

Invariance Principle for the Random Wind-Tree Process

CHRISTOPHER LUTSKO* AND BÁLINT TÓTH*†

*University of Bristol, UK

†Rényi Institute, Budapest, HU

December 6, 2019

Abstract

Consider a point particle moving through a Poisson distributed array of cubes all oriented along the axes - the random wind-tree model introduced in Ehrenfest-Ehrenfest (1912) [6]. We show that, in the joint Boltzmann-Grad and diffusive limit this process satisfies an invariance principle. That is, the process converges in distribution to a Brownian motion in a particular scaling limit. In a previous paper (2019) [13] the authors used a novel coupling method to prove the same statement for the random Lorentz gas with spherical scatterers. In this paper we show that, despite the change in dynamics, the same strategy with some modification can be used to prove an invariance principle for the random wind-tree model.

MSC2010: 60F17; 60K35; 60K37; 60K40; 82C22; 82C31; 82C40; 82C41

KEY WORDS AND PHRASES: wind-tree; Ehrenfest model; invariance principle; scaling limit; coupling; exploration process

1 Introduction

In this paper we consider the motion of a point particle through an array of randomly placed, identically oriented cubes in \mathbb{R}^3 - the so-called random wind-tree process [6]. In a recent paper [13] the authors showed that the random Lorentz gas (i.e the same process with the cubes replaced by spheres [12]) satisfies an invariance principle in a particular scaling limit which is intermediate between the kinetic and purely diffusive time scales. In this paper we prove an invariance principle for the wind-tree process in a similar intermediate regime. The proof will follow similar lines. However there are two key differences: in the Lorentz gas, after collision with a randomly placed scatterer (in 3 dimensions) the velocity is redistributed independently of the initial velocity while for the wind-tree process the velocities form a genuine Markov chain. On the other hand as the collisions are simpler in the wind-tree setting the necessary geometric estimates follow with less effort.

More formally let \mathcal{P} be a Poisson point process of intensity $\rho > 0$ in \mathbb{R}^3 (our results hold for general dimension $d \geq 3$, however to reduce notation we restrict to $d = 3$). Let \mathcal{Q}_r be a cube of side length r oriented parallel with the axes and let $\mathcal{P} + \mathcal{Q}_r$ be an array of *obstacles/scatterers*. We consider the trajectory of a point particle $X^{r,\varrho}(t)$ starting at the origin ($X^{r,\varrho}(0) = 0$) with a fixed initial velocity of unit length. The particle then flies in straight lines, reflecting elastically off of the obstacles. In this setting the origin is in $(\mathcal{P} + \mathcal{Q}_r)^c$ with probability tending to 1, hence such a trajectory is well-defined (see [13, (2)] for more details).

A fundamental open problem for both the random wind-tree model and the random Lorentz gas is to prove an invariance principle in the diffusive limit. That is, in the limit

$$\frac{X^{r,\varrho}(Tt)}{\sqrt{T}} \quad , \quad T \rightarrow \infty, \quad (1)$$

does the scaled process converge weakly to a Wiener process? In our previous paper we showed that the Lorentz gas satisfies an invariance principle in the limit (1) if we *simultaneously* take the low-density limit in a particular scaling limit. The aim for this paper is to replicate that result for the wind-tree model.

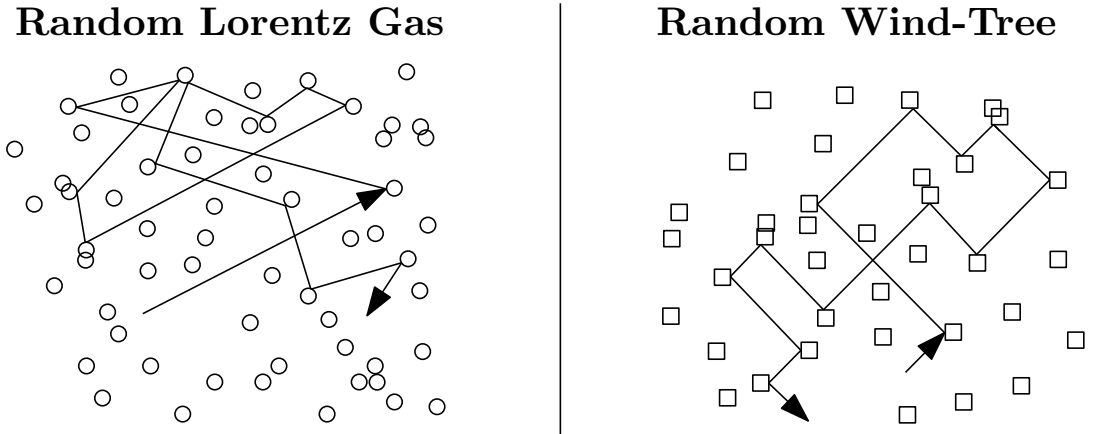


Figure 1: Typical trajectories of the random wind-tree and Lorentz gas models. Note the difference in the dynamics: in the wind-tree model the velocities are restricted to a finite set (in 2 dimensions there are only 4 possible velocities). While in the Lorentz gas the velocities are uniformly distributed on the sphere.

1.1 Scaling and Main Result

Fix a probability vector $\mathbf{p} = (p_1, p_2, p_3)$ with $p_i > 0$ for all i , and let $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. The state-space of velocities is then

$$\Omega := \left\{ v \in S_1^2 : |v_i| = \frac{p_i}{|\mathbf{p}|} \right\} \quad (2)$$

Fix the initial velocity $\dot{X}^{r,\varrho}(0^+) \in \Omega$. We study the process $t \mapsto X^{r,\varrho}(t)$ on $[0, T]$ in the joint Boltzmann-Grad and diffusive scaling limit:

$$r \rightarrow 0 \quad , \quad r^2\varrho \rightarrow |\mathbf{p}|^{-1} \quad , \quad T(r) \rightarrow \infty \quad (3)$$

$$t \mapsto \frac{X(tT)}{\sqrt{T}},$$

note that $|\mathbf{p}|^{-1}$ is the cross-sectional area of the cube as viewed by the particle, and we have dropped the dependence on r and ϱ in the notation (thus $X^{r,\varrho}(t) = X(t)$). With that, the main result of this paper is the following invariance principle:

Theorem 1. *Consider the intermediate scaling limit (3) such that $\lim_{r \rightarrow 0} T(r)r^2 = 0$ then*

$$\left\{t \mapsto T^{-1/2}X(tT)\right\} \Longrightarrow \{t \mapsto W(t)\} \quad (4)$$

as $r \rightarrow 0$ in the averaged-quenched sense (see below). Where $W(t)$ is a Wiener process with covariance matrix $M = \text{diag}(v_1^2, v_2^2, v_3^2)$ in \mathbb{R}^3 .

The proof follows from a joint construction of $t \mapsto X(t)$ and a second Markovian process which we introduce in Section 2.2. In Section 2.4 we state and outline the proof of the main technical theorem of the paper (Theorem 2). Theorem 1 is then a straightforward corollary of that theorem.

Remark. For the Lorentz gas we proved the same theorem with the asymptotic constraint $\lim_{r \rightarrow 0} T(r)r^2 |\log r|^2 = 0$. The reason for this logarithmic correction are those collisions for which the angle between incoming and outgoing velocities is small. In the wind-tree model the velocity of the point particle is restricted to a fixed discrete set, hence the log factor can be removed.

In this context there are two relevant limits one could take:

- (Q) *Quenched limit:* For almost all (i.e. typical) realizations of the underlying Poisson point process, with averaging over the random initial velocity of the particle.
- (AQ) *Averaged-quenched (a.k.a. annealed) limit:* Averaging over the random initial velocity of the particle *and* the random placement of the scatterers.

This paper (and our previous paper [13]) are in the averaged-quenched setting.

1.2 Related work

While we cannot hope to give an exhaustive account we present here some of the related work. The wind-tree model was introduced in the famous monograph by Paul and Tatiana Ehrenfest [6, Appendix to Section 5] as a simplified model to explain the return to equilibrium of the velocity distribution of a gas. It is noteworthy that, in defining the model P. and T. Ehrenfest considered randomly placed scatterers oriented along the axes (as we have done), however the periodic wind-tree model (often referred to as the Ehrenfest model) - where *rectangular* scatterers are centered at the points of a hypercubic lattice - is the better studied model. This owes to the fact that the periodic setting can be studied using methods from parabolic dynamical systems. While the random wind-tree model is less well-understood, it remains of interest as a stochastic process and a model for diffusion in particle systems.

2D Periodic Wind-Tree: The periodic wind-tree model *in 2 dimensions* has been the focus of a lot of recent research. In this setting the billiard flow is parabolic (i.e close orbits diverge polynomially). Thus (unlike for the periodic Lorentz gas - see for example the survey [14]) the tools of hyperbolic dynamics cannot be used. Instead the standard approach is to use the so-called Katok-Zemliakov construction (see [17]), which allows one to replace the billiard flow by linear flow on translation surfaces.

There have not yet been any theorems concerning the diffusive limit or an invariance principle for the periodic wind-tree process. However there have been a number of interesting and contrasting results concerned with the speed of diffusion and exceptional trajectories. Hardy and Weber [10] showed that some specific directions diffuse at a rate of $\log T \log \log T$. While Delecroix-Hubert-Lelièvre [5] showed that typical (with respect to angle) trajectories satisfy the superdiffusive polynomial diffusion rate $T^{2/3}$. Additionally Delecroix [4] showed that for any rectangular scatterer, there is a set of diverging trajectories with positive Hausdorff measure. While Hubert-Lelièvre-Troubetzkoy [11] and then Avila and Hubert [1] showed that the billiard flow is recurrent for almost every direction. Finally Frączek and Ulcigrai [7] proved that generically the billiard flow is not ergodic.

Random Wind-Tree and Lorentz gas: At the moment there are fewer rigorous results about the random wind-tree model and the random Lorentz gas than their periodic counterparts. Gallavotti [8], [9] used classical (probabilistic) methods to show that in the (annealed) Boltzmann-Grad limit (i.e (3) with T constant) both models obey a linear Boltzmann equation with different collision terms. For a wide class of Lorentz gas models with spherically symmetric scattering potentials, Spohn [16] and Boldrighini-Bunimovich-Sinai (for the hard-core random Lorentz gas) [3] showed that in the Boltzmann-Grad limit the Lorentz gas converges in the annealed, respectively quenched sense to a Markovian flight process. To our knowledge all the previous work on these random models has been in the Boltzmann-Grad limit and for *finite* time intervals. The holy grail - the invariance principle in the diffusive limit - remains completely open for both models.

More recently Marklof and Strömbergsson [15] prove convergence to a limiting transport process for a wide class of spherically symmetric potentials and scatterer configurations. In particular this approach subsumes these previous results on the random Lorentz gas [8], [9], [16], [3] and covers many other cases (periodic or quasi-crystals) all with spherically symmetric scattering potentials.

2 Coupling Construction

2.1 State-Space and Notation

Returning now to the random wind-tree model, for the rest of the paper we assume the initial velocity is fixed to be $v_0 \in \Omega$. This will aid in the exposition but can be assumed without loss of generality, since the time taken to reach this velocity is exponentially bounded.

At each collision one component of the velocity changes sign. Let $\vartheta_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be such that $\vartheta_i(v)_j = (-1)^{\delta_{i,j}} v_j$ for $j = 1, 2, 3$. During a collision the probability $\mathbf{P}(v \mapsto \vartheta_i(v)) = p_i$. For any $v \in \Omega$ let Ω_v denote the set of accessible velocities after one collision starting from v , namely

$$\Omega_v = \{w \in \Omega : w = \vartheta_i(v) \text{ for some } 1 \leq i \leq 3\}. \quad (5)$$

Let m_v denote the measure on Ω_v which selects $\vartheta_i(v)$ with probability p_i . Moreover, for $v \in \Omega$ and $w \in \Omega_v$ let $B(v, w)$ be the face of the cube \mathcal{Q}_r such that a particle traveling with velocity v colliding with that face would adopt the velocity w . Formally, for $v \in \Omega$ and $w = \vartheta_k(v)$

$$B(v, w) = \{b \in \partial \mathcal{Q}_r : b_k = -\frac{v_k}{|v_k|} r\}. \quad (6)$$

2.2 Markovian Flight Process

Let $\{u_n\}_{n=0}^\infty$ be a realization of the following Markov chain on Ω : $u_1 = v_0$ and then for all $i \geq 1$, u_{i+1} are independently selected from Ω_{u_i} according to the measure m_{u_i} . For later use let $u_0 \in \Omega_{v_0}$. Let

$$\{\xi_n\}_{n=1}^\infty \sim EXP(1) \quad (7)$$

be i.i.d exponentially distributed *flight times* and let

$$Y_n := \sum_{i=1}^n y_i \quad , \quad y_n := \xi_n u_n \quad (8)$$

denote the *discrete Markovian Flight Process*. To define the continuous process, for $t \in \mathbb{R}$ let

$$\tau_n := \sum_{i=1}^n \xi_i \quad , \quad \nu_t := \max\{n : \tau_n \leq t\} \quad , \quad \{t\} := t - \tau_{\nu_t}, \quad (9)$$

that is τ_n are the scattering times, ν_t is the label of the most recent scattering, and $\{t\}$ is the time since the previous scattering, at time t . Now define

$$Y(t) := Y_{\nu_t} + u_{\nu_t+1}\{t\} \quad (10)$$

to be the (*continuous*) *Markovian Flight Process*. Note that the processes $t \mapsto Y(t)$ and $\{Y_n\}_{n=1}^\infty$ do not depend on r .

