{\sqrt{N_{q_2}^\theta e^u}}.$$

Therefore, we may conclude $e^u \,\mathrm{Err}_2 \ll N^{2-\theta}$. Observe that $E_q^{(B)}(k)$, trivially satisfies,

$$E_q^{(B)}(k) \ll \sqrt{e^u N_q^\theta}. \tag{2.5}$$

Hence, for Γ large, ε small, and $e^u \le q_2^w$, we may infer $e^u \,\mathrm{Err}_1 \ll N^{2-\nu}$ for some $\nu > 0$.

Thus, suppose $e^u > q_2^w$. Now if we apply the bound (2.5) we find $e^u \operatorname{Err}_1 \ll N^{2+\frac{2}{\Gamma}}$. Thus, we need to establish any power-saving estimate on $E_{q_2}^{(B)}(k)$, then we can take Γ large enough, depending on this saving. To this end it suffices to estimate

$$\sum_{r \in \mathcal{R}_q(k)} r^{-\frac{\Theta+1}{2}} e(\beta k^\Theta r^{1-\Theta}).$$

Applying Lemma 4 and Theorem 2 yields a small power power-saving: there exists $\nu > 0$ such that $E_{q_2}^{(B)}(k) = O\big(\sqrt{e^u N_{q_2}^{\theta-\nu}}\big)$ uniformly in $k \in [e^u, e^{u+1})$ and $q_2 \le Q$.

Lastly, the contribution when $e^u \le \max(q_1^w, q_2^w)$ can be neglected, by inserting (2.5) thanks to the choice of w.

□

2.2 The Diagonal

Recall, our goal is to establish (1.7). The main term, $f(0) - \widehat{f}(0)$ will come from the diagonal term. That is, in (2.4) we consider the term $q_1 = q_2 = q$ and $r_1 = r_2 = r$ (where r_i is the r as in (2.3))

$$D_{u,q} = 2\,|c_1|^2 \sum_{k \in [e^u, e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) \sum_{r \in \mathcal{R}_q(k)} \frac{k^\Theta}{r^{\Theta+1}}.$$

Lemma 6. *If $\theta \in (0,1)$, then*

$$\frac{1}{N^2} \sum_{0 \le u \le U} \sum_{q \le Q} D_{u,q} = f(0) - \widehat{f}(0) + o(1) \quad (\text{as } Q \to \infty). \tag{2.6}$$

Proof. By a Riemann integral argument (see for example [Apo76, Theorem 3.2]) the sum

$$\sum_{r \in \mathcal{R}(k)} \frac{1}{r^{\Theta+1}} = \frac{(\alpha\theta k)^{-\Theta} - \left(\alpha\theta k N^{\theta-1}\right)^{-\Theta}}{-\Theta} + O\left(\left(\alpha\theta k N^{\theta-1}\right)^{-\Theta-1}\right).$$

Recall that $|c_1|^2 = \Theta(\alpha\theta)^\Theta$. Thus

$$\frac{1}{N^2} \sum_{0 \le u \le U} \sum_{q \le Q} D_{u,q} = 2 \sum_{k \in [1, N^{1+\varepsilon})} \widehat{f}\Big(\frac{k}{N}\Big) \left(N + O\left(\frac{N^{2-\theta}}{k}\right)\right) = 2N \sum_{k \in [1, N^{1+\varepsilon})} \widehat{f}\Big(\frac{k}{N}\Big) + O(N^{2-\theta+\varepsilon}).$$

Now (2.6) follows by the Poisson summation formula and the fact that f is an even function. □

2.3 Partial Summation

Lemma 5 reduces the problem to estimating

$$\mathcal{E}_{\mathbf{q},u}^{(B)} := \sum_{k \in [e^u, e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) E_{q_1}^{(B)}(k) \overline{E_{q_2}^{(B)}(k)}.$$

The next lemma reduces matters further to bounding the unweighted exponential sums

$$S(\gamma, \mathcal{I}) := \sum_{k \in \mathcal{I}} e(\gamma k^\Theta)$$

requiring the bound to be uniform in the size of γ, and the interval $\mathcal{I}$. Therefore, we introduce

$$S_{\Lambda,\eta}(u) := \sup_{\gamma \in [\Lambda, \eta\Lambda)} \sup_{\mathcal{I} \subseteq [e^u, e^{u+1})} |S(\gamma, \mathcal{I})|.$$

With this maximal operator at hand, we have the following bounds,

Lemma 7. *Let $q_1 \le q_2$ and suppose that $e^u > \max(q_1^w, q_2^w)$. Then, if $q_1 \ne q_2$ we have*

$$\mathcal{E}_{\mathbf{q},u}^{(B)} \ll N^\varepsilon \sum_{j \in \mathcal{J}_{u,q_1}} e^j \sqrt{N_{q_2}^\theta N_{q_1}^{2-\theta}}\, S_{\Lambda(j),\eta}(u) \tag{2.7}$$

where $\Lambda(j) := e^{j-u\Theta} N_{q_1}$, $\eta := C\frac{N_{q_2}}{N_{q_1}}$ for some constant $C > 0$, and $\mathcal{J}_{u,q_1} := [0, u - (1-\varepsilon)(1-\theta)\Gamma \log q_1)$.