For later use we introduce the following *virtual scatterers*:

$$\begin{aligned} Y'_k &:= Y_k + \beta_k \quad , \quad \beta_k \sim UNI(-B(u_k, u_{k+1})) \quad , \quad k \geq 0 \\ \mathcal{S}_n^Y &:= \{Y'_k \in \mathbb{R}^3, \quad 0 \leq k \leq n\} \quad , \quad n \geq 0. \end{aligned} \quad (11)$$

In words Y'_k is the position of a scatterer *if it had caused* the k^{th} collision (of course Y is independent of any scatterers, thus the term virtual). Note also that we assume there is a virtual collision at time 0, this has no effect on the definition of the model however will ease the notation. One difference with the random Lorentz gas is that the position of a scatterer associated to a velocity jump is not uniquely determined. Therefore we select from among the possible virtual scatterers uniformly.

For later use we introduce the sequence of indicators $\epsilon_j = \mathbb{1}\{\xi_j < 1\}$ and the corresponding distributions $EXP(1|1) := \text{distrib}(\xi|\epsilon = 1)$ and similarly $EXP(1|0) = \text{distrib}(\xi|\epsilon = 0)$. We refer to $\underline{\epsilon} := (\epsilon_j)_{j \geq 0}$ as the *signature* of the sequence $(\xi_j)_{j \geq 0}$.

2.3 Joint Construction

Our goal for this section is to construct the physical wind-tree and Markovian processes on the same probability space. We construct the wind-tree process as an *exploration process*: in that the process explores its environment as time moves forward. For convenience for what follows we will also construct a third *auxiliary process*, $\{t \mapsto Z(t)\}$, coupled to the X and Y processes. The auxiliary process, which we call either the *forgetful* or *myopic* process, is only used in Sections 4 - 6. Hence some readers may wish to ignore it until later. Indeed if we only wanted to prove

Theorem 1 for times of order $o(r^{-1})$ (we do this in Section 3) then this myopic process does not play a role and can be ignored.

The construction will proceed inductively on certain (as yet unspecified) time intervals. To simplify the explanation, first we will explain how the processes X and Z are constructed on a given time interval, given certain random data. Then, we will explain how the random data is generated to enable the coupling to $\{t \mapsto Y(t)\}$ and we will explain on which time intervals these processes are defined.

Throughout the construction we label the velocity of $\dot{X}(t) =: V(t)$, $\dot{Y}(t) =: U(t)$ and $\dot{Z}(t) = W(t)$.

2.3.1 Building X on $[\hat{\tau}_{n-1}, \hat{\tau}_n)$

We label the intervals of construction of X by $[\hat{\tau}_{n-1}, \hat{\tau}_n)$. In Subsection 2.3.4 we will make precise what these $\hat{\tau}$ are.

To construct X on an interval $[\hat{\tau}_{n-1}, \hat{\tau}_n)$, given a position $X(\hat{\tau}_{n-1}) = X_{n-1} \in \mathbb{R}^3$, a velocity $V(\hat{\tau}_{n-1}^+) \in \Omega$ and $\mathcal{S}_{n-1}^X \subset \mathbb{R}^{n-1} \cup \{\star\}$ a finite set of points (where \star is a fictitious point at infinity with $\inf_{x \in \mathbb{R}^3} |x - \star| = \infty$ which will aid in the exposition) perform the following steps:

[Step 1] **Mechanical flight on \mathcal{S}_{n-1}^X in $[\hat{\tau}_{n-1}, \hat{\tau}_n)$:** The trajectory $t \mapsto X(t)$ on $t \in [\hat{\tau}_{n-1}, \hat{\tau}_n)$ is defined to be free motion, with initial position X_{n-1} and velocity $V(\hat{\tau}_{n-1}^+)$, and with reflective collisions on $\mathcal{Q}_r + \mathcal{S}_{n-1}^X$.

[Step 2] **Attempt Fresh Collision:** Suppose, we are given a velocity $\hat{w}_{n+1} \in \Omega_{V(\hat{\tau}_n^-)}$ and an impact parameter $\hat{\beta}_n \in -B(V(\hat{\tau}_n^-), \hat{w}_{n+1})$. Set

$$X'' := X(\hat{\tau}_n) + \hat{\beta}_n \tag{12}$$

Now

- If $\exists 0 < s \leq \hat{\tau}_{n-1} : X(s) \in X'' + \mathcal{Q}_r$ then let $X'_n := \star$, and $V(\hat{\tau}_n^+) = V(\hat{\tau}_n^-)$.
- If not, then $X'_n := X''$, and $V(\hat{\tau}_n^+) = \hat{w}_{n+1}$.

Now set $\mathcal{S}_n^X = \mathcal{S}_{n-1}^X \cup \{X'_n\}$.

We say: on the interval $[\hat{\tau}_{n-1}, \hat{\tau}_n)$ the process $\{t \mapsto X(t)\}$ *attempts a fresh collision* at $\hat{\tau}_n$ with data $(\hat{w}_{n+1}, \hat{\beta}_n)$.

We will make precise the distributions of \hat{w}_{n+1} and $\hat{\beta}_n$ in the construction below. Note that if, given a \hat{w}_{n+1} and a $\hat{\beta}_n$, we build X on the interval $[\hat{\tau}_{n-1}, \hat{\tau}_n)$ then, after the construction we have sufficient information to build X on the interval $[\hat{\tau}_n, \hat{\tau}_{n+1})$ (provided we are given another pair $(\hat{w}_{n+2}, \hat{\beta}_{n+1})$).

2.3.2 Building Z on $[\tilde{\tau}_{n-1}, \tilde{\tau}_n)$

We call the process $\{t \mapsto Z(t)\}$ forgetful in that the process only respects *direct mismatches* (see Figure 2 for a diagram). That is, recollisions with the immediately preceding scatterer, or shadowed events where the scattering is shadowed by the immediately preceding path segment.

Suppose that we are given a time interval $[\tilde{\tau}_{n-1}, \tilde{\tau}_n)$. Assume further, we are given a position $Z(\tilde{\tau}_{n-1}) = Z_{n-1}$, velocity $W(\tilde{\tau}_{n-1}^+) \in \Omega$, and a pair $\mathcal{S}_{n-1}^Z = \{Z'_{n-1}, Z'_{n-2}\} \subset \mathbb{R}^3 \cup \{\star\}$.

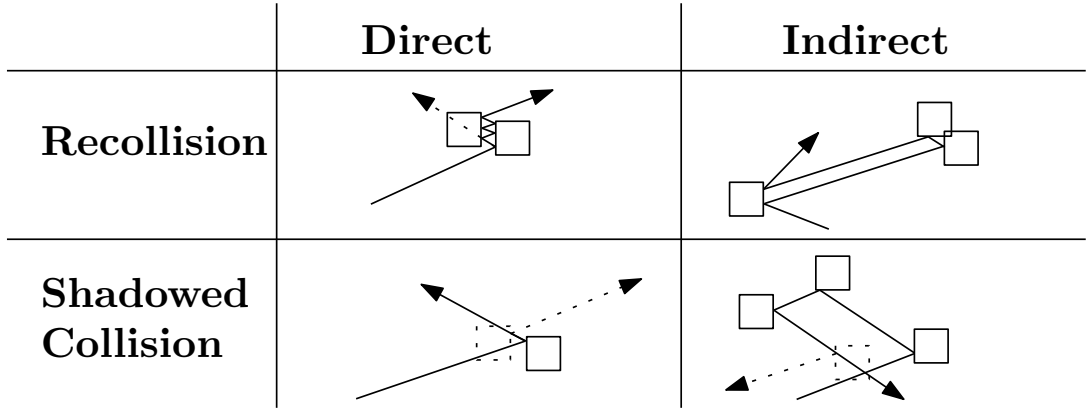


Figure 2: In the above diagram we show examples of direct and indirect, recollisions and shadowed events. In each case the path of the Markovian process is in dotted line while the wind-tree process is in solid line. Additionally, virtual scatterers are in dotted line while actual scatterers for the X process are in solid line.

[Step 1] **Mechanical flight on \mathcal{S}_{n-1}^Z in $[\tilde{\tau}_{n-1}, \tilde{\tau}_n)$:** The trajectory $t \mapsto Z(t)$ on $t \in [\tilde{\tau}_{n-1}, \tilde{\tau}_n)$ is defined to be free motion starting at position $Z(\tilde{\tau}_{n-1})$ and with velocity $W(\tilde{\tau}_{n-1}^+)$ with reflective collisions on $\mathcal{Q}_r + \mathcal{S}_{n-1}^Z$.

[Step 2] **Attempt Fresh Collision:** Suppose that we are given a velocity $\tilde{w}_{n+1} \in \Omega_{W(\tilde{\tau}_n^-)}$ and an impact parameter $\tilde{\beta}_n \in -B(W(\tilde{\tau}_n^-), \tilde{w}_{n+1})$. Set

$$Z'' := Z(\tilde{\tau}_n) + \tilde{\beta}_n \quad (13)$$

Now

- If there exists an $s \in (\tilde{\tau}_{n-2}, \tilde{\tau}_{n-1}] : Z(s) \in Z'' + \mathcal{Q}_r$ then let $Z'_n := \star$, and $W(\tilde{\tau}_n^+) = W(\tilde{\tau}_n^-)$.
- If not, then $Z'_n := Z''$, and $Z(\tilde{\tau}_n^+) = \tilde{w}_{n+1}$.

Now set $\mathcal{S}_n^Z = \{Z'_n, Z'_{n-1}\}$.

Similarly we say that on the interval $[\tilde{\tau}_{n-1}, \tilde{\tau}_n)$ the process $\{t \mapsto Z(t)\}$ *attempts a fresh collision* at $\tilde{\tau}_n$ with data $(\tilde{w}_{n+1}, \tilde{\beta}_n)$.

2.3.3 Parity

Consider just the processes $\{t \mapsto Y(t)\}$ and $\{t \mapsto X(t)\}$, the idea behind the coupling is the following:

- $X(0) = Y(0)$ and the velocities are initially parallel.
- X and Y then run parallel until one of two possible *mismatches* occurs:

- A *recollision*, which corresponds to a collision with a previously placed scatterer during [Step 1] of Subsection 2.3.1.
- A *shadowed collision*, which corresponds to $X'_n = \star$ in [Step 2] of Subsection 2.3.1.
- After a mismatch the two velocity processes proceed independently.
- When the two velocities happen to coincide we recouple the two processes and they run parallel until the next mismatch.

However there is a problem with this setup as we have described it. Note that there are two *parity classes*: $(\mathbf{v}, (\vartheta_i(\vartheta_j(\mathbf{v})))_{i \neq j})$ and $(-\mathbf{v}, (\vartheta_i(\mathbf{v}))_{i=1,2,3})$. The Markov process $(u_n)_{n \in \mathbb{N}}$ alternates between these two classes. The problem is that if there is a parity mismatch between $V(t)$ and $U(t)$ at a given time, then as long as the two processes experience fresh collisions at the same times, only another mismatch can restore the parity. This is too long to wait. Therefore we need to alter the sequence of collision times to restore parity. For this we will make use of Lemma 1. For future use, we define the equivalence relation $u \stackrel{p}{\sim} v$ if u and v are in the same parity class.

Lemma 1. *Let $(\tau_j)_{j \geq 1}$ be the points of a Poisson point process of intensity 1 on \mathbb{R}_+ . Form a new sequence as follows: sample $\xi' \sim EXP(1)$, independently of the sequence $(\tau_j)_{j \geq 1}$. Let the new sequence $(\tau'_j)_{j \geq 1}$ be as follows:*

- *If $\xi' < \tau_1$ then $\tau'_1 = \xi'$, and $\tau'_j = \tau_{j-1}$ for $j \geq 2$. (That is: insert $\xi' < \tau_1$ as the first point and leave the rest as they are.)*
- *If $\xi' > \tau_1$ then $\tau'_j = \tau_{j+1}$ for $j \geq 1$. (That is: delete the first point τ_1 and leave the rest as they are.)*

Proof. Consider the distribution of τ'_1

$$\begin{aligned} \mathbf{P}(\tau'_1 > t) &= \mathbf{P}(\xi > t, \xi < \tau_1) + \mathbf{P}(\tau_2 > t, \xi > \tau_1) \\ &= e^{-t} \mathbf{P}(\tau_1 > \xi \mid \xi > t) + \mathbf{P}(\xi > \tau_1) \mathbf{P}(\tau_2 > t \mid \xi > \tau_1) \end{aligned}$$

where we have used the definition of conditional probability and the fact that ξ is exponentially distributed. Now note that $\mathbf{P}(\tau_2 > t \mid \xi > \tau_1) = \mathbf{P}(\xi > t \mid \xi > \tau_1)$ since τ_2 and ξ are both exponentially distributed conditioned to be larger than τ_1 . Therefore

$$\begin{aligned} \mathbf{P}(\tau'_1 > t) &= e^{-t} \mathbf{P}(\tau_1 > \xi \mid \xi > t) + \mathbf{P}(\xi > \tau_1) \mathbf{P}(\xi > t \mid \xi > \tau_1) \\ &= e^{-t} \mathbf{P}(\tau_1 > \xi \mid \xi > t) + e^{-t} \mathbf{P}(\xi > \tau_1 \mid \xi > t) \\ &= e^{-t} \mathbf{P}(\tau_1 > \xi \mid \xi > t) + e^{-t} (1 - \mathbf{P}(\tau_1 > \xi \mid \xi > t)) \\ &= e^{-t}. \end{aligned}$$

Turning now to the distribution $\tau'_2 - \tau'_1$ (all the other increments are clearly i.i.d exponentially distributed)

$$\begin{aligned}\mathbf{P}(\tau'_2 - \tau'_1 > t) &= \mathbf{P}(\tau_1 - \xi > t, \tau_1 > \xi) + \mathbf{P}(\tau_3 - \tau_2 > t, \tau_1 < \xi) \\ &= e^{-t}\mathbf{P}(\tau_1 > \xi) + e^{-t}\mathbf{P}(\tau_1 < \xi) = e^{-t}.\end{aligned}$$

Finally, we look at the joint distribution

$$\mathbf{P}(\tau'_1 > t, \tau'_2 - \tau'_1 > s) = \mathbf{P}(\xi > t, \tau_1 - \xi > s) + \mathbf{P}(\xi > \tau_1, \tau_2 > t, \tau_3 - \tau_2 > s).$$

By construction $\tau_3 - \tau_2$ is exponentially distributed and independent of τ_1, τ_2, ξ , thus

$$\begin{aligned}\mathbf{P}(\tau'_1 > t, \tau'_2 - \tau'_1 > s) &= \mathbf{P}(\xi > t, \tau_1 - \xi > s) + \mathbf{P}(\xi > \tau_1, \tau_2 > t) e^{-s} \\ &= \mathbf{P}(\xi > t, \tau_1 - \xi > s \mid \xi < \tau_1) \mathbf{P}(\xi < \tau_1) + \mathbf{P}(\xi > \tau_1, \tau_2 > t) e^{-s}.\end{aligned}$$

Conditioned on $\xi < \tau_1$, $\tau_1 - \xi$ is exponentially distributed independently of ξ . Thus

$$\begin{aligned}\mathbf{P}(\tau'_1 > t, \tau'_2 - \tau'_1 > s) &= e^{-s}\mathbf{P}(\xi > t \mid \xi < \tau_1) \mathbf{P}(\xi < \tau_1) + \mathbf{P}(\xi > \tau_1, \tau_2 > t) e^{-s} \\ &= e^{-s}\mathbf{P}(\xi > t, \xi < \tau_1) + \mathbf{P}(\xi > \tau_1, \tau_2 > t) e^{-s} \\ &= e^{-s}\mathbf{P}(\tau'_1 > t) \\ &= e^{-s}e^{-t}.\end{aligned}$$

□

2.3.4 Joint Coupling

Assume $\{t \mapsto Y(t)\}$ is constructed as in Subsection 2.2. We will construct the X and Z processes inductively on the intervals $[\tau_{2n}, \tau_{2n+2})$ as follows: First set

$$\begin{aligned}X(0) = X_0 = 0 \quad , \quad V(0^+) = u_1 \quad , \quad X'_0 = \widehat{\beta}_0 = \beta_0 \quad , \quad \mathcal{S}_0^X = \{X'_0\} \\ Z(0) = Z_0 = 0 \quad , \quad W(0^+) = u_1 \quad , \quad W'_0 = \widetilde{\beta}_0 = \beta_0 \quad , \quad \mathcal{S}_0^Z = \{Z'_0, Z'_{-1}\}\end{aligned}\tag{14}$$

where $Z'_{-1} = \star$. Let $n \in \mathbb{N}$ and sample an exponential time $\zeta_n \sim EXP(1)$ independent of the entire history up to this point. In which case there are 7 possible situations arranged and labelled in the following table:

Parity at time τ_{2n}^+	$\zeta_n \leq \xi_{2n+1}$	$\zeta_n > \xi_{2n+1}$
$U \overset{p}{\sim} V \overset{p}{\sim} W$	A	
$U \overset{p}{\not\sim} V \overset{p}{\sim} W$	B	C
$U \overset{p}{\sim} V \overset{p}{\not\sim} W$	D	E
$U \overset{p}{\sim} W \overset{p}{\not\sim} V$	F	G

For completeness of the construction we define all of these cases, however on our time scales we will (w.h.p) only see situations A, B, and C.

On the interval $[\tau_{2n}, \tau_{2n+2})$ the X and Z processes attempt fresh collisions at the following times:

Situation	X	Z
A	τ_{2n+1}, τ_{2n+2}	τ_{2n+1}, τ_{2n+2}
B	$\tau_{2n} + \zeta_n, \tau_{2n+1}, \tau_{2n+2}$	$\tau_{2n} + \zeta_n, \tau_{2n+1}, \tau_{2n+2}$
C	τ_{2n+2}	τ_{2n+2}
D	τ_{2n+1}, τ_{2n+2}	$\tau_{2n} + \zeta_n, \tau_{2n+1}, \tau_{2n+2}$
E	τ_{2n+1}, τ_{2n+2}	τ_{2n+2}
F	$\tau_{2n} + \zeta_n, \tau_{2n+1}, \tau_{2n+2}$	τ_{2n+1}, τ_{2n+2}
G	τ_{2n+2}	τ_{2n+1}, τ_{2n+2}

In what follows the following **coupling rule** will dictate the random variables $\widehat{\beta}_n, \widehat{w}_n, \widetilde{\beta}_n, \widetilde{w}_n$ used in the attempted fresh collisions.

For the Z -process: If the Z -process is to attempt a fresh collision at time t_a , sample \widetilde{w} from $\Omega_{W(t_a^-)}$ according to the measure $m_{W(t_a^-)}$ and sample $\widetilde{\beta}$ from $-B(W(t_a^-), \widetilde{w})$ both independent of the past. We now attempt to couple W with U at t_a :

- **Couple W to U :** If $W(t_a^-) = U(t_a^-)$ and $t_a = \tau_n$ for some n , attempt a fresh collision at $Z(t_a)$ using data (β_n, u_{n+1}) .
- **W is independent of U :** Otherwise attempt a fresh collision at $Z(t_a)$ using data $(\widetilde{\beta}, \widetilde{w})$.

For the X -process: If the X -process is to attempt a fresh collision at time t_a , sample \widehat{w} from $\Omega_{V(t_a^-)}$ according to the measure $m_{V(t_a^-)}$ and sample $\widehat{\beta}$ from $-B(V(t_a^-), \widehat{w})$ both independent of the past. We now couple V to either U and/or W if possible:

- **Couple V to U :** If $V(t_a^-) = U(t_a^-)$ and $t_a = \tau_n$ for some n attempt a fresh collision at $X(t_a)$ using (β_n, u_{n+1}) .
- **Couple V to W :** If $V(t_a^-) = W(t_a^-)$ and the Z process also attempts a fresh collision *independent of U* at time t_a , attempt a fresh collision at $X(t_a)$ using $(\widetilde{\beta}, \widetilde{w})$.
- **V is independent of U and W :** Otherwise attempt a fresh collision at $X(t_a)$ using $(\widehat{\beta}, \widehat{w})$.

After this construction we have generated two processes. For the wind-tree exploration process $\{t \mapsto X(t)\}$, the *attempted fresh collision* times are $\{\widehat{\tau}_n\}_{n \in \mathbb{N}}$, by Lemma 1 these form a (temporal) Poisson point process on \mathbb{R}_+ ; the scatterers are placed at positions $\{X'_n\} \subset \mathbb{R}^3 \cup \{\star\}$; and the impact parameters are $\{\widehat{\beta}_n\}_{n \in \mathbb{N}}$. Moreover, the *attempted* velocities after collisions are $\{\widehat{w}_n\}_{n \in \mathbb{N}}$, these velocities are attempted since, in [Step 2] the attempted collision may be rejected (i.e $X'_n = \star$). Because of the Poisson distribution of the scatterers in \mathbb{R}^3 this process is distributed like the original wind-tree model as described in the introduction.

For the process $\{t \mapsto Z(t)\}$, the *attempted fresh collision* times are $\{\widetilde{\tau}_n\}_{n \in \mathbb{N}}$, which by Lemma 1 form a (temporal) Poisson point process on \mathbb{R}_+ ; the scatterers are placed at positions $\{Z'_n\} \subset \mathbb{R}^3 \cup \{\star\}$; and the impact parameters are $\{\widetilde{\beta}_n\}_{n \in \mathbb{N}}$. The *attempted* velocities for the Z -process are $\{\widetilde{w}_n\}_{n \in \mathbb{N}}$.

2.4 Main Technical Result and Method Proof

The main result we prove is the following

Theorem 2. *Let $T = T(r)$ be such that $\lim_{r \rightarrow 0} T(r) = \infty$ and $\lim_{r \rightarrow 0} r^2 T(r) = 0$. Then for any $\delta > 0$*

$$\lim_{r \rightarrow 0} \mathbf{P} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)| > \delta \sqrt{T} \right) = 0. \quad (15)$$

From here Theorem 1 follows as a consequence of the classical Donsker's invariance principle [2]: that is, the process $t \mapsto Y(t)$ is a true Markov process, hence Donsker's original invariance principle does not apply directly, however in what follows we will show how to separate Y into i.i.d mean 0 pieces with finite second moment. Thus Donsker's principle will imply that $t \mapsto \frac{Y(tT)}{\sqrt{T}}$ converges to a Wiener process in the diffusive scaling. Therefore the process $t \mapsto X(t)$ does as well. We omit the details of this final step and the rest of the paper is devoted to proving Theorem 2.

The strategy of proof is the same as in [13]. We begin with the joint realization of the Markovian flight process and the wind-tree exploration process described above. During the two mismatch events (recollisions and shadowed scatterings) the two velocity processes diverge. In either case the two processes are decoupled until recoupling is possible. At which point the two processes are recoupled and proceed parallel to each other until the next mismatch.

The proof then follows two steps. In Section 3 we show that such mismatches occur only on time scales of order r^{-1} . Hence until such times both process are (w.h.p) in the the same position and Theorem 2 follows immediately for $T = o(r^{-1})$. Note that this intermediate result is a statement about the Markovian flight process. During the rest of the paper we show that on time scales of order $o(r^{-2})$ only (geometrically) simple mismatches occur. During such mismatches the separation between X and Y is of order $\mathcal{O}(1)$. Hence on the time scales of Theorem 2 there are $o(Tr)$ mismatches. During each mismatch the two processes separate by a distance of order $\mathcal{O}(1)$, hence up to $T = o(r^{-2})$, $\frac{|X(T(r)) - Y(T(r))|}{\sqrt{T}} \rightarrow 0$, thus proving (15). Sections 4-6 are devoted to formalizing this argument.

The reason for introducing the forgetful process $\{t \mapsto Z(t)\}$ is that the forgetful process will satisfy additional independence properties exploited in the proof. Thus during the second stage of the prove, we will in fact show that the forgetful and Markovian processes do not diverge too much. Then we show that with high probability the wind-tree and forgetful processes are in fact the same on these time scales (i.e we show that with probability tending to 1 as $r \rightarrow 0$, the direct mismatches defining the Z -process are the only ones seen by the X -process).

Remark on dimension: As with the Lorentz gas, because of the recurrence of the random walk the same proof does not yield the result in 2 dimensions. For the Lorentz gas the geometry of mismatches imposed another reason that the proof cannot be extended to 2 dimensions. However for the wind-tree model the mismatches have a far simpler geometry and thus this obstruction is not present in 2 dimensions.

2.5 r -consistency and r -compatibility

The proof will hinge on two definitions which we present now for a general process (i.e this could be a segment of any of the above mentioned processes). Let

$$n \in \mathbb{N}, \quad \tau_0 \in \mathbb{R}, \quad \mathcal{Z}_0 \in \mathbb{R}^3, \quad U_0, \dots, U_{n+1} \in \Omega \quad t_1, \dots, t_n \in \mathbb{R}_+,$$

be given, such that either $U_{i+1} \in \Omega_{U_i}$ or $U_{i+1} = U_i$ for all $0 \leq i \leq n$. Moreover fix a set of vectors $\beta_j \in B(U_j, U_{j+1})$ (if $U_j = U_{j+1}$ we set $\beta_j = \star$) and define for $j = 0, \dots, n$,

$$\tau_j := \tau_0 + \sum_{k=1}^j t_k, \quad \mathcal{Z}_j := \mathcal{Z}_0 + \sum_{k=1}^j t_k U_k, \quad \mathcal{Z}'_j := \mathcal{Z}_j + \beta_j$$

and for $t \in [\tau_j, \tau_{j+1}]$, $j = 0, \dots, n$,

$$\mathcal{Z}(t) := \mathcal{Z}_j + (t - \tau_j)U_{j+1}.$$

We call the piece-wise linear trajectory $(\mathcal{Z}(t) : \tau_0^- < t < \tau_n^+)$ mechanically *r-consistent* if

$$\nexists t \in [\tau_0, \tau_n], j \in \{0, \dots, n\} : \mathcal{Z}(t) - \mathcal{Z}'_j \in \mathcal{Q}_r^o \quad (16)$$

(\mathcal{Q}_r^o denotes the interior) and *r-inconsistent* if (16) fails.