Similarly, if $q_1 = q_2 = q$ then

$$\mathcal{E}_{\mathbf{q},u}^{(B)} = D_{u,q} + O\left(N^\varepsilon \sum_{j \in \mathcal{J}_{u,q_1}} e^j \sqrt{N_{q_2}^\theta N_{q_1}^{2-\theta}}\, S_{\Lambda(j),\eta}(u)\right). \tag{2.8}$$

Proof. For brevity, in this proof, let

$$\gamma(\mathbf{r}) = \gamma := \beta(r_2^{1-\Theta} - r_1^{1-\Theta}).$$

Therefore,

$$\mathcal{E}_{\mathbf{q},u}^{(B)} = |c_1|^2 \sum_{k\in[e^u,e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big) \sum_{\substack{r_i\in\mathcal{R}_{q_i}(k)\\ i=1,2}} \frac{k^\Theta}{(r_1r_2)^{\frac{\Theta+1}{2}}} e(\gamma k^\Theta).$$

Thus, the r_i which appear in the overall sum all fall within the ranges

$$\mathcal{R}_{q_i,u} := \left(\frac{\alpha\theta e^u}{N_{q_i+1}^{1-\theta}}, \frac{\alpha\theta e^{u+1}}{N_{q_i}^{1-\theta}}\right) \qquad (i=1,2). \tag{2.9}$$

Now we interchange the r and k summations. For each choice of r_1 and r_2, we have that

$$k \in \mathcal{K}_{\mathbf{q}}(\mathbf{r}) := \frac{1}{\alpha\theta}\left(\max(r_1N_{q_1}^{1-\theta}, r_2N_{q_2}^{1-\theta}, \alpha\theta e^u), \min(r_1N_{q_1+1}^{1-\theta}, r_2N_{q_2+1}^{1-\theta}, \alpha\theta e^{u+1})\right),$$

note, that this interval may sometimes be empty. With that,

$$\mathcal{E}_{\mathbf{q},u}^{(B)} = |c_1|^2 \sum_{\substack{r_i\in\mathcal{R}_{q_i,u}\\ i=1,2}} (r_1r_2)^{-\frac{\Theta+1}{2}} \sum_{k\in\mathcal{K}_{\mathbf{q}}(\mathbf{r})} \widehat{f}\Big(\frac{k}{N}\Big) k^\Theta e(\gamma k^\Theta).$$

Next, we remove the weights via partial summation, Lemma 4. Let $w_k = \widehat{f}(\frac{k}{N})k^\Theta$, we first show

$$|w_k - w_{k+1}| \ll \frac{N^\varepsilon e^{\Theta u}}{k}, \tag{2.10}$$

for any $k \in \mathcal{K}_{\mathbf{q}}(\mathbf{r})$ with the implied constant being absolute. The mean value theorem implies

$$|w_k - w_{k+1}| \le \left|\widehat{f}\Big(\frac{k}{N}\Big)\right| \left|k^\Theta - (k+1)^\Theta\right| + \left|\widehat{f}\Big(\frac{k}{N}\Big) - \widehat{f}\left(\frac{k+1}{N}\right)\right| (k+1)^\Theta \ll \frac{w_k}{k} + \frac{e^{\Theta u}}{N}.$$

Using that $k \ll N^{1+\varepsilon}$, yields (2.10).

Note that $K_{\mathbf{q}}(\mathbf{r}) \subset [e^u, e^{u+1})$. Thus Lemma 4 is applicable and, in combination with (2.10), yields

$$\sum_{k\in\mathcal{K}_{\mathbf{q}}(\mathbf{r})} w_k e(\gamma k^\Theta) \ll N^\varepsilon e^{\Theta u} \max_{K\in\mathcal{K}_{\mathbf{q}}(\mathbf{r})} \left| \sum_{k\in\mathcal{K}_{\mathbf{q}}(\mathbf{r})\cap[1,K]} e(\gamma k^\Theta)\right|.$$

Note that each $r_i \in \mathcal{R}_{q_i,u}$ satisfies $r_i \asymp e^u N_{q_i}^{-(1-\theta)}$ for $i=1,2$ and that the condition $e^u > \max(q_1^w, q_2^w)$ ensures that $\#\mathcal{R}_{q_i,u}$ is at least a small power of N_{q_i}.