Given two finite pieces of mechanically *r-consistent* trajectories $(\mathcal{Z}_a(t) : \tau_{a,0}^- < t < \tau_{a,n_a}^+)$ and $(\mathcal{Z}_b(t) : \tau_{b,0}^- < t < \tau_{b,n_b}^+)$, defined over non-overlapping time intervals: $[\tau_{a,0}, \tau_{a,n_a}] \cap [\tau_{b,0}, \tau_{b,n_b}] = \emptyset$ with $\tau_{a,n_a} \leq \tau_{b,0}$, we will call them mechanically *r-compatible* if

$$\begin{aligned} \nexists t \in [\tau_{a,0}, \tau_{a,n_a}], j \in \{0, \dots, n_b\} : \mathcal{Z}_a(t) - \mathcal{Z}'_{b,j} \in \mathcal{Q}_r^o, \\ \text{and } \nexists t \in [\tau_{b,0}, \tau_{b,n_b}], j \in \{0, \dots, n_a\} : \mathcal{Z}_b(t) - \mathcal{Z}'_{a,j} \in \mathcal{Q}_r^o \end{aligned} \quad (17)$$

mechanical trajectories are *r-incompatible* if (17) fails.

3 No Mismatches Till $T = o(r^{-1})$

3.1 Excursions

Unlike in the 3-dimensional Lorentz gas case the directions of path segments of the Markovian flight process are not independent. To decompose the process $t \mapsto Y(t)$ into i.i.d segments we introduce *excursions*. Let

$$\gamma := \min\{i > 1 : u_{i+1} = v_0\} \quad (18)$$

and define a *pack* to be a collection

$$\varpi := (\gamma; \{u_i\}_{i=1}^\gamma, \{\beta_i\}_{i=1}^\gamma, \{\xi_i\}_{i=1}^\gamma),$$

$u_\gamma \in \Omega_{v_0}$, and for all $i > 1$, $u_i \neq v_0$ and $u_{i-1} \in \Omega_{u_i}$. Given a pack we consider the process $t \mapsto Y(t)$ associated to it via the rules set forth in Section 2.2 - call the process built from such a pack, *an excursion*.

3.2 Concatenation

For $n = 1, 2, 3, \dots$ consider infinitely many independent packs:

$$\varpi_n = (\gamma_n, \{u_{n,i}\}_{i=1}^{\gamma_n}, \{\beta_{n,i}\}_{i=1}^{\gamma_n}, \{\xi_{n,i}\}_{i=1}^{\gamma_n}).$$

For each pack define the associated flight process $t \mapsto Y_n(t)$ together with the discrete process $\{Y_{n,i}\}_{i=0}^{\gamma_n}$. Denote

$$\theta_n := \sum_{i=1}^{\gamma_n} \xi_{n,i}, \quad \bar{Y}_n := Y_{n,\gamma_n}.$$

Define the following variables

$$\begin{aligned} \Gamma_0 &= 0, & \Gamma_n &= \Gamma_{n-1} + \gamma_n, & \text{for } n \geq 1 \\ \nu_n &:= \max\{m : \Gamma_m \leq n\}, & \{n\} &:= n - \Gamma_{\nu_n}. \end{aligned}$$

Likewise

$$\begin{aligned} \Theta_0 &= 0, & \Theta_n &= \Theta_{n-1} + \theta_n, & \text{for } n \geq 1 \\ \nu_t &:= \max\{m : \Theta_m \leq t\}, & \{t\} &:= t - \Theta_{\nu_t}. \end{aligned}$$

Now define the following three processes: the *end-point process* with $\Xi_0 = 0$

$$\Xi_n := \sum_{k=1}^n \bar{Y}_k,$$

the *concatenated discrete Markovian flight process* with $Y_0 = 0$

$$Y_n := \Xi_{\nu_n} + Y_{\nu_n+1,\{n\}},$$

and the continuous *concatenated Markovian flight process* with $Y(0) = 0$

$$Y(t) := \Xi_{\nu_t} + Y_{\nu_t+1}(\{t\}).$$

The advantage of this decomposition is that the different excursions making up the process Y are i.i.d steps with exponentially decaying tails.

3.3 Occupation Measures

Define the following occupation measures for a set $A \subset \mathbb{R}^3$

$$\begin{aligned} G(A) &:= \mathbf{E}(|\{1 \leq k < \infty : Y_k \in A\}|), & H(A) &:= \mathbf{E}(|\{0 < t < \infty : Y(t) \in A\}|), \\ g(A) &:= \mathbf{E}(|\{1 \leq k \leq \gamma_1 : Y_k \in A\}|), & h(A) &:= \mathbf{E}(|\{0 < t < \Theta_1 : Y(t) \in A\}|), \\ R(A) &:= \mathbf{E}(|\{1 \leq k < \infty : \Xi_k \in A\}|). \end{aligned}$$

Lemma 2. *The following upper bounds hold for any measurable set $A \subset \mathbb{R}^3$*

$$R(A) \leq K(A) + L_{v_0}(A), \tag{19}$$

$$g(A) \leq M(A) + L_{v_0}(A), \quad h(A) \leq M(A) + L_{v_0}(A), \tag{20}$$

$$G(A) \leq K(A) + L_{v_0}(A), \quad H(A) \leq K(A) + L_{v_0}(A), \tag{21}$$

where

$$K(dx) := C \min\{1, |x|^{-1}\} dx \quad , \quad M(dx) := C e^{-c|x|} dx$$

$$L_{v_0}(A) := C \int_0^\infty \mathbb{1}\{tv_0 \in A\} e^{-ct} dt$$

This Lemma is slightly different from the Lorentz gas case as L_{v_0} takes into account the discrete state-space of velocities. However the end result (Proposition 1) remains the same.

Proof. To bound $g(A)$ let

$$g_1(A) := \mathbf{P}(Y_1 \in A) = C \int_0^\infty \mathbb{1}\{tv_0 \in A\} e^{-t} dt.$$

We have fixed the initial velocity to be $u_1 = v_0$, therefore the points $\{Y_k - Y_1\}_{k=1}^{\gamma_1}$ are independent of the initial step Y_1 . Therefore write

$$g_2(A) := \mathbf{E}(|\{1 \leq k \leq \gamma_1 : Y_k - Y_1 \in A\}|),$$

and note that

$$g(A) = \int_{\mathbb{R}^3} g_2(A - x) g_1(dx). \quad (22)$$

Similarly we can write

$$h_1(A) := \mathbf{E}(|\{t \leq \tau_1 : Y(t) \in A\}|) = C \int_0^\infty \mathbb{1}\{tv_0 \in A\} e^{-\max\{1,t\}} dt,$$

$$h_2(A) := \mathbf{E}(|\{\tau_1 \leq t \leq \Theta_1 : Y(t) - Y_1 \in A\}|)$$

$$h(A) = \int_{\mathbb{R}^3} h_2(A - x) g_1(dx) + h_1(A). \quad (23)$$

Now the bounds (20) follow by inserting the bounds:

$$g_2(\{x : |x| > s\}) \leq C e^{-cs}, \quad h_2(\{x : |x| > s\}) \leq C e^{-cs}$$

$$g_2(\mathbb{R}^3) = \mathbf{E}(\gamma_1) < \infty, \quad h_2(\mathbb{R}^3) = \mathbf{E}(\Theta_1 - \tau_1) < \infty \quad (24)$$

into (22) and (23). That is,

$$g(A) \leq \int_{A^c} g_2(\{y : |y| > |x|\}) dx + C \int_A g_1(dx) \leq M(A) + L_{v_0}(A) \quad (25)$$

and likewise for $h(A)$.

Now, to achieve (19) note that since $\gamma_1 > 1$

$$\mathbf{P}(\Xi_1 \in A) \leq \mathbf{E}(|\{2 \leq k \leq \gamma_1 : Y_k \in A\}|) \leq g(A) \quad (26)$$

Hence the density of distribution of Ξ_1 is bounded by the density of g . Moreover, because $\mathbf{P}(\theta_1 > s) \leq Ce^{-cs}$ for some $C < \infty$ and $c > 0$, we know that the density of distribution of Ξ_1 has exponentially decaying tails. Therefore Ξ is a random walk, with i.i.d steps, and step distribution bounded by g with exponentially decaying tails. Hence a standard random walk argument implies (19).

(21) then follows by writing (using the fact that the different excursions are i.i.d)

$$G(A) = g(A) + \int_{\mathbb{R}^3} g(A-x)R(dx), \quad H(A) = h(A) + \int_{\mathbb{R}^3} h(A-x)R(dx)$$

and inserting (19) and (20). \square

3.4 Inter-Excursion Mismatches

Let $t \rightarrow Y^*(t)$ denote a Markovian flight process with associated virtual scatterers $Y^{*l} \in \mathcal{S}^{Y^*}$ and initial velocity $u_1^* \in -\Omega_{v_0}$. Let $t \rightarrow Y(t)$ be a second Markovian flight process with associated virtual scatterers \mathcal{S}^Y , and initial velocity v_0 .

We think of Y^* as the process run backwards in time. Define the events

$$\begin{aligned} \widehat{W}_j &:= \{ \{Y(t) - Y'_k : & 0 < t < \Theta_{j-1}, & \Gamma_{j-1} < k \leq \Gamma_j\} \cap \mathcal{Q}_r \neq \emptyset \}, \\ \widetilde{W}_j &:= \{ \{Y'_k - Y(t) : & 0 \leq k < \Gamma_{j-1}, & \Theta_{j-1} < t < \Theta_j\} \cap \mathcal{Q}_r \neq \emptyset \}, \\ \widehat{W}_j^* &:= \{ \{Y^*(t) - Y'_k : & 0 < t < \Theta_{j-1}, & 0 < k \leq \gamma\} \cap \mathcal{Q}_r \neq \emptyset \}, \\ \widetilde{W}_j^* &:= \{ \{Y_k^{*l} - Y(t) : & 0 < k \leq \Gamma_{j-1}, & 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset \}, \\ \widehat{W}_\infty^* &:= \{ \{Y^*(t) - Y'_k : & 0 < t < \infty, & 0 < k \leq \gamma\} \cap \mathcal{Q}_r \neq \emptyset \}, \\ \widetilde{W}_\infty^* &:= \{ \{Y_k^{*l} - Y(t) : & 0 < k < \infty, & 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset \}. \end{aligned}$$

In words \widehat{W}_j is the event that during the $(j-1)^{th}$ excursion, a collision of Y is (virtually) *shadowed* by a previous excursion. And \widetilde{W}_j is the event that during the $(j-1)^{th}$ excursion the process (virtually) *recollides* with a scatterer from an earlier excursion.

It readily follows that

$$\begin{aligned} \mathbf{P}(\widehat{W}_j) &= \mathbf{P}(\widehat{W}_j^*) \leq \mathbf{P}(\widehat{W}_{j+1}^*) \leq \mathbf{P}(\widehat{W}_\infty^*), \\ \mathbf{P}(\widetilde{W}_j) &= \mathbf{P}(\widetilde{W}_j^*) \leq \mathbf{P}(\widetilde{W}_{j+1}^*) \leq \mathbf{P}(\widetilde{W}_\infty^*). \end{aligned} \tag{27}$$

By the union bound

$$\begin{aligned} \mathbf{P}(\widehat{W}_\infty^*) &\leq \sum_{z \in \mathbb{Z}^3} \mathbf{P}(\{1 < k < \infty : Y_k^* \in B_{zr, 2r}\} \neq \emptyset) \mathbf{P}(\{0 < t \leq \theta : Y(t) \in B_{zr, 2r}\} \neq \emptyset) \\ &\leq \sum_{z \in \mathbb{Z}^3} (2r)^{-1} \mathbf{E}(|\{1 < k < \infty : Y_k^* \in B_{zr, 2r}\}|) \cdot \mathbf{E}(|\{0 < t \leq \theta : Y(t) \in B_{zr, 3r}\}|) \\ \mathbf{P}(\widetilde{W}_\infty^*) &\leq \sum_{z \in \mathbb{Z}^3} \mathbf{P}(\{0 < t < \infty : Y^*(t) \in B_{zr, 2r}\} \neq \emptyset) \mathbf{P}(\{1 \leq j \leq \gamma : Y_j \in B_{zr, 2r}\} \neq \emptyset) \\ &\leq \sum_{z \in \mathbb{Z}^3} (2r)^{-1} \mathbf{E}(|\{0 < t < \infty : Y^*(t) \in B_{zr, 3r}\}|) \cdot \mathbf{E}(|\{1 \leq j \leq \gamma : Y_j \in B_{zr, 2r}\}|) \end{aligned} \tag{28}$$

3.5 Computations

(28) implies that

$$\begin{aligned}\mathbf{P}\left(\widetilde{W}_\infty^*\right) &\leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} H^*(B_{zr,3r})g(B_{zr,2r}) \\ \mathbf{P}\left(\widehat{W}_\infty^*\right) &\leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} G^*(B_{zr,3r})h(B_{zr,2r}).\end{aligned}\tag{29}$$

where G^* is defined like G , except that in this instance the initial velocity is chosen from $-\Omega_{v_0}$ rather than fixed to be v_0 .

Lemma 3. *The following bounds hold for some $C < \infty$ and any $v \in \Omega$*

$$\begin{aligned}\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})M(B_{zr,2r}) &\leq Cr^3, & \sum_{z \in \mathbb{Z}^3} L_v(B_{zr,3r})M(B_{zr,2r}) &\leq Cr^3 \\ \sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})L_v(B_{zr,2r}) &\leq Cr^3, & \sum_{z \in \mathbb{Z}^3} L_v(B_{zr,3r})L_w(B_{zr,2r}) &\leq Cr^2.\end{aligned}$$

for $v \neq w \in \Omega$

Proof. The following bounds follow immediately from the definitions of K , M , and L_v

$$\begin{aligned}K(B_{zr,3r}) &\leq Cr^3, \\ M(B_{zr,3r}) &\leq Cr^3 e^{-cr|z|}, \\ L_v(B_{zr,3r}) &\leq Cr^3 \delta_{0,z} + Cr \mathbb{1}\{\exists t > 0 : vt \cap B_{zr,3r}\} (1 - \delta_{0,z}) e^{-cr|z|}.\end{aligned}\tag{30}$$

From here

$$\begin{aligned}\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})M(B_{zr,2r}) &\leq Cr^6 \sum_{z \in \mathbb{Z}^3} e^{-cr|z|} \\ &\leq Cr^3 \int_{\mathbb{R}^3} e^{-c|z|} dz \leq Cr^3\end{aligned}$$

where we use a Riemann integral to go from the first line to the second. Likewise

$$\begin{aligned}\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r})L_v(B_{zr,2r}) &\leq Cr^6 + Cr^4 \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{1}\{\exists t > 0 : vt \cap B_{zr,3r}\} e^{-cr|z|} \\ &\leq Cr^6 + C'r^4 \sum_{z=1}^{\infty} e^{-cr|vz|} \\ &\leq Cr^6 + Cr^3 \int_0^{\infty} e^{-c|vt|} dt \leq Cr^3\end{aligned}\tag{31}$$

where from the first line to the second we approximate the points $zr \in r\mathbb{Z}^3$ close to the line vt by the points rvz for $z \in \mathbb{Z}$.

Similarly

$$\sum_{z \in \mathbb{Z}^3} L_v(B_{zr,3r})M(B_{zr,2r}) \leq Cr^6 + Cr^4 \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{1}\{\exists t > 0 : vt \cap B_{zr,3r}\} e^{-2cr|z|}$$

the bound then follows as it did in (31).

Finally,

$$\begin{aligned} \sum_{z \in \mathbb{Z}^3} L_v(B_{zr,3r})L_w(B_{zr,2r}) &\leq Cr^2 \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{1}\{\exists t > 0 : vt \cap B_{zr,3r}\} \mathbb{1}\{\exists t > 0 : wt \cap B_{zr,3r}\} e^{-2cr|z|} \\ &\leq Cr^2 e^{-cr} \leq Cr^2, \end{aligned}$$

since $v \neq w$ only finitely many $z \in \mathbb{Z}$ contribute to the sum, from which the second line follows. \square

Note that Lemma 2 is stated for G and H and not G^* and H^* . However similar bounds hold for the backwards excursions. Thus (omitting these details), we use Lemma 2 to insert Lemma 3 into (29) to get:

Proposition 1. *There exists a constant $C > 0$ such that for all $j \geq 1$*

$$\mathbf{P}\left(\widehat{W}_j\right) \leq Cr \quad , \quad \mathbf{P}\left(\widetilde{W}_j\right) \leq Cr. \quad (32)$$

3.6 Mismatches within one Excursion

Define the following indicator functions

$$\begin{aligned} \widehat{\eta}_j &= \widehat{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{ \min_{0 \leq t \leq \xi_{j-2}} (tu_{j-2} + y_{j-1} + \beta_{j-1}) \in \mathcal{Q}_r \right\} \\ \widetilde{\eta}_j &= \widetilde{\eta}(y_{j-2}, y_{j-1}, y_j) := \mathbb{1}\left\{ \min_{0 \leq t \leq \xi_j} (y_{j-1} + tu_j - \beta_{j-2}) \in \mathcal{Q}_r \right\} \\ \eta_j &:= \max\{\widetilde{\eta}_j, \widehat{\eta}_j\} \end{aligned} \quad (33)$$

In words, $\widehat{\eta}_j$ is the event that the $(j-1)$ -labelled collision is shadowed by the immediately preceding path (i.e a *direct* shadowing event). And $\widetilde{\eta}_j$ is the event that during the j^{th} path segment there is a recollision with the immediately preceding obstacle (i.e a *direct* recollision) - see the left hand side of Figure 2.

Lemma 4. *For any $i, j < \gamma$ with $i \neq j$ there exists a $C < \infty$ such that*

$$\mathbf{E}(\eta_j) \leq Cr \quad (34)$$

$$\mathbf{E}(\eta_j \eta_i) \leq Cr^2 \quad (35)$$

(35) is not needed to prove the result for $T = o(r^{-1})$ however will be used to prove Theorem 2.

Proof of Lemma 4. Suppose $u_{j-2} = U$. Then throughout the two subsequent collisions we know for some $i = 1, 2, 3$ - $(u_{j-1})_i = (u_j)_i = U_i$ (i.e one coordinate of the velocity remains unchanged). Thus to (directly) recollide with $Y'_{j-2} + \mathcal{Q}_r$ we require $\xi_{j-1} < Cr$ which implies (34). The same is true for shadowing events, that is $\hat{\eta}_j = 1$ implies $\xi_{j-1} > Cr$ for some constant.

(35) follows for the same reason. Suppose $i \neq j$, then for $\eta_j \eta_i = 1$, requires $\max\{\xi_{j-1}, \xi_{i-1}\} < Cr$ for some constant. As these are independent exponentials (35) is immediate. \square

Lemma 4 controls the probability of a *direct* mismatch. However we also need to control indirect mismatches. To that end define

$$\begin{aligned}\hat{\eta}_j^o &:= \mathbb{1} \left\{ \min_{0 \leq t \leq \tau_{j-3}} (Y(t) - Y'_{j-1}) \in \mathcal{Q}_r \right\} \\ \tilde{\eta}_j^o &:= \mathbb{1} \left\{ \min_{\tau_{j-1} \leq t \leq \tau_j} \left(\min_{0 \leq k \leq j-3} (Y(t) - Y'_k) \right) \in \mathcal{Q}_r \right\} \\ \eta_j^o &:= \max\{\tilde{\eta}_j^o, \hat{\eta}_j^o\}\end{aligned}\tag{36}$$

In words $\hat{\eta}_j^o$ is the indicator that an *indirect* (virtual) shadowing event occurs and $\tilde{\eta}_j^o$ is the event an *indirect* (virtual) recollision occurs. That is a mismatch which involves more than the immediately preceding obstacle or path.

Lemma 5. *For any $3 < j \leq \gamma$ there exists a constant $C > 0$ such that*

$$\mathbf{E}(\eta_j^o) \leq C\gamma^2 r^2\tag{37}$$

Proof of Lemma 5. Under time reversal Markovian flight processes remain Markovian flight process while recollisions become shadowed events. Hence recollisions and shadowing events happen with the same probability and thus we may restrict to proving the statement for recollisions.

By the union bound

$$\mathbf{E}(\tilde{\eta}_j^o) \leq \sum_{k \leq j-3} \mathbf{P} \left(\left\{ \min_{\tau_{j-1} \leq t \leq \tau_j} (Y(t) - Y'_k) \in \mathcal{Q}_r \right\} \right).\tag{38}$$

Write $\mathcal{A}_k = \{\min_{\tau_{j-1} \leq t \leq \tau_j} (Y(t) - Y'_k) \in \mathcal{Q}_r\}$ - the event there is an indirect recollision after $k-1$ fresh collisions. To have an indirect recollision, requires at least three distinct velocities along the path, thus

$$\mathbf{P}(\mathcal{A}_k) = \mathbf{P}(\mathcal{A}_k \cap \{\exists i \in [k+1, j-2] : u_i \neq u_j, u_{j-1}\}).$$

Moreover at each collision exactly one of the velocity coordinates changes sign. Hence we know u_j and u_{j-1} differ by a sign change in one coordinate therefore the event in the right hand side of (3.6) implies there is a third velocity which is linearly independent of u_j and u_{j-1} . Therefore

$$(3.6) = \mathbf{P}(\mathcal{A}_k \cap \{\exists i \in [k+1, j-2] : u_i, u_j, u_{j-1} \text{ lin. ind.}\})$$

Moreover note that if we fix i

$$\mathcal{A}_k = \left\{ \min_{0 \leq t \leq \xi_j} (\xi_i u_i + \xi_{j-1} u_{j-1} + t u_j - s_i) \in \mathcal{Q}_r \right\}$$

where

$$s_i = \sum_{\substack{l=k+1 \\ l \neq i}}^{j-2} u_l \xi_l.$$

Let B_i denote the event u_i, u_{j-1}, u_j are linearly independent. In this case

$$\begin{aligned} \mathbf{P}(\mathcal{A}_k) &\leq \sum_{i=k+1}^{j-2} \mathbf{P}(B_i \cap \mathcal{A}_k) \\ &\leq \sum_{i=k+1}^{j-2} \mathbf{E} \left(\mathbf{P} \left(B_i \cap \left\{ \min_{0 \leq t \leq \xi_j} (\xi_i u_i + \xi_{j-1} u_{j-1} + t u_j - s_i) \in \mathcal{Q}_r \right\} \mid s_i \right) \right). \end{aligned}$$

Lemma 6 (below) implies that the probability inside the expectation is bounded by Cr^2 . As $j-2-k \leq \gamma$ this implies

$$\mathbf{P}(\mathcal{A}_k) \leq C\gamma r^2.$$

Inserting this into (38) then implies (37) □

Lemma 6. *Suppose $U_1, U_2, U_3 \in \Omega$ are linearly independent and $\xi_1, \xi_2, \xi_3 \sim EXP(1)$ are i.i.d exponentials. Then there exists a constant $C < \infty$ such that for any $s \in \mathbb{R}^3$*

$$\mathbf{P} \left(\min_{0 \leq t \leq \xi_3} (U_1 \xi_1 + U_2 \xi_2 + U_3 t - s) \in \mathcal{Q}_r \right) \leq Cr^2. \quad (39)$$

Proof. We can assume

$$U_1 = (\nu_1, \nu_2, \nu_3) \quad , \quad U_2 = (-\nu_1, \nu_2, \nu_3) \quad , \quad U_3 = (-\nu_1, -\nu_2, \nu_3)$$

in which case for any $t \leq \xi_3$

$$U_1 \xi_1 + U_2 \xi_2 + U_3 t = ((\xi_1 - \xi_2 - t)\nu_1, (\xi_1 + \xi_2 - t)\nu_2, (\xi_1 + \xi_2 + t)\nu_3). \quad (40)$$

Therefore the event on the left hand side of (39) is the event that there exists a $t \leq \xi_3$ satisfying the system of inequalities

$$\begin{aligned} s_1 - \frac{r}{2} &\leq (\xi_1 - \xi_2 - t)\nu_1 &\leq s_1 - \frac{r}{2} \\ s_2 - \frac{r}{2} &\leq (\xi_1 + \xi_2 - t)\nu_2 &\leq s_2 - \frac{r}{2} \\ s_3 - \frac{r}{2} &\leq (\xi_1 + \xi_2 + t)\nu_3 &\leq s_3 - \frac{r}{2} \end{aligned}$$

solving these equations, we find that regardless of t there exist c_1, c_2, C_1, C_2 such that

$$\xi_1 \in [c_1 - C_1 r, c_1 + C_1 r] \quad , \quad \xi_2 \in [c_2 - C_2 r, c_2 + C_2 r]$$

since ξ_1 and ξ_2 are i.i.d exponentials (39) follows immediately. \square

4 Beyond the Naïve Coupling

In the following sections we extend the results of Section 3 to times on the order $o(r^{-2})$. In order to reduce the amount of notation we will use the same notation for the *analogous* objects and will give the redefinitions explicitly. Recall the definition of the process $\{t \mapsto Z(t)\}$ given in Subsection 2.3. We will split the process $\{t \mapsto Z(t)\}$ into legs (similar to the excursions of the previous section).

4.1 Legs

Similar to Subsection 3.1 we split $t \mapsto Z(t)$ into legs. However to ensure that the different legs are independent we impose the restriction that each leg begins and ends with two path segments of length greater than 1. Let $\tilde{\xi}_n = \tilde{\tau}_n - \tilde{\tau}_{n-1}$ for all $n \geq 1$. Let

$$\gamma := \min\{i > 1 : \tilde{\xi}_{i-1}, \tilde{\xi}_i, \tilde{\xi}_{i+1}, \tilde{\xi}_{i+1} > 1, \tilde{w}_{i+1} = \tilde{w}_1 = v_0\}. \quad (41)$$

Note that the condition on $\tilde{\xi}_i$ implies that $\gamma \in \{2\} \cup \{5, \dots\}$. If we define $\theta := \sum_{i=1}^{\gamma} \tilde{\xi}_i$ then

$$\mathbf{P}(\gamma > s) \leq C e^{-cs} \quad , \quad \mathbf{P}(\theta > s) \leq C e^{-cs}. \quad (42)$$

The definition of a pack is then similar to Subsection 3.1: a *pack* is a collection

$$\varpi := \left(\gamma; \{\tilde{\xi}_i\}_{i=1}^{\gamma}, \{\tilde{\beta}_i\}_{i=1}^{\gamma}, \{\tilde{w}_i\}_{i=1}^{\gamma} \right),$$

Given a pack we consider the process $t \mapsto Z(t)$ associated to it via the rules set forth in Subsection 2.3 and call such a segment a *leg*. Note that, in order to have a direct mismatch at step n requires that $\tilde{\xi}_{n-1} < Cr$ for some constant $C < \infty$. Hence the beginning and end of a leg are Markovian steps.