To reduce matters to exponential sums requires control of the difference $\gamma(\mathbf{r})$, thus let $\mathcal{D}(j)$ denote the set of pairs $(r_1, r_2) \in \mathcal{R}_{q_1,u} \times \mathcal{R}_{q_2,u}$ satisfying $e^j \le |r_2 - r_1| < e^{j+1}$. Since $q_1 \le q_2$, after relabeling we can assume $r_2 < r_1$. Hence

$$\gamma(\mathbf{r}) = \beta(1-\Theta)\int_{r_2}^{r_1} \tau^{-\Theta}\mathrm{d}\tau \gg (r_1 - r_2)r_1^{-\Theta} \gg (r_1 - r_2)\left(\frac{N_{q_1}^{1-\theta}}{\alpha\theta e^{u+1}}\right)^\Theta \gg (r_1 - r_2)e^{-u\Theta}N_{q_1}.$$

Similarly, we can bound $\gamma(\mathbf{r})$ from above by $(r_1 - r_2)e^{-u\Theta}N_{q_2}$. Thus, $\gamma(\mathbf{r}) \in [\Lambda(j), \eta\Lambda(j)]$.

Moreover, the range of j can be constrained by

$$e^j \asymp r_1 - r_2 \ll r_1 \ll e^u N_{q_1+1}^{-(1-\theta)}$$

which implies that $j \in \mathcal{J}_{u,q_1}$. Thus, we have the following bound

$$\begin{aligned}\mathcal{E}_{\mathbf{q},u}^{(B)} &\ll N^\varepsilon \sum_{j\in\mathcal{J}_{u,q_1}} \left(\frac{e^u}{N_{q_1}^{1-\theta}}\frac{e^u}{N_{q_2}^{1-\theta}}\right)^{-\frac{\Theta+1}{2}} e^{\Theta u}\#\mathcal{D}(j)S_{\Lambda(j),\eta}(u)\\ &\ll N^\varepsilon \sum_{j\in\mathcal{J}_{u,q_1}} \left(\frac{e^u}{N_{q_1}^{1-\theta}}\frac{e^u}{N_{q_2}^{1-\theta}}\right)^{-\frac{\Theta+1}{2}} e^{\Theta u}\frac{e^u}{N_{q_2}^{1-\theta}}e^j S_{\Lambda(j),\eta}(u).\end{aligned}$$

The summand on the right simplifies to

$$\left(\frac{e^u}{N_{q_1}^{1-\theta}}\frac{e^u}{N_{q_2}^{1-\theta}}\right)^{-\frac{\Theta+1}{2}} e^{\Theta u}\frac{e^u}{N_{q_2}^{1-\theta}}e^j \ll e^j\sqrt{N_{q_2}^\theta N_{q_1}^{2-\theta}}.$$

Giving the overall bound

$$\mathcal{E}_{\mathbf{q},u}^{(B)} \ll N^\varepsilon \sum_{j\in\mathcal{J}_{u,q_1}} e^j\sqrt{N_{q_2}^\theta N_{q_1}^{2-\theta}}S_{\Lambda(j),\eta}(u).$$

If $q_1 = q_2$, we remove the diagonal $r_1 = r_2$ and apply exactly the same bound to the off-diagonal. ☐

3 Proof of the Main Theorem

3.1 Exponential Sum Bounds

Recall, from Lemma 7 that $\Lambda(j) = e^{j-u\Theta}N_{q_1}$. Thanks to Lemmas 5, 6, and 7, we will show that Theorem 1 can be deduced from the next lemma.

Lemma 8. *Assume $q_1 < q_2$, we have the following bound*

$$S_{\Lambda(j),\eta}(u) \ll e^{8u/15}N_{q_2}^{11\theta/30} + e^{u-j/2}N_{q_1}^{-1/2}. \tag{3.1}$$

Proof. First we apply the B-process (Theorem 3) in the k variable. k is in an interval $[A, B]$ of size e^u, and the phase function $\phi(k) \asymp \gamma e^{u\Theta}$ satisfies $\phi'(k) \asymp \gamma e^{u(\Theta-1)}$ and $\phi''(k) \asymp \gamma e^{u(\Theta-2)}$, where $\gamma \in [\Lambda(j), \eta\Lambda(j)]$. Applying Theorem 3 and a trivial estimate gives one bound (which will suffice for $\theta < 1/3$).