Furthermore given a pack ϖ a *backwards leg* is defined to be

$$(\theta; Z^*(t); 0 \leq t \leq \theta)$$

where

$$Z^*(t) = Z(\theta - t, \varpi^*) - \bar{Z}(\varpi^*)$$

(we use the notation $Z(t, \varpi)$ to denote the forward forgetful process built from the pack ϖ) where

$$\varpi^* := (\gamma; \{\tilde{\xi}_{\gamma-j}\}_{j=0}^{\gamma-1}, \{\tilde{\beta}_{\gamma-j}\}_{j=0}^{\gamma-1}, \{\tilde{w}_{\gamma-j}\}_{j=0}^{\gamma-1})$$

As before denote

$$Z_j^* := Z^*(\tilde{\tau}_j), \quad 0 \leq j \leq \gamma \quad , \quad \overline{Z^*} = Z_{\gamma}^*.$$

Note the processes $t \mapsto Z(t)$ and $t \mapsto Z^*(t)$ do not have the same distribution.

4.2 Concatenation

Let $\varpi_n = \left(\gamma_n; \{\tilde{\xi}_{n,j}\}_{j=1}^{\gamma_n}, \{\tilde{\beta}_{n,j}\}_{j=1}^{\gamma_n}, \{\tilde{w}_{n,j}\}_{j=1}^{\gamma_n} \right)$, $n \geq 1$, be a sequence of i.i.d *packs* and consider the associated forwards legs $(Z_n(t) : 0 \leq t \leq \theta_n)$, $(Z_{n,j} : 1 \leq j \leq \gamma_n)$ and backwards legs $(Z_n^*(t) : 0 \leq t \leq \theta_n)$, $(Z_{n,j}^* : 1 \leq j \leq \gamma_n)$.

To construct the concatenated forward and backward processes $t \mapsto Z(t)$, $t \mapsto Z^*(t)$, $0 \leq t < \infty$, define for $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$

$$\begin{aligned} \Gamma_n &:= \sum_{k=1}^n \gamma_k, \quad \nu_n := \max\{m : \Gamma_m \leq n\}, \quad \{n\} := n - \Gamma_{\nu_n}, \\ \Theta_n &:= \sum_{k=1}^n \theta_k, \quad \nu_t := \max\{m : \Theta_m < t\}, \quad \{t\} := t - \Theta_{\nu_t}. \end{aligned} \tag{43}$$

The concatenated (multi-leg) forward and backward Z -processes are

$$\begin{aligned} \Xi_n &:= \sum_{k=1}^n \overline{Z}_k, & Z_n &:= \Xi_{\nu_n} + Z_{\nu_n+1, \{n\}}, & Z(t) &:= \Xi_{\nu_t} + Z_{\nu_t+1}(\{t\}), \\ \Xi_n^* &:= \sum_{k=1}^n \overline{Z}_k^*, & Z_n^* &:= \Xi_{\nu_n}^* + Z_{\nu_n+1, \{n\}}^*, & Z^*(t) &:= \Xi_{\nu_t}^* + Z_{\nu_t+1}^*(\{t\}). \end{aligned} \tag{44}$$

4.3 Mismatches in a Leg

Let $\varpi = (\gamma; \{\tilde{\xi}_j\}_{j=1}^{\gamma}, \{\tilde{\beta}_j\}_{j=1}^{\gamma}, \{\tilde{w}_j\}_{j=1}^{\gamma})$ be a pack. Let $u \in \Omega_{v_0}$ a velocity and $\beta_0 \in B(u, v_0)$ an impact parameter.

Let $t \mapsto \mathcal{X}(t)$ be the wind-tree process coupled to the pack ϖ . That is, given the processes $t \mapsto Y(t)$ and $t \mapsto Z(t)$ follow the rules in Subsection 2.3 until time τ_γ .

Consider the jointly realized triple $((Y(t), \mathcal{X}(t), Z(t)) : 0^- < t < \theta^+)$ - a Markovian flight process, a wind-tree exploration process and a forgetful process all coupled to ϖ . The time interval $0^- < t < \theta^+$ indicates that the velocity immediately prior to the position at 0 is u , there is a collision with a scatterer at β_0 , and at θ^+ the velocity of Y and Z is w .

Proposition 2. *There exists a $C < \infty$ such that for all $w \in \Omega$ and $u \in \Omega_w$ and $\beta_0 \in B(u, w)$*

$$\mathbf{P}(\mathcal{X}(t) \neq Z(t) : 0^- < t < \theta^+) \leq r^2. \tag{45}$$

This proposition will be proved in Section 6.

4.4 Inter-Leg Mismatches

Consider a forgetful process $t \mapsto Z(t)$ built from legs. Define the following events

$$\begin{aligned}\widehat{W}_j &:= \{\{Z(t) - Z'_k : 0 < t < \Theta_{j-1}, \Gamma_{j-1} < k \leq \Gamma_j\} \cap \mathcal{Q}_r \neq \emptyset\}, \\ \widetilde{W}_j &:= \{\{Z'_k - Z(t) : 0 \leq k < \Gamma_{j-1}, \Theta_{j-1} < t < \Theta_j\} \cap \mathcal{Q}_r \neq \emptyset\},\end{aligned}\tag{46}$$

i.e \widehat{W}_j is the event a collision during the j^{th} leg is (virtually) shadowed by a path segment in a previous leg. \widetilde{W}_j is the event that during the j^{th} leg the process (virtually) collides with an obstacle placed during a previous leg.

Proposition 3. *There exists a $C < \infty$ such that for all $j \geq 1$,*

$$\mathbf{P}(\widetilde{W}_j) \leq Cr^2 \quad , \quad \mathbf{P}(\widehat{W}_j) \leq Cr^2.\tag{47}$$

The proof of this proposition is the content of Section 5.

5 Proof of Proposition 3

The proof of Proposition 3 follows the similar lines to that of Proposition 1. However as we have redefined legs we shall go through the full proof. In this section we redefine the Green's functions g, h, G , and H .

5.1 Occupation Measures

Let $t \mapsto Z(t)$ be a forward forgetful process with initial velocity v_0 and $t \mapsto Z^*(t)$ a backward process with initial velocity in $\Omega_{-\tilde{w}_1}$ (distributed according to m_{-v_0}). Define the events

$$\begin{aligned}\widehat{W}_j^* &:= \{\{Z^*(t) - Z'_k : 0 < t < \Theta_{j-1}, 0 < k \leq \gamma\} \cap \mathcal{Q}_r \neq \emptyset\}, \\ \widetilde{W}_j^* &:= \{\{Z_k^* - Z(t) : 0 < k \leq \Gamma_{j-1}, 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset\}, \\ \widehat{W}_\infty^* &:= \{\{Z^*(t) - Z'_k : 0 < t < \infty, 0 < k \leq \gamma\} \cap \mathcal{Q}_r \neq \emptyset\}, \\ \widetilde{W}_\infty^* &:= \{\{Z_k^* - Z(t) : 0 < k < \infty, 0 < t < \theta\} \cap \mathcal{Q}_r \neq \emptyset\}.\end{aligned}$$

The same calculation as (27), (28), and (29) implies

$$\begin{aligned}\mathbf{P}(\widetilde{W}_j) &\leq \mathbf{P}(\widetilde{W}_\infty^*) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} H^*(B_{zr, 3r})g(B_{zr, 2r}), \\ \mathbf{P}(\widehat{W}_j) &\leq \mathbf{P}(\widehat{W}_\infty^*) \leq (2r)^{-1} \sum_{z \in \mathbb{Z}^3} G^*(B_{zr, 3r})h(B_{zr, 2r}),\end{aligned}\tag{48}$$

where the right hand side is in terms of the following Green's functions: for $A \subset \mathbb{R}^3$

$$\begin{aligned}g(A) &:= \mathbf{E}(|\{1 \leq k \leq \gamma : Z_k \in A\}|), & g^*(A) &:= \mathbf{E}(|\{1 \leq k \leq \gamma : Z_k^* \in A\}|), \\ h(A) &:= \mathbf{E}(|\{0 < t \leq \theta : Z(t) \in A\}|), & h^*(A) &:= \mathbf{E}(|\{0 < t \leq \theta : Z^*(t) \in A\}|), \\ R^*(A) &:= \mathbf{E}(|\{1 \leq n < \infty : \Xi_n^* \in A\}|), & & \\ G^*(A) &:= \mathbf{E}(|\{1 \leq k < \infty : Z_k^* \in A\}|), & H^*(A) &:= \mathbf{E}(|\{0 < t < \infty : Z^*(t) \in A\}|).\end{aligned}$$

Note that

$$\begin{aligned} G^*(A) &= g^*(A) + \int_{\mathbb{R}^3} g^*(A-x)R^*(dx), \\ H^*(A) &= h^*(A) + \int_{\mathbb{R}^3} h^*(A-x)R^*(dx). \end{aligned} \tag{49}$$

5.2 Bounds

Lemma 7. *The following bounds hold for any Borel set $A \subset \mathbb{R}^3$*

$$g(A) \leq M(A) + \tilde{L}_{v_0}(A), \quad g^*(A) \leq M(A) + \tilde{L}_{v_0}^\perp(A), \tag{50}$$

$$h(A) \leq M(A) + L_{v_0}(A), \quad h^*(A) \leq M(A) + L_{v_0}^\perp(A), \tag{51}$$

$$R^*(A) \leq K(A) + \tilde{L}_{v_0}^\perp(A), \tag{52}$$

$$G^*(A) \leq K(A) + \tilde{L}_{v_0}^\perp(A), \quad H^*(A) \leq K(A) + L_{v_0}^\perp(A), \tag{53}$$

where K , L_{v_0} , and M are as defined in Lemma 2 and

$$L_{v_0}^\perp(A) := C \sum_{w \in \Omega_{-v_0}} \int_0^\infty \mathbb{1}\{tw \cap A\} e^{-ct} dt,$$

$$\tilde{L}_{v_0}(A) := C \int_1^\infty \mathbb{1}\{tv_0 \cap A\} e^{-ct} dt, \quad \tilde{L}_{v_0}^\perp(A) := C \sum_{w \in \Omega_{-v_0}} \int_1^\infty \mathbb{1}\{tw \cap A\} e^{-ct} dt.$$

Proof. The proof of this Lemma follows the same lines as the proof of Lemma 2 however the legs in this section are conditioned to have the first step longer than 1. (52) follows from the fact that the steps of Ξ_n^* are i.i.d with exponentially decaying tails and the density of each step is bounded by $g^*(dx)$.

To bound $g(A)$ write:

$$\begin{aligned} g(A) &= \int_{\mathbb{R}^3} g_2(A-x)g_1(dx), \\ g_1(A) &:= \mathbf{P}(Z_1 \in A) = C \int_1^\infty \mathbb{1}\{tv_0 \in A\} e^{-t} dt, \\ g_2(A) &:= \mathbf{E}(|\{1 \leq k \leq \gamma_1 : Z_k - Z_1 \in A\}|). \end{aligned}$$

This follows since $Z_k - Z_1$ is independent of Z_1 for every $k \geq 2$. (50) then follows in the same way as did (20) in Lemma 2 from the bounds

$$g_2(\{x : |x| > s\}) \leq C e^{-cs} \quad , \quad g_2(\mathbb{R}^3) = \mathbf{E}(\gamma) < \infty.$$

For $g^*(A)$ write

$$\begin{aligned}
g^*(A) &= \mathbf{E}(|\{1 \leq k \leq \gamma_1 : Z_k^* \in A\}|) \\
&\leq \sum_{w \in \Omega_{-v_0}} \mathbf{E}(|\{1 \leq k \leq \gamma_1 : Z_k^* \in A\}| \mid \tilde{w}_1^* = w) =: \sum_{w \in \Omega_{-v_0}} g_w^*(A),
\end{aligned}$$

where $\tilde{w}_1^* := \dot{Z}^*(0^+)$. As for $g(A)$ we now split

$$\begin{aligned}
g_w^*(A) &= \int_{\mathbb{R}^3} g_{2,w}^*(A-x) g_{1,w}^*(dx) \\
g_{1,w}^*(A) &:= \mathbf{P}(Z_1^* \in A \mid \tilde{w}_1^* = w) \\
g_{2,w}^*(A) &:= \mathbf{E}(|\{1 \leq k \leq \gamma_1 : Z_k - Z_1 \in A\}| \mid \tilde{w}_1^* = w)
\end{aligned}$$

Our bound for $g^*(A)$ now follows the same lines as for $g(A)$. $h^*(A)$ is very similar.