In fact, we require slightly more than the B-process to move past $\theta = 1/3$. Thus we will apply the B-process precisely, and then use partial summation and the A-process to bound the resulting sum. First, set $\vartheta = 1/(1-\Theta)$ and apply Theorem 3, to conclude

$$\sum_{k\in[e^u,e^{u+1})} e(\gamma k^\Theta) = c_3 \sum_{h\in[a,b)} \sqrt{\frac{\gamma^\vartheta}{h^{\vartheta+1}}}\, e(c_4\gamma^\vartheta h^{1-\vartheta}) + O(e^{u-j/2}N_{q_1}^{-1/2} + \log(N_{q_2})) \tag{3.2}$$

where $a < b$ are positive integers of size $\gamma e^{u(\Theta-1)}$ and c_3, c_4 are (complex, respectively real) nonzero constants depending only on θ and α. A trivial estimate implies we can assume $b - a \geq 10$.

By exploiting partial summation, we may apply Lemma 4 to the main term in (3.2). Thus, to prove (3.1), it suffices to bound:

$$\frac{e^u}{\gamma^{1/2}e^{\Theta u/2}} \sum_{h\in[a,b)} e(\gamma^\vartheta h^{1-\vartheta}) \tag{3.3}$$

for $a < b$ being of size $\gamma e^{u(\Theta-1)}$ and such that $b-a \geq 10$. To that end, we use Theorem 2 for an arbitrary integer l. In that notation, let $F \asymp \gamma e^{u\Theta}$, and $M = \gamma e^{u(\Theta-1)}$. Thus, we conclude that (recall $L = 2^l$)

$$\sum_{h\in[a,b)} e(\gamma^\vartheta h^{1-\vartheta}) \ll (\gamma e^{u\Theta})^{\frac{1}{4L-2}}(\gamma e^{u(\Theta-1)})^{1-\frac{l+2}{4L-2}} + e^{-u}. \tag{3.4}$$

Inserting this into (3.3) and using that $\gamma \ll e^{u(1-\Theta)}N_{q_2}^\theta$

$$\frac{e^u}{\gamma^{1/2}e^{\Theta u/2}} \sum_{h\in[a,b)} e(\gamma^\vartheta h^{1-\vartheta}) \ll e^u(e^u N_{q_2}^\theta)^{\frac{1}{4L-2}-1/2}(N_{q_2}^\theta)^{1-\frac{l+2}{4L-2}},$$

which, on rearranging and choosing $l = 3$, gives (3.1).

□

3.2 Proof of Theorem 1

Recall, after applying Lemma 5, our goal is to find an asymptotic for

$$\mathcal{E}_{\mathbf{q},u}^{(B)}(N) := \sum_{k\in[e^u,e^{u+1})} \widehat{f}\Big(\frac{k}{N}\Big)E_{q_1}^{(B)}(k)\overline{E_{q_2}^{(B)}(k)}$$

when $q_1 = q_2$ and a bound for $q_1 < q_2$ where $e^u > \max(q_1^w, q_2^w)$.

First apply partial summation and isolate the diagonal, $q_1 = q_2$, contribution as done in Lemma 7 giving the off-diagonal, (2.7), and the diagonal term (2.8). This relates $\mathcal{E}(N)$ to the exponential sums $S_{\Lambda(j),\eta}(u)$ from Subsection 2.3.

Now assume $q_1 \neq q_2$, in which case using (3.1) (a similar calculation shows that the second error term in (3.1) gives a negligible contribution) gives

$$\begin{aligned} N^\varepsilon\sqrt{N_{q_2}^\theta N_{q_1}^{2-\theta}} \sum_{j\in\mathcal{J}_{u,q_1}} e^j S_{\Lambda(j),\eta}(u) &\ll N^\varepsilon\sqrt{N_{q_2}^\theta N_{q_1}^{2-\theta}} \sum_{j\in\mathcal{J}_{u,q_1}} e^j e^{8u/15}N_{q_2}^{11\theta/30} \\ &\ll N^\varepsilon\sqrt{N_{q_2}^\theta N_{q_1}^\theta}\, e^u e^{8u/15}N_{q_2}^{11\theta/30} \\ &\ll N^\varepsilon N^{\frac{46}{30}}N^{\frac{41\theta}{30}}. \end{aligned}$$

Thus, for $\theta < 14/41$, we may choose Γ large enough, and ε small enough such that $\sum_{\mathbf{q}}\sum_u \mathcal{E}_{\mathbf{q},u} = o(N^2)$.

Finally, in the diagonal case, we again apply (2.8) to reduce to an exponential sum, and then we can bound the error term using (3.1) in exactly the same way. □

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