The bounds on G^* and H^* follow by inserting the bounds for g^*, h^*, R^* into (49). □

5.3 Computations

Lemma 8. *The following bounds hold for some $C < \infty$ and r small enough*

$$\begin{aligned}
\sum_{z \in \mathbb{Z}^3} \tilde{L}_{v_0}^\perp(B_{zr,3r}) L_{v_0}(B_{zr,2r}) &= 0, & \sum_{z \in \mathbb{Z}^3} L_{v_0}^\perp(B_{zr,3r}) \tilde{L}_{v_0}(B_{zr,2r}) &= 0, \\
\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r}) \tilde{L}_{v_0}(B_{zr,2r}) &\leq Cr^3, & \sum_{z \in \mathbb{Z}^3} \tilde{L}_{v_0}^\perp(B_{zr,3r}) M(B_{zr,2r}) &\leq Cr^3, \\
\sum_{z \in \mathbb{Z}^3} L_{v_0}^\perp(B_{zr,3r}) M(B_{zr,2r}) &\leq Cr^3.
\end{aligned}$$

Proof. These bounds follow by observing

$$\begin{aligned}
\tilde{L}_{v_0}(B_{zr,3r}) &\leq C \mathbb{1}\{\exists t \geq 1 : B_{zr,3r} \cap v_0 t\} r e^{-cr|z|}, \\
\tilde{L}_{v_0}^\perp(B_{zr,3r}) &\leq C \sum_{w \in \Omega_{-v_0}} \mathbb{1}\{\exists t \geq 1 : B_{zr,3r} \cap wt\} r e^{-cr|z|}, \\
L_{v_0}^\perp(B_{zr,3r}) &\leq C \delta_{0,z} r^3 + C \sum_{w \in \Omega_{-v_0}} \mathbb{1}\{\exists t \geq 3r : B_{zr,3r} \cap wt\} r e^{-cr|z|},
\end{aligned} \tag{54}$$

and (30). With that the first two bounds are trivial. The third bound follows from:

$$\begin{aligned}
\sum_{z \in \mathbb{Z}^3} K(B_{zr,3r}) \tilde{L}_{v_0}(B_{zr,2r}) &\leq Cr^6 + Cr^4 \sum_{w \in \Omega_{-v_0}} \sum_{z \in (\mathbb{Z}^3)^*} \mathbb{1}\{\exists t \geq 3r : B_{zr,3r} \cap wt\} e^{-cr|z|} \\
&\leq Cr^6 + Cr^4 \sum_{w \in \Omega_{-v_0}} \sum_{z \in \mathbb{Z}^*} e^{-cr|vz|} \leq Cr^3,
\end{aligned}$$

where in the last line we approximate the sum by an integral in the same way as we did in (31).
 Note that by (54)

$$\sum_{z \in \mathbb{Z}^3} \tilde{L}_{v_0}^\perp(B_{zr,3r})M(B_{zr,2r}) \leq \sum_{z \in \mathbb{Z}^3} L_{v_0}^\perp(B_{zr,3r})M(B_{zr,2r}).$$

Moreover by (30) and (54)

$$\sum_{z \in \mathbb{Z}^3} L_{v_0}^\perp(B_{zr,3r})M(B_{zr,2r}) \leq Cr^6 + Cr^4 \sum_{w \in \Omega_{-v_0}} \mathbb{1}\{\exists t \geq 3r : B_{zr,3r} \cap wt\} e^{-2cr|z|} \leq Cr^3.$$

□

Proposition 3. The proof of Proposition 3 follows by inserting the bounds in Lemma 7 into (48) and then applying Lemma 8.

□

6 Proof of Proposition 2

In the setting of Section 4.3 the proof of Proposition 2 will follow from considering the following indicator functions

$$\begin{aligned} \tilde{\eta}_j &:= \mathbb{1} \left\{ \min_{\tilde{\tau}_{j-1} < t < \tilde{\tau}_j} (Z(t) - Z'_{j-2}) \in \mathcal{Q}_r \right\} \\ \hat{\eta}_j &:= \mathbb{1} \left\{ \min_{\tilde{\tau}_{j-3} < t < \tilde{\tau}_{j-2}} (Z(t) - Z(\tilde{\tau}_{j-1}) - \tilde{\beta}_{j-1}) \in \mathcal{Q}_r \right\} \\ \eta_j &:= \max\{\tilde{\eta}_j, \hat{\eta}_j\} \end{aligned} \tag{55}$$

In particular, η_j is the probability of a mismatch for the Z -process in immediately before the j^{th} leg. It is important to note, the simple geometric fact (which follows simply from the fact that the collision angles are bounded) that $\eta_j^* = 1$ implies $\xi_{j-1} < Cr$ for some constant $C < \infty$. This fact will make the geometric estimates vastly easier than for the Lorentz gas, where the equivalent statement is false.

The following statements will provide the proof of Proposition 2

$$\mathbf{P} \left(\left\{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \right\} \cap \left\{ \sum_{j=1}^{\gamma} \eta_j > 1 \right\} \right) \leq Cr^2, \tag{56}$$

$$\mathbf{P} \left(\left\{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \right\} \cap \left\{ \sum_{j=1}^{\gamma} \eta_j = 0 \right\} \right) \leq Cr^2, \tag{57}$$

$$\mathbf{P} \left(\left\{ \mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+ \right\} \cap \left\{ \sum_{j=1}^{\gamma} \eta_j = 1 \right\} \right) \leq Cr^2. \tag{58}$$

6.1 Proof of (56)

The simple geometric fact stated in the previous section implies

$$\mathbf{P} \left(\sum_{j=1}^{\gamma} \eta_j > 1 \right) \leq \frac{\gamma^2}{2} \max_{1 \leq j < k \leq \gamma} \mathbf{P}(\eta_j \eta_k = 1) \leq C\gamma^2 r^2.$$

(56) now follows from the exponential tail bounds (42). □

6.2 Proof of (57)

On $\left\{ \sum_{j=1}^{\gamma} \eta_j = 0 \right\}$, the process $\{t \mapsto Z(t)\}$ is distributed like a Markovian flight process. Hence the event in (57) can be written

$$\{\mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+\} \cap \left\{ \sum_{j=1}^{\gamma} \eta_j = 0 \right\} = \{\exists 3 \leq j \leq \gamma : \eta_j^o = 1\} \cap \left\{ \sum_{j=1}^{\gamma} \eta_j = 0 \right\}$$

where η_j^o is the indicator of an indirect mismatch, as defined in (36). Therefore using Lemma 5

$$\begin{aligned} \mathbf{P} \left(\{\mathcal{X}(t) \neq Z(t) : 0^- \leq t \leq \theta^+\} \cap \left\{ \sum_{j=1}^{\gamma} \eta_j = 1 \right\} \right) &\leq \mathbf{P}(\{\exists 3 \leq j \leq \gamma : \eta_j^o = 1\}) \\ &\leq \gamma \max_{3 \leq j \leq \gamma} \mathbf{P}(\eta_j^o = 1) \\ &\leq C\gamma^3 r^2. \end{aligned}$$

Thus (57) again follows from the exponential tail bounds (42). □

6.3 Proof of (58)

Given a $\gamma \in \{2\} \cup \{5, \dots\}$, a signature $\underline{\epsilon}$ (recall the definition of a signature given at the end of Subsection 2.2) compatible with the definition of a pack, and a fixed label $3 < k < \gamma$. Let $V_1, V_2 \in \Omega$ and let ϖ be a pack with signature $\underline{\epsilon}$ and $\tilde{w}_{k-2} = V_1$ and $\tilde{w}_{k+1} = V_2$ (we assume V_1 and V_2 are compatible with this definition).

- On $0^- < t \leq \tilde{\tau}_{k-1}$ - $Z^{(k)}(t) = Y(t)$, conditioned such that $\tilde{w}_{k-2} = V_1$.
- On $\tilde{\tau}_{k-1} < t \leq \tilde{\tau}_k$ - $Z^{(k)}(t)$ is constructed like the Z -process, conditioned such that the final velocity is $\tilde{w}_k \in \Omega_{V_2}$
- On $\tilde{\tau}_k < t < \tilde{\tau}_\gamma$ - $Z^{(k)}(t) = Y(t)$ a Markovian flight process starting at $Z^{(k)}(\tilde{\tau}_k)$, conditioned such that $\tilde{w}_{k+1} = V_2$.

On $\{\eta_j = \delta_{j,k} : 1 \leq j \leq \gamma\}$ - $Z^{(k)}$ is distributed like Z . The reason for conditioning on V_1 and V_2 is to ensure the following three parts are independent:

$$\begin{aligned} (Z^{(k)}(t) : 0^- < t \leq \tilde{\tau}_{k-3}) &= (Y(t) : 0^- < t \leq \tilde{\tau}_{k-3}), \\ (Z^{(k)}(\tilde{\tau}_{k-3} + t) - Z^{(k)}(\tilde{\tau}_{k-3}) : 0 \leq t \leq \tilde{\tau}_k - \tilde{\tau}_{k-3}), \\ (Z^{(k)}(\tilde{\tau}_k + t) - Z^{(k)}(\tilde{\tau}_k) : 0 \leq t < \theta^+ - \tilde{\tau}_k). \end{aligned} \quad (59)$$

Let $A_{a,a}^{(k)}$, $1 \leq a \leq 3$ be the event that the a -th part of the trajectory is r -inconsistent. For $1 \leq a < b \leq 3$ we denote $A_{a,b}^{(k)}$ the event that the a and b -th parts are r -incompatible. Therefore to prove (58) we will bound

$$\begin{aligned} \max_{\epsilon, k, V_1, V_2} \mathbf{P} \left(\{\hat{\eta}_k = 1\} \cap A_{a,b}^{(k)} \mid \epsilon, V_1, V_2 \right), \\ \max_{\epsilon, k, V_1, V_2} \mathbf{P} \left(\{\tilde{\eta}_k = 1\} \cap \{\hat{\eta}_k = 0\} \cap A_{a,b}^{(k)} \mid \epsilon, V_1, V_2 \right), \end{aligned} \quad a, b = 1, 2, 3. \quad (60)$$

6.4 Bounds

First notice that $A_{1,1}^{(k)}$, $A_{3,3}^{(k)}$ and $A_{1,3}^{(k)}$ involve only Markovian segments hence the following estimates follow readily from Lemmas 2, 3, 4, and 5:

$$\begin{aligned} \max_{\epsilon, k, V_1, V_2} \mathbf{P} \left(\{\hat{\eta}_k = 1\} \cap A_{a,b}^{(k)} \mid \epsilon, V_1, V_2 \right) &\leq C\gamma^3 r^2, \\ \max_{\epsilon, k, V_1, V_2} \mathbf{P} \left(\{\tilde{\eta}_k = 1\} \cap \{\hat{\eta}_k = 0\} \cap A_{a,b}^{(k)} \mid \epsilon, V_1, V_2 \right) &\leq C\gamma^3 r^2, \end{aligned} \quad a, b = 1, 3. \quad (61)$$

Therefore there remain 6 bounds.

Note that during middle segment in (59) the velocity of $Z^{(k)}(t)$ is restricted to only three possible velocities. Thus one component of the velocity remains unchanged throughout this segment. Therefore the middle segment can only be r -inconsistent if two of the path segments are shorter than Cr for some constant $C < \infty$. Thus

$$\begin{aligned} \mathbf{P} \left(\{\hat{\eta}_k = 1\} \cap A_{2,2}^{(k)} \mid \epsilon, V_1, V_2 \right) &\leq Cr^2, \\ \mathbf{P} \left(\{\tilde{\eta}_k = 1\} \cap \{\hat{\eta}_k = 0\} \cap A_{2,2}^{(k)} \mid \epsilon, V_1, V_2 \right) &\leq Cr^2. \end{aligned} \quad (62)$$

It remains to prove

$$\begin{aligned} \mathbf{P} \left(\{\hat{\eta}_k = 1\} \cap A_{b,2}^{(k)} \mid \epsilon, V_1, V_2 \right) &\leq C\gamma r^2, \\ \mathbf{P} \left(\{\tilde{\eta}_k = 1\} \cap \{\hat{\eta}_k = 0\} \cap A_{b,2}^{(k)} \mid \epsilon, V_1, V_2 \right) &\leq C\gamma r^2, \end{aligned} \quad b = 1, 3. \quad (63)$$

We will only prove (63) for $b = 3$ as the proof for $b = 1$ is the same. Given a set $A \subset \mathbb{R}^3$ define the following occupation measures for the third part of (59)

$$\begin{aligned}
G_{\underline{\epsilon}}^{(k)}(A) &:= \mathbf{E} \left(\#\{1 \leq j \leq \gamma - k : Z^{(k)}(\tilde{\tau}_{j+k}) - Z^{(k)}(\tilde{\tau}_k) \in A\} \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2 \right), \\
&\quad \mathbf{E} \left(\#\{1 \leq j \leq \gamma - k : \tilde{Y}(\tilde{\tau}_j) \in A\} \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2 \right), \\
H_{\underline{\epsilon}}^{(k)}(A) &:= \mathbf{E} \left(\left| \{\tau_j \leq \theta : Z^{(k)}(t) - Z^{(k)}(\tilde{\tau}_k) \in A\} \right| \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2 \right), \\
&\quad \mathbf{E} \left(\left| \{0 \leq t \leq \tau_{\gamma-k} : \tilde{Y}(t) \in A\} \right| \mid \epsilon_{k+j} : 1 \leq j \leq \gamma - k, V_2 \right),
\end{aligned}$$

where $t \mapsto \tilde{Y}(t)$ is a Markovian flight process with initial velocity in Ω_{V_2} . Similarly

$$\begin{aligned}
\hat{G}_{\underline{\epsilon}}^{(k)}(A) &:= \mathbf{E} \left(\#\{1 \leq j \leq 3 : Z^{(k)}(\tilde{\tau}_{k-j}) - Z^{(k)}(\tilde{\tau}_k) \in A\} \cdot \hat{\eta}_k \mid \underline{\epsilon}, V_1, V_2 \right), \\
\hat{H}_{\underline{\epsilon}}^{(k)}(A) &:= \mathbf{E} \left(\left| \{\tilde{\tau}_{k-3} \leq t \leq \tilde{\tau}_k : Z^{(k)}(t) - Z^{(k)}(\tilde{\tau}_k) \in A\} \right| \cdot \hat{\eta}_k \mid \underline{\epsilon}, V_1, V_2 \right), \\
\tilde{G}_{\underline{\epsilon}}^{(k)}(A) &:= \mathbf{E} \left(\#\{1 \leq j \leq 3 : Z^{(k)}(\tilde{\tau}_{k-j}) - Z^{(k)}(\tilde{\tau}_k) \in A\} \cdot \tilde{\eta}_k \cdot (1 - \hat{\eta}_k) \mid \underline{\epsilon}, V_1, V_2 \right), \\
\tilde{H}_{\underline{\epsilon}}^{(k)}(A) &:= \mathbf{E} \left(\left| \{\tilde{\tau}_{k-3} \leq t \leq \tilde{\tau}_k : Z^{(k)}(t) - Z^{(k)}(\tilde{\tau}_k) \in A\} \right| \cdot \tilde{\eta}_k \cdot (1 - \hat{\eta}_k) \mid \underline{\epsilon}, V_1, V_2 \right).
\end{aligned}$$

As the middle and last parts in (59) are independent the following bounds apply

$$\begin{aligned}
\mathbf{P} \left(\{\hat{\eta}_k = 1\} \cap A_{3,2}^{(k)} \mid \underline{\epsilon}, V_1, V_2 \right) &\leq Cr^{-1} \left(\int_{\mathbb{R}^3} G_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \hat{H}_{\underline{\epsilon}}^{(k)}(dx) + \int_{\mathbb{R}^3} H_{\underline{\epsilon}}^{(k)}(B_{x,3r}) \hat{G}_{\underline{\epsilon}}^{(k)}(dx) \right), \\
\mathbf{P} \left(\{\tilde{\eta}_k = 1\} \cap \{\hat{\eta}_k = 0\} \cap A_{3,2}^{(k)} \mid \underline{\epsilon}, V_1, V_2 \right) &\leq \\
&\leq Cr^{-1} \left(\int_{\mathbb{R}^3} G_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \tilde{H}_{\underline{\epsilon}}^{(k)}(dx) + \int_{\mathbb{R}^3} H_{\underline{\epsilon}}^{(k)}(B_{x,3r}) \tilde{G}_{\underline{\epsilon}}^{(k)}(dx) \right).
\end{aligned} \tag{64}$$

By (21) there exists a constant $C < \infty$ such that

$$G_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \leq CF(x), \quad H_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \leq CF(x) \tag{65}$$

where $F : \mathbb{R}^3 \rightarrow \mathbb{R}_+$

$$F(x) = r\{|x| \leq r\} + \frac{r^3}{|x|^2}\{r < |x| \leq 1\} + \frac{r^3}{|x|}\{|x| > 1\} + re^{-c|x|} \mathbf{1}\{\exists t > 0 : B_{x,2r} \cap tV_2\}\{|x| > r\}.$$

For simplicity we will only treat the first term on the right hand side in the second line of (64) (this is the most difficult), the other terms can be dealt with similarly.

Since during the middle section of (59) one component of the velocity does not change sign we can conclude

$$\hat{G}_{\underline{\epsilon}}^{(k)}(B_{0,s}), \tilde{G}_{\underline{\epsilon}}^{(k)}(B_{0,s}) \leq Crs, \quad \hat{H}_{\underline{\epsilon}}^{(k)}(B_{0,s}), \tilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,s}) \leq Crs, \tag{66}$$

and

$$\widehat{G}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3), \widetilde{G}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3) \leq Cr, \quad \widehat{H}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3), \widetilde{H}_{\underline{\epsilon}}^{(k)}(\mathbb{R}^3) \leq Cr. \quad (67)$$

First note that by (66)

$$\begin{aligned} \int_{|x|>r} r e^{-c|x|} \mathbb{1}\{\exists t > 0 : B_{x,2r} \cap tV_2\} \widetilde{H}_{\underline{\epsilon}}^{(k)}(dx) &\leq Cr^2 \int_{|x|>r} e^{-c|x|} \mathbb{1}\{\exists t > 0 : B_{x,2r} \cap tV_2\} dx \\ &\leq Cr^4 \int_{t>r} e^{-c|tV_2|} dt \leq Cr^4 \end{aligned}$$

and

$$\int_{|x|>1} \frac{r^3}{|x|} \widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{x,2r}) \leq Cr^4.$$

Finally let $\widetilde{F}(u) = r\{u \leq r\} + \frac{r^3}{u^2}\{r < u \leq 1\}$, then by applying integration by parts

$$\begin{aligned} \int_{\{|x|<1\}} \widetilde{F}(|x|) \widetilde{H}_{\underline{\epsilon}}^{(k)}(dx) &\leq C \int_0^1 \widetilde{F}(u) d\widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,u}) \\ &= Cr^3 \widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,1}) - C \int_0^1 \widetilde{H}_{\underline{\epsilon}}^{(k)}(B_{0,u}) \widetilde{F}'(u) du \\ &\leq Cr^4 + Cr^4 \int_r^1 u^{-2} du \\ &\leq Cr^4 + Cr^3. \end{aligned}$$

(63) follows by inserting these bounds into (64).

6.5 Proof of Theorem 2 - concluded

The proof of Theorem 2 now follows the same lines as [13, Section 7] repeated here for completeness.

Let $\{t \mapsto Y(t)\}$ be a Markovian flight process. Let $\{t \mapsto Z(t)\}$ be a coupled forgetful process. We split $\{t \mapsto Z(t)\}$ into i.i.d legs $(Z_n(t) : 0 \leq t \leq \theta_n)$, each associated to an i.i.d pack $\varpi_n = \left(\gamma_n; \{\widetilde{\xi}_{n,j}\}_{j=1}^\gamma, \{\widetilde{\beta}_{n,j}\}_{j=1}^\gamma, \{\widetilde{w}_{n,j}\}_{j=1}^\gamma\right)$. In addition, to each leg $(Z_n(t) : 0 \leq t \leq \theta_n)$ we associate a wind-tree process coupled to that leg $(\mathcal{X}_n(t) : 0 \leq t \leq \theta_n)$. From these components we construct the concatenated auxilliary process

$$\mathcal{X}(t) = \sum_{k=1}^{\nu_t} \mathcal{X}(\theta_n) + \mathcal{X}_{\nu_t+1}(\{t\}). \quad (68)$$

Note that $t \mapsto \mathcal{X}(t)$ is *not* a physical process. Each leg is independent of the others. Finally let $t \mapsto X(t)$ be the true wind-tree process, coupled to $t \mapsto Y(t)$ and $t \mapsto Z(t)$ as in Section 2.3.

We will use Propositions 2 and 3 to prove that until time $T = T(r) = o(r^{-2})$ the processes $t \mapsto X(t)$, $t \mapsto \mathcal{X}(t)$, and $t \mapsto Z(t)$ coincide with high probability.

For this define the (discrete) stopping times

$$\begin{aligned}\rho &:= \min\{n : \mathcal{X}_n(t) \neq Z_n(t), 0 \leq t \leq \theta_n\} \\ \sigma &:= \min\{n : \max\{\mathbb{1}_{\widetilde{W}_n}, \mathbb{1}_{\widehat{W}_n} > 0\} = 1\},\end{aligned}$$

and note that by construction

$$\inf\{t : Z(t) \neq X(t)\} \geq \Theta_{\min\{\rho, \sigma\}-1}.$$

Lemma 9. *Let $T = T(r)$ such that $\lim_{r \rightarrow \infty} T(r) = \infty$ and $\lim_{r \rightarrow \infty} r^2 T(r) = 0$. Then*

$$\lim_{r \rightarrow 0} \mathbf{P} \left(\Theta_{\min\{\rho, \sigma\}-1} < T \right) = 0. \quad (69)$$

Lemma 10. *Let $T = T(r)$ such that $\lim_{r \rightarrow \infty} T(r) = \infty$ and $\lim_{r \rightarrow \infty} r^2 T(r) = 0$. Then for any $\delta > 0$*

$$\lim_{r \rightarrow 0} \mathbf{P} \left(\max_{0 \leq t \leq T} |Y(t) - Z(t)| > \delta \sqrt{T} \right) = 0. \quad (70)$$

Proof of Lemma 9.

$$\begin{aligned}\mathbf{P} \left(\Theta_{\min\{\rho, \sigma\}-1} < T \right) &\leq \mathbf{P} \left(\rho \leq 2\mathbf{E}(\theta)^{-1} T \right) + \mathbf{P} \left(\sigma \leq 2\mathbf{E}(\theta)^{-1} T \right) + \mathbf{P} \left(\sum_{j=1}^{2\mathbf{E}(\theta)^{-1} T} \theta_j < T \right) \\ &\leq Cr^2 T + Cr^2 T + Ce^{-cT},\end{aligned} \quad (71)$$

where $C < \infty$ and $c > 0$. The first term on the right hand side of (71) is bounded by union bound and (45) from Proposition 2. Likewise the second term is bounded by union bound Proposition 3. In bounding the third term we use a large deviation upper bound for the sum of independent θ_j -s.

Finally (69) readily follows from (71). \square

Proof of Lemma 10. Note first that

$$\max_{0 \leq t \leq T} |Y(t) - Z(t)| \leq \sum_{j=1}^{\nu_T+1} \eta_j \left(\sum_{i=j}^{\gamma_{\nu'_j}} \xi_i \right),$$

with ν_T and η_j defined in (9), respectively, (33) and ν'_j is ν_j from (43) (the label of the leg containing j). Hence,

$$\begin{aligned}\mathbf{P} \left(\max_{0 \leq t \leq T} |Y(t) - Z(t)| > \delta \sqrt{T} \right) &\leq \mathbf{P} \left(\sum_{j=1}^{2T} \eta_j \left(\sum_{i=j}^{\gamma_{\nu'_j}} \xi_i \right) > \delta \sqrt{T} \right) + \mathbf{P}(\nu_T > 2T) \\ &\leq C\delta^{-1} \sqrt{T} r + e^{-cT},\end{aligned} \quad (72)$$

with $C < \infty$ and $c > 0$. The first term on the right hand side of (72) is bounded by Markov's inequality and the bound

$$\mathbf{E} \left(\eta_j \left(\sum_{i=j}^{\gamma_{\nu'_j}} \xi_i \right) \right) \leq Cr.$$

To see this recall the exponential tail bound for γ (42). The bound on the second term follows from a straightforward large deviation estimate on $\nu_T \sim POI(T)$.

Finally (70) readily follows from (72). □

(15) is a direct consequence of Lemmas 9 and 10 and this concludes the proof of Theorem 2. □

Acknowledgements

The work of BT was supported by EPSRC (UK) Fellowship EP/P003656/1 and by NKFI (HU) K-129170. CL was supported by EPSRC Studentship EP/N509619/1 1793795. We would like to thank Jens Marklof for helping identify some of the relevant literature.

References

- [1] A. Avila, P. Hubert: Recurrence for the wind-tree model. *Ann. I. H. Poincaré - AN* (2017)
- [2] P. Billingsley: *Convergence of Probability Measures* Wiley, New York, 1968
- [3] C. Boldrighini, L.A. Bunimovich, Y.G. Sinai: On the Boltzmann equation for the Lorentz gas. *J. Stat. Phys.* **32**: 477-501 (1983)
- [4] V. Delecroix: Divergent trajectories in the periodic wind-tree model. *Journal of Modern Dynamics* **7**: (2013)
- [5] V. Delecroix, P. Hubert, S. Lelièvre: Diffusion for the wind tree model. *Ann. Sci. Ec. Norm. Supér.* (4) **47**: no. 6, 1085-1110 (2014)
- [6] P. Ehrenfest, T. Ehrenfest: Begriffliche Grundlagen der statistischen Auffassung in der Mechanik *Encykl. d. Math. Wissensch.* **IV 2 II**, Heft 6, 90 S (1912) (Translated:) The conceptual foundations of the statistical approach in mechanics. *Dover Books on Physics* 9780486662503 (1959)
- [7] K. Frączek, C. Ulcigrai: Non-ergodic \mathbb{Z} -periodic billiards and infinite translation surfaces. *Invent. Math.* **197**: no. 2, 241-298 (2014)
- [8] G. Gallavotti: Divergencies and the approach to equilibrium in the Lorentz and the wind-tree models. *Phys. Rev.* **185**: 308-322 (1969)
- [9] G. Gallavotti: Rigorous theory of the Boltzmann equation in the Lorentz gas. *Nota Interna Univ di Roma* **358** (1970)

- [10] J. Hardy, J. Weber: Diffusion in a periodic wind-tree model. *Journal of Math. Phys.* **21**: 1802 (1980)
- [11] P. Hubert, S. Lelièvre, S. Troubetzkoy: The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion. *J. Reine Angew. Math.* **656**: 223-244 (2011)
- [12] H.A. Lorentz: The motion of electrons in metallic bodies. *Proc. Amstredam Acad.* **7**: 438, 585, 604 (1905)
- [13] C. Lutsko, B. Tóth: Invariance principle for the random Lorentz gas - beyond the Boltzmann-Grad limit. *arXiv:1812.11325 [math.PR]* (2019)
- [14] J. Marklof: The low-density limit of the Lorentz gas: periodic, aperiodic and random. In: *Proceedings of the International Congress of Mathematicians – 2014 Seoul* Vol. 3, 623-646, Kyung Moon Sa, Seoul, 2014.
- [15] J. Marklof, A. Strömbergsson: Kinetic theory for the low density Lorentz gas. *arXiv:1910.04982 [math.DS]* (2019)
- [16] H. Spohn: The Lorentz process converges to a random flight process. *Commun. Math. Phys.* **60**: 277-290 (1978)
- [17] S. Tabachnikov: *Billiards*. Panoramas et Synthèses, Société mathématique de France, 1995.

AUTHORS' ADDRESS:
School of Mathematics
University of Bristol
Bristol, BS8 1TW
United Kingdom
`chris.lutsko@bristol.ac.uk`
`balint.toth@bristol.ac.uk`